Differential Forms in Algebraic Topology: Chern-Weil Construction of Characteristic Classes. Pontryagin Classes

Sichuan University, Spring 2024

References

Main References

- Sections 20.4, 22.10, 23, 24, 26 & Appendix B of Tu2017.
- Section 22 of Bott-Tu.

Pullback Bundles

Setup

- $f : N \to N$ is a smooth map between smooth manifolds.
- $E \xrightarrow{\pi} M$ is a smooth vector bundle of rank k.

Definition

As a set the pullback bundle is

$$f^*E := \{(q, v) \in N \times E;\ f(q) = \pi(v)\}.$$

Lemma

f^*E is regular submanifold of $N \times E$ of dimension $\dim N + k$. In particular, this is a smooth manifold.

Remark

Let $\pi_1 : N \times E \to N$ and $\pi_2 : N \times E \to E$ the projections on the 1st and 2nd factors.

- By restriction we get smooth maps $\eta := \pi_{1|f^*E} : f^*E \to N$ and $\tilde{f} := \pi_{2|f^*E} : f^*E \to E$.
- We then have a commutative diagram,

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M. \end{array}$$

- That is, $\tilde{f}$ is a lift of f.

Pullback Bundles

Consequence

If $s : M \to E$ is a smooth section, then the map

$$f^*s : N \longrightarrow f^*E, \quad q \longrightarrow (q, s(f(q))),$$

is smooth and $\eta \circ (f^*s) = \mathbb{1}_N$.

Pullback Bundles

Facts

- The map $\eta : f^*E \to N$ is a surjective submersion.
- Given any $q \in N$, we have
$$\eta^{-1}(q) = \{(q, v);\ v \in E \text{ s.t. } \pi(v) = f(q)\}.$$
- As $\pi(v) = f(q) \Leftrightarrow v \in \pi^{-1}(f(q)) = E_{f(q)}$, we get
$$\eta^{-1}(q) = \{q\} \times E_{f(q)}.$$
- In particular, $\eta^{-1}(q)$ is a vector space that is naturally identified with $E_{f(q)}$.

Remark

At the set theoretic level, we then have
$$f^*E = \bigsqcup_{q \in N} \eta^{-1}(q) \simeq \bigsqcup_{q \in N} E_{f(q)}.$$

Pullback Bundles

Proposition

$f^*E \xrightarrow{\eta} f^*E$ *is a smooth vector bundle.*

Remark

The smooth vector bundle structure is such that, if $q \in N$ and $(e_1, \ldots, e_k)$ is a smooth frame of E over an open $U \ni f(q)$, then $(f^*e_1, \ldots, f^*e_k)$ is smooth frame of f^*E over $f^{-1}(U)$.

Pullback Bundles

Proposition

Let $\nabla : \mathscr{X}(M) \times \Gamma(E) \to \Gamma(E)$ *be a connection on* E.

1. *There is a unique connection* $f^*\nabla : \mathscr{X}(N) \times \Gamma(f^*E) \to \Gamma(f^*E)$ *on* f^*E *such that*
$$(f^*\nabla)_X (f^*s) = f^* (\nabla_{f_*X} s) \qquad \forall X \in \mathscr{X}(N)\ \forall s \in \Gamma(E).$$
2. *If* $\omega = (\omega^i_j)$ *is the connection 1-form of* ∇ *relative to a* C^∞*-fram* $(e_1, \ldots, e_k)$ *of* E*, then the connection 1-form of* $f^*\nabla$ *relative to the frame* $(f^*e_1, \ldots, f^*e_k)$ *is* $f^*\omega = (f^*\omega^i_j)$.

Isomorphisms of Vector Bundles

Setup

- $E_1 \overset{\pi}{:\to} M$ and $E_2 \overset{\pi}{:\to} N$ are smooth vector bundles over M.

Reminder (Bundle Maps)

A smooth bundle map $\varphi : E_1 \to E_2$ is a smooth map such that:

(i) $\pi \circ \varphi = \pi$, i.e., $\varphi(E_{1,p}) \subseteq E_{2,p}$ for all $p \in M$.

(ii) For every $p \in M$, the induced map $\varphi_p : E_{1,p} \to E_{2,p}$ is linear.

Definition

A bundle map $\varphi : E_1 \to E_2$ is an isomorphism if this is a bijection and its inverse is a (smooth) bundle map.

Remark

Let $\varphi : E_1 \to E_2$ be a bundle map.

- This is an isomorphism if and only if, for each $p \in M$, the induced map $\varphi_p : E_{1,p} \to E_{2,p}$ is a linear isomorphism.
- In particular, if φ is an isomorphism, then E_1 and E_2 must have the same rank.

Proposition

1. *If $f : N \to M$ and $g : P \to N$ are smooth maps, then we have a canonical vector bundle isomorphism,*

$$g^* (f^* E) \simeq (f \circ g)^* E.$$

2. *In particular, if $f : M \to M$ is an isomorphism, then $f^* E \simeq E$.*

Notation

Let M be a smooth manifold.

1. $\mathrm{Vect}(M)$ is the set of isomorphism classes of smooth vector bundles over M.
2. $\mathrm{Vect}_k(M)$, $k \geq 1$, is the subset of isomorphism classes of vector bundles of rank k.

Remark

- By pullback any smooth map $f : N \to M$ gives rise to a natural map,
$$f^* : \mathrm{Vect}(M) \longrightarrow \mathrm{Vect}(N).$$
- It follows that $M \to \mathrm{Vect}(M)$ is a functor from the category of manifolds to the category of sets.

Definition

A characteristic class is the datum for each smooth manifold M of a map

$$c_M : \mathrm{Vect}(M) \longrightarrow H^*(M)$$

which is natural in the sense that, for every smooth map $f : N \to M$ between smooth manifolds, we have

$$c_N(f^*E) = f^* c_M(E) \qquad \forall E \in \mathrm{Vect}(M).$$

Remark

In other words, a characteristic class is a natural transformation $c : \mathrm{Vect}(\cdot) \to H^*(\cdot)$ from the Vect-functor to the de Rham cohomology functor.

Invariant Polynomials

Setup

- $\mathfrak{gl}_r(\mathbb{R})$ is the Lie algebra of $n \times n$ matrices with real entries.
- $\mathrm{GL}_r(\mathbb{R})$ is the Lie group of invertible $n \times n$ matrices with real entries.

Fact

$\mathrm{GL}_r(\mathbb{R})$ acts on $\mathfrak{gl}_r(\mathbb{R})$ by the adjoint action,

$$\mathrm{Ad} : \mathrm{GL}_r(\mathbb{R}) \times \mathfrak{gl}_r(\mathbb{R}) \longrightarrow \mathfrak{gl}_r(\mathbb{R}),$$
$$\mathrm{Ad}_A(X) = A^{-1}XA, \qquad (A, X) \in \mathrm{GL}_r(\mathbb{R}) \times \mathfrak{gl}_r(\mathbb{R}).$$

Setup

- $\mathbb{R}[x^i_j]$ is the algebra of polynomials in the indeterminate $r \times r$ matrix $X = (x^i_j)$.

Remark

- We have an adjoint action of $\mathrm{GL}_r(\mathbb{R})$ on $\mathbb{R}[x^i_j]$.

$$\mathrm{Ad} : \mathrm{GL}_r(\mathbb{R}) \times \mathbb{R}[x^i_j] \longrightarrow \mathbb{R}[x^i_j],$$
$$\mathrm{Ad}_A(P) = P(A^{-1}XA), \qquad (A, P) \in \mathrm{GL}_r(\mathbb{R}) \times \mathbb{R}[x^i_j].$$

- We then have

$$(\mathrm{Ad}_A P)(X) = P(A^{-1}XA) = P(\mathrm{Ad}_A X) \quad \forall X \in \mathfrak{gl}_r(\mathbb{R}).$$

Invariant Polynomials

Definition

A polynomial $P(X) \in \mathbb{R}[x^i_j]$ is invariant (or $\mathrm{Ad}\,\mathrm{GL}_r(\mathbb{R})$-invariant) if

$$\mathrm{Ad}_A(P) = P \qquad \forall A \in \mathrm{GL}_r(\mathbb{R}).$$

Remark

Equivalently, polynomial $P(X) \in \mathbb{R}[x^i_j]$ is invariant if

$$P(A^{-1}XA) = P(X) \qquad \forall A \in \mathrm{GL}_r(\mathbb{R}) \quad \forall X \in \mathfrak{gl}_r(\mathbb{R}).$$

Definition

The set of $\mathrm{Ad}\,\mathrm{GL}_r(\mathbb{R})$-invariant polynomials is denoted $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$.

Remark

$\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ is a sub-ring of $\mathbb{R}[x^i_j]$.

Invariant Polynomials

Example

The trace $\mathrm{Tr}[X]$ and the determinant $\det(X)$ are invariant polynomials.

Example

Given any $k \geq 1$, the polynomial $\Sigma_k(X) := \mathrm{Tr}[X^k]$ is an invariant polynomial. It is called the k-th trace polynomial.

Example

- The characteristic polynomial of $-X$ is
$$\det(\lambda + X) = \lambda^r + f_1(X)\lambda^{r-1} + \cdots + f_{r-1}(X)\lambda + f_r(X),$$
where $f_1(X), \ldots, f_r(X)$ are polynomials in $\mathbb{R}[x^i_j]$ with $f_r(X) = \det(X)$.
- If $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of X counted with multiplicity, then
$$f_k(X) = \sum_{i_1 \leq \cdots \leq i_r} \lambda_{i_1} \cdots \lambda_{i_r}.$$
- The $\mathrm{Ad}\,\mathrm{GL}_r(\mathbb{R})$-invariance of $\det(\lambda + X)$ implies the invariance of the coefficients $f_1(X), \ldots, f_r(X)$.
- The invariants polynomials $f_1(X), \ldots, f_r(X)$ are called the coefficients of the characteristic polynomials of $-X$.

Invariant Polynomials

Theorem

1. *As a ring* $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ *is generated by the trace polynomials* $\mathrm{Tr}[X], \ldots, \mathrm{Tr}[X^r]$.
2. *It is also generated by the coefficients* $f_1(X), \ldots, f_r(X)$ *of the characteristic polynomial* $\det(\lambda + X)$.

Remark

- A polynomial $P \in \mathbb{R}[x^i_j]$ is a linear combinations of monomials,
$$E^I_J(X) := x^{i_1}_{j_1} \cdots x^{i_k}_{j_k},$$
where $I = (i_1, \ldots, j_k)$ and $J = (j_1, \ldots, j_k)$ range over $\{1, \ldots, r\}^k$ and k ranges over non-negative integers.
- If $\lambda \in \mathbb{R}$, then $\lambda X = (\lambda x^i_j)$, and so we have
$$E^I_J(\lambda X) := (\lambda x^{i_1}_{j_1}) \cdots (\lambda x^{i_k}_{j_k}) = \lambda^k E^I_J(X).$$

Invariant Polynomials

Definition

A polynomial $P \in \mathbb{R}[x^i_j]$ is homogeneous of degree k, $k \geq 0$, if

$$P(\lambda X) = \lambda^k P(X) \qquad \forall \lambda \in \mathbb{R}.$$

Remark

k must be a non-negative integer.

Lemma

Let $P \in \mathbb{R}[x^i_j]$.

(i) *P has a unique decomposition,*

$$P(X) = P_0(X) + P_1(X) + \cdots + P_m(X),$$

where $P_k(X) \in \mathbb{R}[x^i_j]$ is homogeneous of degree k.

(ii) *If $P(X)$ is $\mathrm{Ad}\,\mathrm{GL}_r(\mathbb{R})$-invariant, then all the homogeneous components $P_0(X), \ldots, P_m(X)$ are $\mathrm{Ad}\,\mathrm{GL}_r(\mathbb{R})$-invariant.*

Invariant Polynomials

Setup

- $\mathscr{A}$ is an $\mathbb{R}$-algebra with unit $1_{\mathscr{A}}$.
- We have a natural embedding $\mathbb{R} \ni \lambda \hookrightarrow \lambda 1_{\mathscr{A}} \in \mathscr{A}$.
- $\mathfrak{gl}_r(\mathscr{A})$ is the algebra of $r \times r$-matrices with entries in $\mathscr{A}$.

Remark

- The embedding $\mathbb{R} \hookrightarrow \mathscr{A}$ gives rise to an algebra embedding $\mathfrak{gl}_r(\mathbb{R}) \hookrightarrow \mathfrak{gl}_r(\mathscr{A})$.
- We thus have an $\operatorname{Ad} \mathrm{GL}_r(\mathbb{R})$-action on $\mathfrak{gl}_r(\mathscr{A})$,

$$\operatorname{Ad} : \mathrm{GL}_r(\mathbb{R}) \times \mathfrak{gl}_r(\mathscr{A}) \longrightarrow \mathfrak{gl}_r(\mathscr{A}), \qquad \operatorname{Ad}_A(X) = A^{-1}XA.$$

Theorem

If $P \in \operatorname{Inv}(\mathfrak{g}_r(\mathbb{R}))$, *then*

$$P(A^{-1}XA) = P(X) \qquad \forall A \in \mathrm{GL}_r(\mathbb{R})\ \forall X \in \mathfrak{gl}_r(\mathscr{A}).$$

Setup

- $P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ is a homogeneous polynomial of degree k.

Setup

- M is a smooth manifold of dimension n.
- $E \xrightarrow{\pi} M$ is a smooth vector bundle of rank r.
- ∇ is a connection on E.

Chern-Weil Construction of Characteristic Classes

Setup

- $(e_1, \ldots, e_r)$ is a C^∞-frame of E over an open $U \subseteq M$.
- $\Omega = (\Omega^i_j)$ is the curvature matrix of ∇ relative to $(e_1, \ldots, e_r)$.

Facts

- The exterior algebra $\Omega^*(U) = \oplus \Omega^j(U)$ is an $\mathbb{R}$-algebra.
- Each curvature form $\Omega^i_j \in \Omega^2(U)$, and so $\Omega \in \mathfrak{gl}_r(\Omega^*(U))$.
- Therefore, we may define

$$P(\Omega) \in \Omega^*(U).$$

Facts

- As $P(X)$ is a homogeneous of degree k it is linear combinations of monomials,
$$x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_k}^{i_k},$$
where $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ range over $\{1, \ldots, r\}^k$.
- Thus, $P(\Omega)$ is a linear combination of monomials,
$$\Omega_{j_1}^{i_1} \wedge \Omega_{j_2}^{i_2} \wedge \cdots \wedge \Omega_{j_k}^{i_k} \in \Omega^{2k}(U).$$
- It then follows that
$$P(\Omega) \in \Omega^{2k}(U).$$
- In particular, $P(\Omega) = 0$ for $k > \frac{1}{2} n$.

Facts

- If $p \in U$, then $\Lambda^* T_p^* M$ is an $\mathbb{R}$-algebra as well.
- $\Omega_p = ((\Omega_j^i)_p) \in \mathfrak{gl}_r(\Lambda^* T_p^* M)$.
- Therefore, we may also define $P(\Omega_p)$, and we then have

$$P(\Omega_p) = P(\Omega)_p \in \Lambda^{2k} T_p^* M.$$

Setup

- $(\overline{e}_1, \ldots, \overline{e}_r)$ is another C^∞-frame of E over U.
- We then may write $e_j = \sum a^i_j e_i$, with $a = (a^i_j) \in C^\infty(U, \mathrm{GL}_r(\mathbb{R}))$.
- $\overline{\Omega}$ is the curvature matrix of ∇ relative to $(\overline{e}_1, \ldots, \overline{e}_r)$.

Reminder

- We have
$$\overline{\Omega} = a^{-1}\Omega a.$$
- In particular, at p we have
$$\overline{\Omega}_p = a(p)^{-1}\Omega_p a_p = \mathrm{Ad}_{a(p)}\,\Omega_p.$$

Lemma

$P(\overline{\Omega})_p = P(\Omega)_p$.

Proof.

Thanks to the $\mathrm{Ad}\, \mathrm{GL}_r(\mathbb{R})$-invariance of P, we have

$$P(\overline{\Omega})_p = P(\overline{\Omega}_p) = P(\mathrm{Ad}_{a(p)}\, \Omega_p) = P(\Omega_p) = P(\Omega)_p.$$

□

Consequence

$P(\Omega)_p$ does not depend on the choice of the frame $(e_1, \ldots, e_r)$ near p.

To sum up we have proved:

Proposition

If $P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ is homogeneous of degree k, then there is a unique differential form $P(\Omega) \in \Omega^k(U)$, such that, given any $p \in M$, we have

$$P(\Omega)_p = P(\Omega_p),$$

where Ω is the curvature matrix of ∇ relative to any C^∞-frame of E near p.

Remark

If $\mathscr{A}$ is an $\mathbb{R}$-algebra with unit, then, given any $X \in \mathfrak{gl}_r(\mathscr{A})$, the map $\mathbb{R}[x^i_j] \ni P \to P(X) \in \mathfrak{gl}_r(\mathscr{A})$ is an algebra map.

Lemma

If $P \in \mathrm{Inv}(\mathfrak{g}_r(\mathbb{R}^n))$ is homogeneous of degree k and $Q \in \mathrm{Inv}(\mathfrak{g}_r(\mathbb{R}^n))$ is homogeneous of degree ℓ, then

$$(PQ)(\Omega) = P(\Omega) \wedge Q(\Omega) \in \Omega^{k+\ell}(M).$$

Remark

- The above considerations can be extended to non-homogeneous polynomials in $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$.
- If $P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ and we put $P = P_0 + \cdots + P_m$, where P_k is homogeneous of degree k, then we define

$$P(\Omega) := P_0(\Omega) + \cdots + P_m(\Omega) \in \Omega^*(M).$$

Therefore, we arrive at the following result:

Proposition

The data of (E, ∇) define a graded algebra map,

$$\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R})) \longrightarrow \Omega^*(M), \qquad P \longrightarrow P(\Omega).$$

Proposition

Let $P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ be a homogeneous of degree k.

1. *The differential form $P(\Omega) \in \Omega^{2k}(M)$ is closed.*
2. *Its cohomology class $[P(\Omega)] \in H^{2k}(M)$ does not depend on ∇.*

Remark

If $P \in \mathrm{Inv}(\mathfrak{gl}_r(M))$ and we put $P = P_0 + \cdots + P_m$, with P_k homogeneous of degree k, then

$$[P(\Omega)] = [P_0(\Omega)] + \cdots + [P_m(\Omega)] \in H*(M)$$

does not depend on ∇.

Therefore, we arrive at the following result:

Theorem

The datum of E uniquely define a graded algebra map,

$$c_E : \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R})) \longrightarrow H^*(M), \qquad P \longrightarrow [P(\Omega)].$$

This map is called the Chern-Weil homomorphism.

Remark

Let $\varphi : F \to E$ be a vector bundle isomorphism.

- We have a connection $\varphi^*\nabla$ on F given by
$$(\varphi^*\nabla)_X s = \varphi^{-1}(\nabla_X(\varphi \circ s)), \qquad X \in \mathscr{X}(M),\ s \in \Gamma(F).$$
- If $(f_1, \ldots, f_r)$ is a local frame of E, then $\varphi(f_1), \ldots, \varphi(f_r)$.
- If ω is a connection matrix of ∇ relative to $(\varphi(f_1), \ldots, \varphi(f_r))$, then this is also the connection matrix of $\varphi^*\nabla$ relative to $(f_1, \ldots, f_r)$.
- It follows that $\varphi^*\nabla$ and ∇ have the same curvature matrix.
- We then deduce that, for all $P \in \operatorname{Inv}(\mathfrak{gl}_r(\mathbb{R}))$, we have
$$c_F(P) = [P(\Omega)] = c_E(P).$$
- This shows that the Chern-Weil homomorphism c_E depends only on the isomorphism class of E

Lemma

If $f : N \to M$ *be a smooth map, then*

$$c_{f^*E}(P) = f^* c_E(P) \qquad \forall P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R})).$$

Proof.

- Let ∇ be a connection on E. Then $f^*\nabla$ is a connection on f^*E.
- Let $(e_1, \ldots, e_r)$ be a C^∞-frame of E over an open $U \subseteq M$.
- If ω is the connection matrix of ∇ relative to this frame, then the connection matrix of $f^*\nabla$ is $\overline{\omega} = f^*\omega$.
- If Ω and $\overline{\Omega}$ be the curvature matrices of ∇ and $f^*\nabla$, then

$$\overline{\Omega} = d\overline{\omega} + \overline{\omega} \wedge \overline{\omega} = f^*\omega + f^*\omega \wedge f^*\omega = f^*(d\omega + \omega \wedge \omega) = f^*\Omega.$$

- Therefore, on $f^{-1}(U)$ we have

$$P(\overline{\Omega}) = P(f^*\Omega) = f^*P(\Omega).$$

- It then follows that

$$c_{f^*E}(P) = [P(\overline{\Omega})] = [f^*P(\Omega)] = f^*[P(\Omega)] = f^*c_E(P).$$

The result is proved. □

Combining all this we then arrive at the following result:

Theorem

The datum of any invariant polynomial $P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ *defines a characteristic class,*

$$c_M : \mathrm{Vect}_r(M) \longrightarrow H^*(M), \qquad E \longrightarrow c_E(P).$$

Vanishing of Characteristic Classes

Reminder

- The algebra $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ is generated by the trace polynomials $\mathrm{Tr}[X^k]$, $k = 1, \ldots, r$.

Lemma

Suppose that $\mathscr{A}$ be an $\mathbb{R}$-algebra with unit. Let $A = (a^i_j) \in \mathfrak{gl}_r(\mathscr{A})$ be skew-symmetric, i.e., $A^T = -A$.

(i) $\mathrm{Tr}[A] = a^1_1 + \cdots + a^r_r = 0$.

(ii) *If k is odd, then A^k is skew-symmetric, and hence $\mathrm{Tr}[A^k] = 0$.*

Proof.

- We have $\mathrm{Tr}[A] = \mathrm{Tr}[A^T] = -\mathrm{Tr}[A] = 0$.
- If k is odd, then

$$(A^k)^T = (A^T)^k = (-A)^k = (-1)^k A^k = -A^k.$$

- That is, A^k is skew-symmetric, and hence $\mathrm{Tr}[A^k] = 0$. □

Vanishing of Characteristic Classes

Lemma

If k is odd, then the characteristic class $[\mathrm{Tr}[\Omega^k]]$ vanishes.

Proof.

- Every vector bundle admits a Riemannian metric and a connection compatible with that metric.
- We may assume that E has a Riemannian metric and ∇ is a metric connection.
- If $(e_1, \ldots, e_r)$ is an orthonormal frame of E over an open U, then the curvature matrix $\Omega = (\Omega^i_j)$ is skew-symmetric.
- By the previous lemma, for all $p \in U$, we then have
$$\mathrm{Tr}[\Omega^k]_p = \mathrm{Tr}[(\Omega_p)^k] = 0.$$
- It then follows that $\mathrm{Tr}[\Omega^k] = 0$.
- As the class $[\mathrm{Tr}[\Omega^k]] \in H^{2k}(M)$ does not depend on the connection, this gives the result.

□

Vanishing of Characteristic Classes

Facts

- Each trace polynomial $\Sigma_k(X) = \mathrm{Tr}[X^k]$, $k = 1, \ldots, r$, is homogeneous of degree k.
- These polynomials generate $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$.
- Therefore, if $P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ is homogeneous of degree k, then it is a linear combination of products,

$$\Sigma_1^{\alpha_1}\Sigma_2^{\alpha_2}\cdots\Sigma_r^{\alpha_r}, \qquad \alpha_1 + 2\alpha_2 + \cdots + r\alpha_r = k.$$

- If k is odd, then there must be $j \in \{1, \ldots r\}$ such that j is odd and $\alpha_j \neq 0$.
- As $[\Sigma_j(\Omega)] = [\mathrm{Tr}[\Omega^j]] = 0$ since j is odd, we then have

$$\left[\Sigma_1^{\alpha_1}\cdots\Sigma_r^{\alpha_r}(\Omega)\right] = [\Sigma_j(\Omega)]^{\alpha_j}\left[\Sigma_1^{\alpha_1}\cdots\Sigma_{j-1}^{\alpha_{j-1}}\Sigma_{j+1}^{\alpha_{j+1}}\cdots\Sigma_r^{\alpha_r}(\Omega)\right] = 0.$$

Therefore, we arrive at the following result:

Proposition

If $P \in \mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ is homogeneous of odd degree, then the corresponding characteristic class $[P(\Omega)] \in H^{2k}(M)$ vanishes.

Vanishing of Characteristic Classes

Reminder

The algebra $\mathrm{Inv}(\mathfrak{gl}_r(\mathbb{R}))$ is generated by the following homogeneous polynomials:

- The trace polynomials $\mathrm{Tr}[X], \ldots, \mathrm{Tr}[X^r]$.
- The coefficients $f_k(X), \ldots, f_r(X)$ of the characteristic polynomial of $-X$.

Consequence

Set $\ell = [r/2]$. The characteristic classes of E are generated by the following classes:

- The even degree trace polynomials,

$$\big[\,\mathrm{Tr}[\Omega^2]\big],\ \big[\,\mathrm{Tr}[\Omega^4]\big],\ \ldots,\big[\,\mathrm{Tr}[\Omega^{2\ell}]\big].$$

- The even degree coefficients of the characteristic polynomial $\det(\lambda I + \Omega)$,

$$f_2(\Omega),\ f_4(\Omega),\ \ldots, f_{2\ell}(\Omega).$$

Pontryagin Classes

Definition

The k-th Pontryagin class of E is

$$p_k(E) := \left[f_{2k}\left(\frac{i}{2\pi}\Omega\right)\right] \in H^{4k}(M).$$

Definition

Set $\ell = [r/2]$. The total Pontryagin class of E is

$$\begin{aligned} p(E) &:= \det\left[I + \frac{i}{2\pi}\Omega \right] \\ &= 1 + p_1(E) + \cdots + p_\ell(E). \end{aligned}$$

Theorem (Whitney Product Formula)

If E and E' are smooth vector bundles over M, then

$$p(E \oplus E') = p(E)p(E').$$

Pontryagin Numbers

Setup

- M is a compact oriented manifold of dimension $4m$.
- E is a smooth vector bundle over M of rank r.

Facts

Set $\ell = [r/2]$.

- If $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are non-negative integers, then the product $f_2^{\alpha_1} f_4^{\alpha_2} \cdots f_{2\ell}^{\alpha_\ell}$ is homogeneous of degree
$$k = 2\alpha_1 + 4\alpha_2 + \cdots + 2\ell\alpha_\ell.$$
- Thus,
$$p_1(E)^{\alpha_1} \cdots p_\ell(E)^{\alpha_\ell} = \left[\left(f_2^{\alpha_1} \cdots f_{2\ell}^{\alpha_\ell} \right) \left(\frac{i}{2\pi} \Omega \right) \right] \in H^{2k}(M).$$
- In particular, if $\alpha_1 + 2\alpha_2 + \cdots + \ell\alpha_\ell = m$, then $2k = 4m$, and hence
$$p_1(E)^{\alpha_1} p_2(E)^{\alpha_2} \cdots p_\ell(E)^{\alpha_\ell} \in H^{4m}(M).$$

Pontryagin Numbers

Definition

The integrals,

$$\int_M p_1(E)^{\alpha_1} p_2(E)^{\alpha_2} \cdots p_\ell(E)^{\alpha_\ell}, \qquad \alpha_1 + 2\alpha_2 + \cdots \ell\alpha_\ell = m,$$

are called Pontryagin numbers of E.

Remark

1. If $\alpha_1 + 2\alpha_2 + \cdots \ell\alpha_\ell = m$, then we must have $\alpha_j = 0$ for $j > m$.
2. Therefore, only the Pontryagin classes $p_k(E)$ with $k \leq \min(\ell, m)$ comes into play in the definition of the Pontryagin numbers.

Pontryagin Numbers

Remark

- If $E = TM$, then $r = 4m$, and hence $\ell = 2m$.
- We then have $\min(\ell, m) = m$.

Definition

The Pontryagin numbers of M are the Pontryagin numbers of TM, i.e., the integrals,

$$\int_M p_1(TM)^{\alpha_1} p_2(TM)^{\alpha_2} \cdots p_m(TM)^{\alpha_m}, \quad \alpha_1 + 2\alpha_2 + \cdots m\alpha_m = m.$$

Pontryagin Numbers

Example

If $\dim M = 4$ (i.e., $m = 1$), then the only Pontryagin number is

$$\int_M p_1(TM).$$

Example

If $\dim M = 8$ (i.e., $m = 2$), then the only Pontryagin numbers are

$$\int_M p_1(TM)^2 \qquad \text{and} \qquad \int_M p_2(TM).$$

The Cobordism Problem

Question

What manifolds appear as boundaries of manifolds with boundary?

Theorem

Assume that $M = \partial N$, where N is a compact oriented manifold with boundary. Then all the Pontryagin numbers of M vanish.

Corollary

If M has a non-zero Pontryagin number, then M cannot be the boundary of a compact oriented manifold.

Idea of Proof.

- By Stokes' theorem, for any closed form $\omega \in \Omega^{4m}(N)$, we have
$$\int_{\partial N} [\omega] = \int_{\partial N} \omega = \int_N d\omega = 0.$$
- It can be shown that the Pontryagin classes on ∂N are restrictions of Pontryagin classes on N.
- The Pontryagin classes are represented by closed forms, and so are their products.
- Thus, if $\alpha_1 + \cdots + m\alpha_m = m$, then
$$\int_{\partial N} p_1^{\alpha_1} \cdots p_m^{\alpha_m} = \int_M d(p_1^{\alpha_1} \cdots p_m^{\alpha_m}) = 0.$$

This gives the result. □

The Cobordism Problem

Remark

If a manifold M is oriented, then $-M$ is the manifold M with the opposite orientation.

Definition

If M_1 and M_2 are compact oriented manifolds, then we say that M_1 and M_2 are cobordant if there is a compact oriented manifold with boundary N such that $\partial N = M_1 \sqcup (-M_2)$.

Remark

More precisely, this means there are smooth embeddings $i_1 : M_1 \to \partial N$ such that

- $i_1(M_1)$ and $i_2(M_2)$ are disjoint open subsets of ∂N such that $i_1(M_1) \cup i_2(M_2) = \partial N$.
- i_1 is orientation-preserving and i_2 is orientation-reversing.

Example

Any compact oriented manifold M is cobordant to itself, since for $N = M \times [0, 1]$, we have

$$\partial N = M \sqcup (-M).$$

The Cobordism Problem

Theorem

If M_1 and M_2 are compact oriented manifolds of dimension $4m$ that are cobordant, then their Pontryagin numbers agree.

Corollary

If M_1 and M_2 don't have the same Pontryagin numbers, then they cannot be cobordant.

The Cobordism Problem

Proof.

- Suppose that $M_1 \sqcup (-M_2) = \partial N$, where N is a compact oriented manifold with boundary.
- If $\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m = m$, then

$$\int_{M_1} p_1^{\alpha_1} \cdots p_m^{\alpha_m} - \int_{M_2} p_1^{\alpha_1} \cdots p_m^{\alpha_m} = \int_{M_1 \sqcup (-M_2)} p_1^{\alpha_1} \cdots p_m^{\alpha_m}$$
$$= \int_N p_1^{\alpha_1} \cdots p_m^{\alpha_m}$$
$$= 0$$

This gives the result. □

The Embedding Problem

Theorem (Whitney)

Any smooth manifold of dimension n has an embedding into $\mathbb{R}^{2n+1}$.

Question

What manifolds of dimension n can be embedded into $\mathbb{R}^{n+1}$?

Theorem

If M is a compact oriented manifold of dimension $4m$ that can be embedded into $\mathbb{R}^{4m+1}$, then all its Pontryagin classes vanish.

Corollary

If one of the Pontryagin classes of M is non-zero, then M cannot be embedded into $\mathbb{R}^{4m+1}$.

The Embedding Problem

Sketch of Proof.

- By definition, the total Pontryagin class of M is $p(TM) = 1 + p_1(TM) + \cdots + p_m(TM)$.
- Thus, the Pontryagin classes vanish if and only if $p(TM) = 1$.
- If M is an oriented submanifold of codimension 1 in $\mathbb{R}^{4m+1}$, then it can be shown there is a nowhere vanishing vector field ν along M such that

$$T(\mathbb{R}^{4m+1})_{|M} = TM \oplus N, \qquad \text{where } N = \mathbb{R}\nu.$$

- Whitney product formula then gives

$$p\big(T(\mathbb{R}^{4m+1})_{|M}\big) = p(TM)p(N).$$

- As N is a line bundle $p(N) = 1$.
- By naturality $p(T(\mathbb{R}^{4m+1})_{|M}) = p(T(\mathbb{R}^{4m+1})) = 0$.
- It then follows that $p(TM) = p(T(\mathbb{R}^{4m+1})_{|M}) = 1$, which gives the result.

□