Differential Forms in Algebraic Topology: Review: Bump Functions and Partitions of Unity

Sichuan University, Spring 2024

References

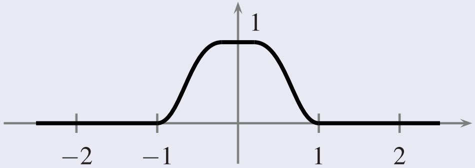
Main References

- Section 13 of Tu2011.
- Section 2 of Bott-Tu.

Definition

Given a point q in a manifold M and an open neighborhood U of q in M, a bump function at q supported in U is any continuous function $\rho : M \to \mathbb{R}$ such that:

$$0 \le \rho \le 1, \qquad \rho = 1 \text{ near } q, \qquad \operatorname{supp}(\rho) \subseteq U.$$



C^∞ Bump Function on $\mathbb{R}$

Step 1 (Example 1.3 and Problem 1.2)

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} e^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

- By definition $0 \leq f \leq 1$ and $f(t) = 0$ if and only if $t \leq 0$.
- It can also be shown that f is smooth (see Example 1.3 and Problem 1.2 in Tu2011).

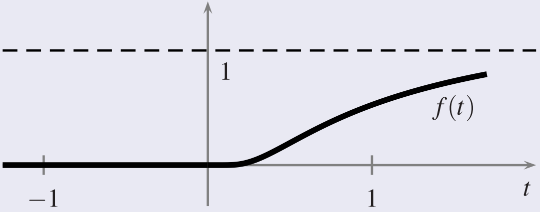


Fig. 13.3. The graph of $f(t)$.

C^∞ Bump Function on $\mathbb{R}$

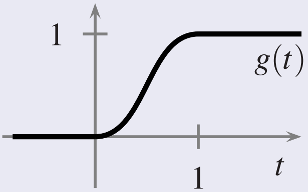
Step 2

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}, \qquad t \in \mathbb{R}.$$

- $g(t)$ is well defined and C^∞, since $f(t) + f(1-t) > 0$.
- We have

$$0 \leq g \leq 1, \qquad g^{-1}(0) = (-\infty, 0], \qquad g^{-1}(1) = [1, \infty).$$



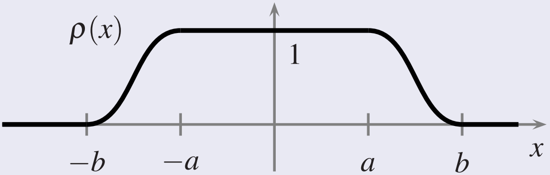
Step 3

Let $0 < a < b$, and define $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2}\right), \qquad x \in \mathbb{R}.$$

- $\rho(x)$ is a C^∞ function.
- We have

$$0 \leq \rho \leq 1, \quad \rho^{-1}(1) = [-a, a], \quad \rho^{-1}(0) = (-\infty, -b] \cup [b, \infty).$$



In particular, ρ is a C^∞ bump function at the origin on $\mathbb{R}$.

C^∞ Bump Function on $\mathbb{R}^n$

Notation

If $q \in \mathbb{R}^n$ and $r > 0$, then $B(q, r)$ is the open ball about q with radius r in $\mathbb{R}^n$ and $\overline{B}(q, r)$ is its closure.

Lemma

Let $q \in \mathbb{R}^n$, and define $\sigma : \mathbb{R}^n \to \mathbb{R}$ by

$$\sigma(x) = \rho\big(\|x - q\|\big), \qquad x \in \mathbb{R}^n.$$

where $\rho(t)$ is as in Step 3. Then $\sigma(x)$ is C^∞-function such that

$$0 \leq \sigma \leq 1, \quad \sigma^{-1}(1) = \overline{B}(q, a), \quad \sigma^{-1}(0) = \mathbb{R}^n \setminus B(q, b).$$

In particular, σ is a C^∞ bump function at q supported in $\overline{B}(q, b)$.

C^∞ Bump Functions on Manifolds

Setup

M is a smooth manifold of dimension n.

Lemma (Extension Lemma)

Assume $U \subseteq M$ is open, and let $f \in C^\infty(U)$ have compact support. Define $\tilde{f} : M \rightarrow \mathbb{R}$ by

$$\tilde{f}_{|U} = f \qquad \text{and} \qquad \tilde{f}_{M \setminus U} = 0.$$

Then f is a smooth function on M s.t. $\text{supp}(\tilde{f}) = \text{supp}(f)$.

Remarks

1. The result continues to hold if $\text{supp}(f)$ is a closed subset of M contained in U. In general, the result may fail.
2. This gives a natural identification,

$$C_c^\infty(U) \simeq \{f \in C_c^\infty(M);\ \text{supp}(f) \subseteq U\}.$$

C^∞ Bump Functions on Manifolds

Proposition

Let $p \in M$. Given any open neighborhood U of p in M, there exists a C^∞ bump function at p with compact support in U.

Sketch of Proof.

- Let (V, ϕ) be a chart near p. Set $q = \phi(p)$, and let $r > 0$ be s.t. $\overline{B}(q, r) \subseteq \phi(U \cap V)$.
- Let $\sigma \in C_c^\infty(\mathbb{R}^n)$ be a C^∞-bump function at q with (compact) support in $\overline{B}(q, r)$.
- Set $\psi = \sigma \circ \phi$. Then ψ is a C^∞ bump function on V at p with compact support in $U \cap V$.
- By the Extension Lemma it uniquely extends to $\tilde{\psi} \in C_c^\infty(M)$ such that $\tilde{\psi} = \psi$ on $U \cap V$ and $\operatorname{supp}(\tilde{\psi}) = \operatorname{supp}(\psi)$.
- Thus, $\tilde{\psi}$ is a C^∞ bump function on M near p with compact support in U.

□

Application: Extension of Smooth Functions

Remark

Let $f : U \to \mathbb{R}$ be a C^∞ function on an open U of a manifold M.

- If f is constant outside some compact set (or even some closed subset of M contained in U), then in a similar way as on Slide 8 we may extend f into a C^∞-function on the whole manifold M.
- In general this is not possible.

Application: Extension of Smooth Functions

Facts

Assume $U \subseteq M$ is open, $p \in U$. Let $\psi \in C_c^\infty(U)$ be a bump function near p (so that $\psi = 1$ near p).

- If $f \in C^\infty(U)$, then ψf is a smooth function on U with compact support.
- By the Extension Lemma it uniquely extends to a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\text{supp}(\tilde{f}) = \text{supp}(\psi f)$.
- In particular, $\tilde{f} = \psi f$ on U, and hence $\tilde{f} = \psi f = f$ near p.

Therefore, we have proved the following result.

Proposition (Proposition 13.2; Extension of smooth functions)

Let $p \in M$ and U an open neighborhood of p. Then, for every smooth function $f : U \to \mathbb{R}$, there exists a smooth function $\tilde{f} : M \to \mathbb{R}$ with compact support in U such that $\tilde{f} = f$ near p.

Locally Finite Families of Functions

Definition

Let $(f_\alpha)_{\alpha \in A}$ be a family of continuous functions on M.

1. We say that this family is finite if $\{\alpha \in A;\ f_\alpha \neq 0\}$ is finite.
2. We say that this family is locally finite if every point $p \in M$ has an open neighborhood V such that the family of restrictions $((f_\alpha)_{|V})_{\alpha \in A}$ is finite, i.e., $\{\alpha \in A;\ (f_\alpha)_{|V} \neq 0\}$ is finite.

Remark

- If V is open, then $(f_\alpha)_{|V} = 0$ iff $V \subseteq M \setminus (\mathrm{supp}(f_\alpha))$.
- Equivalently, $(f_\alpha)_{|V} \neq 0$ if and only if $V \cap \mathrm{supp}(f_\alpha) \neq \emptyset$.
- Thus, $(f_\alpha)_{\alpha \in A}$ is locally finite if and only if each $p \in M$ has an open neighborhood V s.t. $\{\alpha \in A;\ V \cap \mathrm{supp}(f_\alpha) \neq \emptyset\}$ is finite.

Locally Finite Families of Functions

Remark

Let $(f_\alpha)_{\alpha\in A}$ be a locally finite family of continuous functions. Let $K \subseteq M$ be compact.

- For every $q \in K$ there is an open neighborhood V_q of q such that $J(V_q) := \{\alpha; (f_\alpha)_{|V_q} \neq 0\}$ is finite.
- Here $K \subseteq \cup_{q\in K} V_q$. As K is compact, we can find a finite family $\{q_1, \ldots, q_m\}$ in K such that $K \subseteq \cup_{i=1}^m V_{q_i}$.
- If $f_\alpha \neq 0$ on K, then there is $q \in K$ and $i \in \{1, \ldots, m\}$ such that $f_\alpha(q) \neq 0$ and $q \in V_{q_i}$, and hence $\alpha \in J(V_{q_i})$.
- Thus,
$$J(K) := \{\alpha; (f_\alpha)_{|K} \neq 0\} \subseteq \bigcup_{i=1}^m J(V_{q_i}).$$
- In particular, $J(K)$ is finite.

Locally Finite Families of Functions

In fact, we have the following result:

Lemma

Let $(f_\alpha)_{\alpha\in A}$ be a family of continuous functions on M. TFAE:

1. *The family $(f_\alpha)_{\alpha\in A}$ is locally finite.*
2. *For every compact subset $K \subseteq M$, the set $J(K) := \{\alpha \in A; (f_\alpha)_{|K} \neq 0\}$ is finite.*

Remarks

1. Specializing this to $K = \{pt\}$ shows that, for every $p \in M$, the set $J(p) := \{\alpha \in A; f_\alpha(p) \neq 0\}$ is finite.
2. If M is compact, then every locally finite family of continuous functions is finite.

Locally Finite Families of Functions

Facts

Let $(f_\alpha)_{\alpha\in A}$ be a finite family in $C^\infty(M)$.

- The index set $J := \{\alpha \in A;\ f_\alpha \neq 0\}$ is finite.
- We then can set
$$\sum_{\alpha\in A} f_\alpha := \sum_{\alpha\in J} f_\alpha.$$
- This defines a smooth function on M.

Facts

Let $(f_\alpha)_{\alpha \in A}$ be a locally finite family in $C^\infty(M)$.

- Given any $p \in M$, the set $J(p) := \{\alpha \in A;\ f_\alpha(p) \neq 0\}$ is finite.
- We then set
$$\sum_{\alpha \in A} f_\alpha(p) := \sum_{\alpha \in J(p)} f_\alpha(p).$$
- This defines a function $\sum_{\alpha \in A} f_\alpha : M \to \mathbb{R}$.
- Such a function is called a locally finite sum.

Facts (continued)

- Let V be an open neighborhood of p such that $J(V) := \{\alpha; V \cap \operatorname{supp}(f_\alpha) \neq \emptyset\}$ is finite.
- If $q \in V$, then $J(q) \subseteq J(V)$ and $f_\alpha(q) = 0$ if $\alpha \in J(V) \setminus J(q)$.
- Thus, $$\sum_{\alpha \in A} f_\alpha(q) = \sum_{\alpha \in J(q)} f_\alpha(q) = \sum_{\alpha \in J(V)} f_\alpha(q).$$
- This shows that $$\sum_{\alpha \in A} f = \sum_{\alpha \in J(V)} f_\alpha \quad \text{on } V.$$
- As $J(V)$ is finite, we then see that $\sum_{\alpha \in A} f$ is C^∞ on V.
- Thus, $\sum_{\alpha \in A} f_\alpha$ is C^∞ near every point $p \in M$, and so this is a smooth function on M

Partitions of Unity

Reminder

An *open cover of* M is a family of open sets $(U_\alpha)_{\alpha \in A}$ such that $M = \cup_{\alpha \in A} U_\alpha$.

Definition

1. A C^∞ partition of unity on M is a locally finite family $(\rho_\alpha)_{\alpha \in A}$ in $C^\infty(M)$ such that
   - (i) $\rho_\alpha \geq 0$ for all $\alpha \in A$.
   - (ii) $\sum \rho_\alpha = 1$ on M.
2. If $(U_\alpha)_{\alpha \in A}$ is an open cover of M such that $\text{supp}(\rho_\alpha) \subseteq U_\alpha$, then we say that the partition of unity is subordinate to $(U_\alpha)_{\alpha \in A}$.

Proposition (Proposition 13.6)

Assume that M is a compact. Given any open cover $(U_\alpha)_{\alpha \in A}$ of M, there exists a (finite) C^∞ partition of unity $(\rho_\alpha)_{\alpha \in A}$ which is subordinate to $(U_\alpha)_{\alpha \in A}$.

Remark

If M is compact, then every C^∞-partition of unity is a finite family in $C^\infty(M)$.

Proof of Proposition 13.6 (Part 1)

- For every $q \in M$ there is $\alpha(q) \in A$ be such that $q \in U_{\alpha(q)}$.
- Let ψ_q be a C^∞ bump function at q supported in U_α. In particular, $0 \leq \psi_q \leq 1$ and $\psi_q = 1$ near q.
- Set $W_q = \{\psi_q > 0\}$. Then W_q is an open neighborhood of q.
- Here $M = \cup_{q \in M} W_q$. As M is compact there are $q_1, \ldots, q_m$ in M such that $M = \cup_{i=1}^m W_{q_i}$.
- Set $\psi = \sum_{i=1}^m \psi_{q_i} \in C^\infty(M)$. Given any $q \in M$, there i such that $q \in W_{q_i} = \{\psi_{q_i} > 0\}$.
- Thus,
$$\psi(q) = \sum \psi_{q_j}(q) \geq \psi_{q_i}(q) > 0.$$
This shows that ψ is > 0 on M.

Proof of Proposition 13.6 (Part 1, continued)

- For $i = 1, \ldots, m$, set
$$\varphi_i = \frac{\psi_{q_i}}{\psi} \in C^\infty(M).$$
- We have $\varphi_i \geq 0$, and
$$\sum_{1 \leq i \leq m} \varphi_i = \frac{1}{\psi} \sum_{i=1}^{m} \psi_{q_i} = \frac{1}{\psi}\psi = 1.$$
- For $i = 1, \ldots, m$, set $\tau(i) = \alpha_{q_i}$ (so that $\operatorname{supp}(\psi_{q_i}) \subseteq U_{\alpha_{q_i}}$).
- We then have
$$\operatorname{supp}(\varphi_i) = \operatorname{supp}(\psi_{q_i}) \subseteq U_{\alpha_{q_i}} = U_{\tau(i)}.$$
- Therefore, $(\varphi_i)_{i=1}^{m}$ is a (finite) C^∞ partition of unity such that, for each i there is $\alpha = \tau(i)$ in A so that $\operatorname{supp} \varphi_i \subseteq U_\alpha$.

Proof of Proposition 13.6 (Part 2)

- Set $I = \{1, \ldots, m\}$. We then have a map $\tau : i \to \tau(i)$ from I to A such that $\operatorname{supp} \varphi_i \subseteq U_{\tau(i)}$.
- For $\alpha \in A$, set
$$\rho_\alpha = \sum_{i \in \tau^{-1}(\alpha)} \varphi_i \quad \text{if } \alpha \in \tau(I), \qquad \rho_\alpha = 0 \quad \text{if } \alpha \notin \tau(I).$$
- As $\tau(I)$ is finite, $\{\rho_\alpha\}_{\alpha \in A}$ is a finite family in $C^\infty(M)$.
- As $I = \cup_{\alpha \in \tau(I)} \tau^{-1}(\alpha)$ is a partition of I, we have
$$\sum_{\alpha \in A} \rho_\alpha = \sum_{\alpha \in \tau(I)} \rho_\alpha = \sum_{\alpha \in \tau(I)} \sum_{i \in \tau^{-1}(\alpha)} \varphi_i = \sum_{i \in I} \varphi_i = 1.$$
- Thus, $\{\rho_\alpha\}_{\alpha \in A}$ is a C^∞-partition of the unity on M.

Proof of Proposition 13.6 (Part 2, continued).

- If $\alpha \notin \tau(I)$, then $\text{supp}(\rho_\alpha) = \emptyset \subseteq U_\alpha$.
- If $\alpha \in \tau(I)$, then $\text{supp}(\varphi) \subseteq U_\alpha$ for $i \in \tau^{-1}(i)$, and hence

$$\text{supp}\, \rho_\alpha = \text{supp}\,\Big(\sum_{i \in \tau^{-1}(\alpha)} \varphi_i \Big) \subseteq U_\alpha.$$

- Therefore, the partition of the unity $\{\rho_\alpha\}_{\alpha \in A}$ is subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$.

The proof is complete. □

In general, we have the following result:

Theorem (Theorem 13.7)

Let $\{U\}_{\alpha \in A}$ be an open cover of M.

(i) *There is a C^∞ partition of unity $(\varphi_k)_{k\geq 1}$ such that, for each integer $k \geq 1$, the support of φ_k is compact and contained in some U_α.*

(ii) *If we do not require compact support, then there is a C^∞-partition of the unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$.*

Integration of Differential Forms

Setup

U is an open subset of $\mathbb{R}^n$ with coordinates $x^1, \ldots, x^n$.

Definition

Let $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(U)$. The *integral* of ω over a Borel set $A \subseteq U$ is defined by

$$\int_A \omega = \int_A f(x)dx^1 \wedge \cdots \wedge dx^n := \int_A f(x)dx.$$

Lemma

Let $\phi : V \to U$ an orientation-preserving diffeomorphism. For every $\omega \in \Omega_c^n(U)$, we have

$$\int_V \phi^*\omega = \int_U \omega.$$

Integration of Differential Forms

Setup

M is an oriented manifold of dimension n.

Definition

Let (U, ϕ) be an oriented chart. The integral of any $\omega \in \Omega_c^n(U)$ is defined by

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^* \omega.$$

Integration of Differential Forms

Setup

- $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is an oriented atlas.
- $\{\rho_\alpha\}$ is a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}$.

Facts

Let $\omega \in \Omega_c^n(M)$.

- Here $\text{supp}(\omega)$ is compact and the family (ρ_α) is locally finite.
- Thus, $J := \{\alpha \in A;\ \rho_\alpha \neq 0 \text{ on } \text{supp}(\omega)\}$ is finite.
- If $\alpha \notin J$, then $\rho_\alpha \omega = 0$.
- As $\sum_{\alpha \in A} \rho_\alpha = 1$, we get

$$\omega = \sum_{\alpha \in J} \rho_\alpha \omega = \sum_{\alpha \in A} \rho_\alpha \omega \qquad \text{(finite sum)}.$$

Facts (Continued)

- $\text{supp}(\rho_\alpha\omega) \subseteq \text{supp}(\rho_\alpha) \subseteq U_\alpha$.
- Therefore, the integral $\int_{U_\alpha} \rho_\alpha\omega$ is well defined.
- We have $\int_{U_\alpha} \rho_\alpha\omega = 0$ if $\alpha \notin J$.

Definition

The integral of ω over M is defined by

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha\omega \qquad \text{(finite sum)}.$$

Remark

The integral $\int_M \omega$ doesn't depend on the choice of the partition of unity $\{\rho_\alpha\}$, and hence is well defined (see Tu's book).