Differential Forms in Algebraic Topology: The Mayer-Vietoris Sequence

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References

Main References

- Section 26 of Tu2011.
- Section 2 of Bott-Tu.

Direct Sums of Cochain Complexes

Definition

Let $\mathscr{A} = (A^*, d)$ and $\mathscr{B} = (B^*, d')$ be cochain complexes. Their direct sum $\mathscr{A} \oplus \mathscr{B}$ is the cochain complex such that:

- The space of k-cochains is $A^k \oplus B^k$.
- The differential in degree k is

$$
\begin{aligned}
d \oplus d' : A^k \oplus B^k &\longrightarrow A^{k+1} \oplus B^{k+1} \\
(a, b) &\longrightarrow (da, db).
\end{aligned}
$$

Remark

- We have
$$Z^k(\mathscr{A} \oplus \mathscr{B}) = Z^k(\mathscr{A}) \oplus Z^k(\mathscr{B}),$$
$$B^k(\mathscr{A} \oplus \mathscr{B}) = B^k(\mathscr{A}) \oplus B^k(\mathscr{B})$$
- We then have an exact sequence,
$$0 \longrightarrow B^k(\mathscr{A} \oplus \mathscr{B}) \longrightarrow Z^k(\mathscr{A} \oplus \mathscr{B}) \longrightarrow H^k(\mathscr{A}) \oplus H^k(\mathscr{B}) \longrightarrow 0.$$

This gives the following result:

Lemma

We have a canonical isomorphism,
$$H^k(\mathscr{A} \oplus \mathscr{B}) \simeq H^k(\mathscr{A}) \oplus H^k(\mathscr{B}).$$

Direct Sums of Cochain Complexes

Lemma

Let $\varphi : \mathscr{C} \rightarrow \mathscr{A}$ *and* $\psi : \mathscr{C} \rightarrow \mathscr{B}$ *be cochain maps.*

1. $\varphi \oplus \psi : \mathscr{C} \rightarrow \mathscr{A} \oplus \mathscr{B}$ *is a cochain map.*
2. *This induces linear maps,*
$$(\varphi \oplus \psi)^* : H^k(\mathscr{C}) \longrightarrow H^k(\mathscr{A} \oplus \mathscr{B}).$$
3. *Under the identification* $H^k(\mathscr{A} \oplus \mathscr{B}) = H^k(\mathscr{A}) \oplus H^k(\mathscr{B})$, *we have*
$$(\varphi \oplus \psi)^* = \varphi^* \oplus \psi^* : H^k(\mathscr{C}) \longrightarrow H^k(\mathscr{A}) \oplus H^k(\mathscr{B}).$$

Lemma

Let $\varphi_1 : \mathscr{A} \to \mathscr{C}$ *and* $\varphi_2 : \mathscr{B} \to \mathscr{C}$ *be cochain maps. Define* $\varphi : \mathscr{A} \oplus \mathscr{B} \to \mathscr{C}$ *by*

$$\varphi(a, b) = \varphi_1(a) + \varphi_2(b), \qquad a \in A^k, \; b \in B^k.$$

Then:

1. φ *is a cochain map.*
2. *It induces linear maps,*

$$\varphi^* : H^k(\mathscr{A}) \oplus H^k(\mathscr{B}) \longrightarrow H^k(\mathscr{C}).$$

The Mayer-Vietoris Sequence

Setup

- M is a smooth manifold.
- U and V are open sets such that $M = U \cup V$.

Facts

- By pullback the inclusion maps $i_U : U \hookrightarrow M$ and $i_V : V \hookrightarrow M$ give rise to cochain maps,

$$i_U^* : \Omega^*(M) \longrightarrow \Omega^*(U), \qquad i_V^* : \Omega^*(M) \longrightarrow \Omega^*(V).$$

- We thus get a cochain map,

$$i := i_U^* \oplus i_V^* : \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V).$$

Remark

- We know (see Prop. 17.14, Tu2011) that the pullback maps $i_U^*: \Omega^k(M) \to \Omega^k(U)$ and $i_V^*: \Omega^k(M) \to \Omega^k(V)$ agree with the restriction maps,

$$i_U^*\omega = \omega_{|U}, \qquad i_V^*\omega = \omega_{|V}, \qquad \omega \in \Omega^k(M).$$

- Thus,

$$i(\omega) = (i_U^*\omega, i_V^*\omega) = \left(\omega_{|U}, \omega_{|V}\right), \qquad \omega \in \Omega^k(M).$$

Facts

- We also have inclusion maps $j_U : U \cap V \hookrightarrow U$ and $j_V : U \cap V \hookrightarrow V$.
- They give rise to cochain maps,

$$\begin{aligned} j_U^* &: \Omega^*(U) \longrightarrow \Omega^*(U \cap V), \quad j_U^*\omega = \omega_{|U\cap V}, \\ j_V^* &: \Omega^*(V) \longrightarrow \Omega^*(U \cap V), \quad j_V^*\tau = \tau_{|U\cap V}. \end{aligned}$$

- We thus get a cochain map,

$$j : \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U \cap V),$$
$$j(\omega, \tau) = j_V^*\tau - j_U^*\omega = \tau_{|U\cap V} - \omega_{|U\cap V}.$$

The Mayer-Vietoris Sequence

Proposition

We have a short-exact sequence of cochain complexes,

$$0 \longrightarrow \Omega^*(M) \xrightarrow{i} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j} \Omega^*(U \cap V) \longrightarrow 0.$$

Proof of exactness at $\Omega^*(M)$.

- Recall that $M = U \cup V$. Thus, if $\sigma \in \Omega^k(M)$, then

$$i(\sigma) = 0 \Longleftrightarrow \big(\sigma_{|U} = 0 \text{ and } \sigma_{|V} = 0\big) \Longleftrightarrow \sigma = 0 \text{ on } U \cup V = M.$$

- This shows that i is injective and gives exactness at $\Omega^*(M)$.

□

Proof of exactness at $\Omega^k(U) \oplus \Omega^k(V)$.

- Let $(\omega, \tau) \in \operatorname{im}(i)$, i.e., there is $\sigma \in \Omega^k(M)$ such that $\omega = \sigma_{|U}$ and $\tau = \sigma_{|V}$.
- Thus,

$$j(\omega, \tau) = \tau_{|U\cap V} - \omega_{|U\cap V} = \sigma_{|U\cap V} - \sigma_{|U\cap V} = 0.$$

- This shows that $\operatorname{im}(i) \subseteq \ker(j)$.
- Let $(\omega, \tau) \in \ker(j)$, i.e., $\tau_{|U\cap V} - \omega_{|U\cap V} = 0$.
- As $M = U \cup V$, it follows there is a unique $\sigma \in \Omega^k(M)$ such that $\sigma_{|U} = \omega$ and $\sigma_{|V} = \tau$.
- That is, $i(\sigma) = (\omega, \tau)$, and so $(\omega, \tau) \in \operatorname{im}(i)$.
- This shows that $\ker(j) \subseteq \operatorname{im}(i)$, and hence $\operatorname{im}(i) = \ker(j)$.
- This proves exactness at $\Omega^k(U) \oplus \Omega^k(V)$.

$\square$

The Mayer-Vietoris Sequence

Proof of exactness at $\Omega^*(U \cap V)$.

- This amounts to show that j is surjective.
- Let $\omega \in \Omega^k(U \cap V)$, and let (ρ_U, ρ_V) be a C^∞ partition of unity on M subordinate to the open cover $\{U, V\}$.
- Define $\omega_U : U \to \Lambda^k(T^*U)$ by

$$\omega_U(x) = \begin{cases} \rho_V(x)\omega(x) & \text{if } x \in U \cap V, \\ 0 & \text{if } x \in U \setminus (U \cap V). \end{cases}$$

- This is a smooth k-form on U (see next slide).
- Likewise, we define a smooth k-form $\omega_V : V \to \Lambda^k(T^*V)$ by

$$\omega_V(x) = \begin{cases} \rho_U(x)\omega(x) & \text{if } x \in U \cap V, \\ 0 & \text{if } x \in V \setminus (U \cap V). \end{cases}$$

- On $U \cap V$, we then have

$$j(-\omega_U, \omega_V) = \rho_V\omega - (-\rho_U\omega) = (\rho_U + \rho_V)\omega = \omega.$$

- This shows that j is surjective.

□

Proof of the smoothness of ω_U and ω_V.

- As $\omega_U = \rho_V \omega$ on $U \cap V$, we see that ω_U is C^∞ on $U \cap V$.
- We also see that $\omega_U = 0$ on $(U \cap V) \setminus \operatorname{supp}(\rho_V)$.
- As $\omega_U = 0$ on $U \setminus (U \cap V)$, this implies that $\omega_U = 0$ on $U \setminus \operatorname{supp}(\rho_V)$.
- As $U \setminus \operatorname{supp}(\rho_V)$ is an open set, we see that ω_U is C^∞ there.
- It follows that ω_U is C^∞ on $(U \cap V) \cup (U \setminus \operatorname{supp}(\rho_V)) = U$.
- Likewise, ω_V is a C^∞ form on V.

□

The Mayer-Vietoris Sequence

By the Zig-Zag Lemma the short exact sequence gives rise to a long exact sequence in cohomology. Namely:

Theorem (Mayer-Vietoris Sequence)

If U and V are open subsets of M such that $M = U \cup V$, then we have a long exact sequence,

$$\cdots \longrightarrow H^k(M) \xrightarrow{i^*} H^k(U) \oplus H^k(V) \xrightarrow{j^*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \longrightarrow \cdots .$$

Remark

1. The map $i^* : H^k(M) \to H^k(U) \oplus H^k(V)$ is given by

$$i^*[\sigma] = [i(\sigma)] = \left[(\sigma_{|U}, \sigma_{|V})\right] = \left([\sigma_{|U}], [\sigma_{|V}]\right).$$

2. The map $j^* : H^k(U) \oplus H^k(V) \to H^k(U \cap V)$ is given by

$$j^*\left([\omega], [\tau]\right) = [j(\omega, \tau)] = \left[\tau_{|U \cap V} - \omega_{|U \cap V}\right].$$

Proposition

Let $\{\rho_U, \rho_V\}$ be a C^∞-partition of unity subordinate to $\{U, V\}$. Given any closed form $\omega \in \Omega^k(U \cap V)$, we have

$$\delta([\omega]) = [\sigma],$$

where $\sigma \in \Omega^{k+1}(M)$ is the unique (closed) form on M such that

$$\sigma = d\rho_U \wedge \omega \quad \text{on } U \cap V, \qquad \sigma = 0 \quad \text{on } M \setminus (U \cap V).$$

Proof.

- Let $\sigma : M \to \Lambda^{k+1}(T^*M)$ be the $(k+1)$-form such that
$$\sigma = d\rho_U \wedge \omega \quad \text{on } U \cap V, \qquad \sigma = 0 \quad \text{on } M \setminus (U \cap V).$$
- As $\sigma = d\rho_U \wedge \omega$ on $U \cap V$, we see that σ is C^∞ on $U \cap V$. □

Proof.

- As $\rho_U + \rho_V = 1$, we have $d\rho_U + d\rho_V = 0$, i.e., $d\rho_V = -d\rho_U$.
- As $\text{supp}(d\rho_U) \subseteq U$ and $\text{supp}(d\rho_V) \subseteq V$, we have
$$\text{supp}(d\rho_U) = \text{supp}(-d\rho_V) \subseteq U \cap V.$$
- Thus, $\sigma = 0$ on $(U \cap V) \setminus \text{supp}(d\rho_U)$ and $\sigma = 0$ on $M \setminus (U \cap V)$, i.e., $\sigma = 0$ on $M \setminus \text{supp}(d\rho_U)$.
- As $M \setminus \text{supp}(d\rho_U)$ is open, we see that σ is C^∞ there.
- Thus, σ is C^∞ on $(U \cap V) \cup (M \setminus \text{supp}(d\rho_U)) = M$, i.e., $\sigma \in \Omega^{k+1}(M)$.
- Here $d\sigma = 0$ on $M \setminus \text{supp}(d\rho_U)$, since $\sigma = 0$ there.
- As $d\omega = 0$, on $U \cap V$ we have
$$d\sigma = d(d\rho_U \wedge \omega) = -d\rho_U \wedge d\omega = 0.$$
- It follows that σ is a closed $(k+1)$-form on M.

□

Proof.

- We know that $\omega = j(-\omega_U, \omega_V)$, with $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ such that

$$\omega_U = \rho_V \omega \quad \text{and} \quad \omega_V = \rho_U \omega \quad \text{on } U \cap V,$$
$$\omega_U = 0 \quad \text{on } U \setminus (U \cap V), \qquad \omega_V = 0 \quad \text{on } V \setminus (U \cap V).$$

- As $d\omega = 0$ and $d\rho_V = -d\rho_U$, on $U \cap V$ we have

$$d\omega_U = d(\rho_V \omega) = d\rho_V \wedge \omega = -d\rho_U \wedge \omega = -\sigma,$$
$$d\omega_V = d(\rho_U \omega) = d\rho_U \wedge \omega = \sigma.$$

- We also have

$$d\omega_U = 0 = -\sigma \quad \text{on } U \setminus (U \cap V), \quad d\omega_V = 0 = \sigma \quad \text{on } V \setminus (U \cap V).$$

- Thus, $\sigma_{|U} = -d\omega_U$ and $\sigma_{|V} = d\omega_V$, and hence

$$d(-\omega_U, \omega_V) = (-d\omega_U, d\omega_V) = i(\sigma).$$

- We then have $\delta[\omega] = [\sigma]$.

□

The Mayer-Vietoris Sequence

Proposition

Assume that U, V *and* $U \cap V$ *are connected with* $U \cap V \neq \emptyset$. *Then:*

1. M *is connected.*
2. *We have an exact sequence,*

$$0 \longrightarrow H^0(M) \xrightarrow{i^*} H^0(U) \oplus H^0(V) \xrightarrow{j^*} H^0(U \cap V) \longrightarrow 0.$$

3. *In the Mayer-Vietoris sequence the connected map* δ *vanishes in degree* 0.
4. *The Mayer-Vietoris sequence may start with*

$$0 \longrightarrow H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V) \xrightarrow{j^*} H^1(U \cap V) \xrightarrow{\delta} \cdots .$$

Proof.

- A topological space X is connected if and only if every continuous function $f : X \to \{0,1\}$ is constant.
- If $f : M \to \{0,1\}$ is continuous, then it is constant on U, V and $U \cap V$ and it takes the same constant value on these sets.
- Therefore, f is constant on M. This shows that M is connected.
- As M is connected, we know that

$$H^0(M) = \{\text{constant functions } f : M \to \mathbb{R}\} \simeq \mathbb{R}1.$$

- Likewise,

$$H^0(U) \simeq \mathbb{R}1, \qquad H^0(V) \simeq \mathbb{R}1, \qquad H^0(U \cap V) \simeq \mathbb{R}1.$$

- With these identifications $i^* : H^0(M) \to H^0(U) \oplus H^0(V)$ becomes

$$\mathbb{R} \ni \lambda \longrightarrow (\lambda, \lambda) \in \mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2.$$

Proof (Continued).

- The map $j*: H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V)$ becomes
$$\mathbb{R}^2 \ni (\lambda, \mu) \longrightarrow \mu - \lambda \in \mathbb{R}.$$
- The map i^* is injective, the map j^* is surjective, and
$$\ker j^* = \{(\lambda, \lambda);\ \lambda \in \mathbb{R}\} = \mathrm{im}(i^*).$$
- Therefore, we have an exact sequence,
$$0 \longrightarrow H^0(M) \xrightarrow{i^*} H^0(U) \oplus H^0(V) \xrightarrow{j^*} H^0(U \cap V) \longrightarrow 0.$$
- We then have $\ker \delta = \mathrm{im}\, j^* = H^0(U \cap V)$, and so $\delta = 0$ on $H^0(U \cap V)$.
- Therefore, we may start the Mayer-Vietoris sequence with
$$0 \longrightarrow H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V) \xrightarrow{j^*} H^1(U \cap V) \xrightarrow{\delta} \cdots.$$

□

De Rham Cohomology of $\mathbb{S}^1$

Lemma (Alternating Sum of Dimensions; Exercise 26.2)

Suppose that we have an exact sequence of vector spaces,

$$0 \longrightarrow A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} \cdots \longrightarrow A^m \longrightarrow 0.$$

Then, we have

$$\sum_{i=0}^{m} (-1)^j \dim A^j = 0.$$

Reminder

If $M = I_1 \cup \cdots \cup I_m$ is a disjoint unions of intervals, then

$$H^k(M) = \begin{cases} \mathbb{R}^r & \text{if } k = 0, \\ 0 & \text{for } k \geq 1. \end{cases}$$

De Rham Cohomology of $\mathbb{S}^1$

Facts

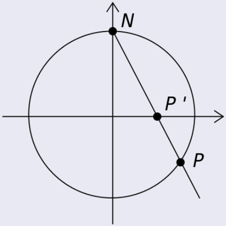
- As $\mathbb{S}^1$ is a connected 1-dimensional manifold, we know that

$$H^0(\mathbb{S}^1) = \mathbb{R}, \qquad H^k(\mathbb{S}^1) = 0 \quad \text{for } k \geq 2.$$

- Therefore, we only need to compute $H^1(\mathbb{S}^1)$.

Lemma

Set $N = (0,1) \in \mathbb{S}^1$.



1. *In Cartesian coordinates, the stereographic projection* $\varphi : \mathbb{S}^1 \setminus \{N\} \to \mathbb{R}$ *is given by*
$$\varphi(x,y) = \frac{x}{1-y}, \qquad (x,y) \in \mathbb{S}^1.$$
2. *This is a smooth diffeomorphism with inverse,*
$$\varphi^{-1}(t) = \frac{1}{t^2+1}(2t, t^2-1), \qquad t \in \mathbb{R}.$$

De Rham Cohomology of $\mathbb{S}^1$

Lemma

$H^1(\mathbb{S}^1) = \mathbb{R}$.

Proof.

Set $N = (0, 1)$ and $S = (0, 1)$.

- We have an open covering $\mathbb{S}^1 = U \cup V$, where
$$U = \mathbb{S}^1 \setminus \{N\}, \quad V = -U = \mathbb{S}^1 \setminus \{S\}, \quad U \cap V = \mathbb{S}^1 \setminus \{N, S\}.$$
- The stereographic projection gives an isomorphism $U \simeq \mathbb{R}^n$. Thus,
$$H^0(U) = H^0(\mathbb{R}) = \mathbb{R}, \qquad H^1(U) = H^1(\mathbb{R}) = 0.$$
- As V is diffeomorphic to U under the involution $z \to -z$, we also have
$$H^0(V) = H^0(U) = \mathbb{R}, \qquad H^1(V) = H^1(U) = 0.$$

□

Proof (continued).

- The stereographic projection of $S \in \mathbb{S}^1$ is the origin $0 \in \mathbb{R}$.
- We thus get a diffeomorphism of $U \cap V = \mathbb{S}^1 \setminus \{N, S\} \simeq \mathbb{R} \setminus 0$.
- As $\mathbb{R} \setminus 0 = (-\infty, 0) \cup (0, \infty)$ is the union of two disjoint open intervals, we get

$$H^0(U \cap V) = H^0((-\infty, 0) \cup (0, \infty)) = \mathbb{R}^2.$$

- As $H^1(U) \oplus H^1(V) = 0$ the Mayer-Vietoris sequence induces the exact sequence,

$$0 \longrightarrow H^0(\mathbb{S}^1) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V) \longrightarrow H^1(\mathbb{S}^1) \longrightarrow 0.$$

- Thus, by taking the alternating sum of dimensions, we get

$$\dim H^0(\mathbb{S}^1) - \dim \left(H^0(U) \oplus H^0(V)\right) + \dim H^0(U \cap V) - \dim H^1(\mathbb{S}^1) = 0.$$

□

Proof (continued).

- We have

$$\dim H^0(\mathbb{S}^1) = 1, \qquad \dim H^0(U \cap V) = 2,$$
$$\dim(H^0(U) \oplus H^0(V)) = \dim H^0(U) + \dim H^0(V) = 2.$$

- Therefore, we get

$$1 - 2 + 2 - \dim H^1(\mathbb{S}^1) = 0.$$

- Thus, $\dim H^1(\mathbb{S}^1) = 1$, and hence $H^1(\mathbb{S}^1) = \mathbb{R}$.

The proof is complete. □

De Rham Cohomology of $\mathbb{S}^1$

To sum up we have:

Proposition

$$H^k(\mathbb{S}^1) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1, \\ 0 & \text{for } k \geq 2. \end{cases}$$

Remark

We will see later that, for all $n \geq 2$, we have

$$H^k(\mathbb{S}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Ω_c^* and Inclusions of Open Sets

Setup

$i : V \to U$ is an inclusion of open subsets of M.

Reminder

By pullback we get a cochain map,

$$i^* : \Omega^*(U) \longrightarrow \Omega^*(V), \qquad i^*\omega = \omega_{|V}.$$

Remark

- As the inclusion $i : V \to U$ need not be a proper map, it does not map $\Omega_c^*(U)$ to $\Omega_c^*(V)$.
- For instance, if $f \in C_c^\infty(U)$ and $V \subseteq \operatorname{supp}(f)$, then $i^*f = f_{|V}$ does not have compact support.

Ω_c^* and Inclusions of Open Sets

Reminder (Extension by zero)

Let $\omega \in \Omega_c^k(V)$ and define $\tilde{\omega} : U \to \Lambda^k(T^*U)$ by

$$\tilde{\omega} = \omega \quad \text{on } V, \qquad \tilde{\omega} = 0 \quad \text{on } U \setminus V.$$

Then $\tilde{\omega}$ is a smooth k-form on U such that $\text{supp}(\tilde{\omega}) = \text{supp}(\omega)$, and hence $\tilde{\omega} \in \Omega_c^k(U)$. It is called the extension by zero of ω to U.

Fact

This gives rise to a pushforward map,

$$\begin{aligned} i_* : \Omega_c^k(V) &\longrightarrow \Omega_c^k(U), \\ \omega &\longrightarrow \text{extension by zero of } \omega. \end{aligned}$$

Ω_c^* and Inclusions of Open Sets

Proposition

1. *The linear map* $i_* : \Omega_c^k(V) \to \Omega_c^k(U)$ *is injective.*
2. *Its image is*

$$\Omega_{c,V}^k(U) := \left\{ \omega \in \Omega_c^k(U);\ \operatorname{supp}(\omega) \subseteq V \right\}.$$

3. *The inverse* $(i_*)^{-1} : \Omega_{c,V}^k(U) \to \Omega_c^k(V)$ *is the pullback map* $\sigma \to i^*\sigma = \sigma_{|V}$ *on* $\Omega_{c,V}^k(U)$.

Proof.

- If $\omega \in \Omega_c^k(V)$, then $(i_*\omega)_{|V} = \omega$, i.e., $i^* \circ i_* = \mathrm{id}$ on $\Omega_c^k(V)$.
- This implies that i_* is injective (since it has a right-inverse). □

Proof, Continued.

- If $\omega \in \Omega_c^k(V)$, then $\text{supp}(i_*\omega) = \text{supp}(\omega) \subseteq V$, and hence $\text{im}(i_*) \subseteq \Omega_{c,V}^k(U)$.
- Let $\sigma \in \Omega_c^k(U)$ be such that $\text{supp}(\sigma) \subseteq V$, and set $\omega = \sigma_{|V}$.
- Then $\omega \in \Omega^k(V)$ and $\text{supp}(\omega) \subseteq \text{supp}(\sigma)$ is compact, and hence $\omega \in \Omega_c^k(V)$.
- Here $\sigma = \omega$ on V and $\sigma = 0$ on $U \setminus V$ (since $\text{supp}(\sigma) \subseteq V$).
- Thus, σ is the extension by zero of ω, and hence $\sigma = i_*\omega \in \text{im}(i_*)$.
- It follows that $\text{im}(i_*) \subseteq \Omega_{c,V}^k(U)$.
- This also shows that $\sigma = i_*(\sigma_{|V}) = i_* \circ i^*(\sigma)$, and hence $i_* \circ i^* = \text{id}$ on $\Omega_{c,V}^k(U)$.
- As $i^* \circ i_* = \text{id}$ on $\Omega_c^k(V)$,we see that $(i_*)^{-1} = i^*$ on $\Omega_{c,V}^k(U)$.

The proof is complete. □

Ω_c^* and Inclusions of Open Sets

Lemma

The pushforward map $i_* : \Omega_c^*(V) \to \Omega_c^*(U)$ is a cochain map.

Proof.

- Let $\omega \in \Omega_c^k(V)$. As $i_*\omega = \omega$ on V, we see that $d(i_*\omega) = d\omega$ on V.
- As $\text{supp}(i_*\omega) = \text{supp}(\omega)$, we see that $\omega = 0$ on the open set $U \setminus \text{supp}(\omega)$.
- Thus $d(i_*\omega) = 0$ on $U \setminus \text{supp}(\omega)$, and hence $d(i_*\omega) = 0$ on $U \setminus V$.
- This shows that $d(i_*\omega)$ is the extension by zero of $d\omega$, i.e., $d(i_*\omega) = i_*(d\omega)$.

This proves the result. □

Ω_c^* and Inclusions of Open Sets

Lemma

Let $j : W \to V$ be the inclusion of an open set W into V. Then $i \circ j : W \to U$ is the inclusion of W into U, and we have

$$i_* \circ j_* = (i \circ j)_* \qquad \text{on } \Omega_c^*(W).$$

Proof.

- It's immediate that $i \circ j$ is the inclusion of W into U.
- Let $\omega \in \Omega_c^k(W)$. We have

$$i_*(j_*\omega)_{|W} = (i_*(j_*\omega)_V)_{|W} = (j_*\omega)_{|W} = \omega.$$

- We have $\mathrm{supp}(i_*(j_*\omega)) = \mathrm{supp}(j_*\omega) = \mathrm{supp}(\omega)$, and hence $i_*(j_*\omega) = 0$ on $U \setminus (\mathrm{supp}(\omega))$.
- As $\mathrm{supp}(\omega) \subseteq W$, we see that $i_*(j_*\omega) = 0$ on $U \setminus W$.
- This shows that $i_*(j_*\omega)$ is the extension by zero to U of ω, i.e., $i_*(j_*\omega) = (i \circ j)_*\omega$.

This proves the result. □

Mayer-Vietoris Sequence for Ω_c^*

Setup

U and V are open subsets of M such that $M = U \cup V$.

Facts

- The inclusions $i_U : U \to M$ and $i_V : V \to M$ give rise to cochain maps,

$$(i_U)_* : \Omega_c^*(U) \longrightarrow \Omega_c^*(M), \qquad (i_V)_* : \Omega_c^*(V) \longrightarrow \Omega_c^*(M).$$

- Therefore, we get a cochain map,

$$\begin{aligned} i : \Omega_c^*(U) \oplus \Omega_c^*(V) &\longrightarrow \Omega_c^*(M), \\ (\omega, \tau) &\longrightarrow (i_U)_*\omega + (i_V)_*\tau. \end{aligned}$$

Mayer-Vietoris Sequence for Ω_c^*

Facts

- The inclusions $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ also give rise to cochain maps,

$$(j_U)_* : \Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(U), \qquad (j_V)_* : \Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(V).$$

- We thus get a cochain map $j := (-(j_U)_*) \oplus (j_V)_*$, i.e.,

$$\begin{aligned} j : \Omega_c^*(U \cap V) &\longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V), \\ \omega &\longrightarrow (-(j_U)_*\omega, (j_V)_*\omega) . \end{aligned}$$

Mayer-Vietoris Sequence for Ω_c^*

Proposition (see Bott-Tu)

We have an exact sequence of cochain complexes,

$$0 \longrightarrow \Omega_c^*(U \cap V) \xrightarrow{j} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{i} \Omega_c^*(M) \longrightarrow 0.$$

Exactness at $\Omega_c^*(U \cap V)$.

- The pushforward maps $(j_U)_* : \Omega_c^*(U \cap V) \to \Omega_c^*(U)$ and $(j_V)_* : \Omega_c^*(U \cap V) \to \Omega_c^*(V)$ are injective.
- Therefore the direct sum $j = (-(j_U)_*) \oplus (j_V)_*$ is injective.
- This gives exactness at $\Omega_c^*(U \cap V)$. □

Mayer-Vietoris Sequence for Ω_c^*

Exactness at $\Omega_c^*(M)$.

- By definition $i(\omega, \tau) = (i_U)_*\omega + (i_V)_\tau$, where i_U and i_V are the inclusions of U and V into M.
- Thus,

$$\mathrm{im}(i) = \mathrm{im}\left((i_U)_*\right) + \mathrm{im}\left((i_V)_*\right) = \Omega^*_{c,U}(M) + \Omega^*_{c,V}(M).$$

- Let $\omega \in \Omega_c^*(M)$ and let $\{\rho_U, \rho_V\}$ be a C^∞ partition of unity subordinate to the cover $\{U, V\}$.
- We have $\omega = \rho_U\omega + \rho_V\omega$ (since $\rho_U + \rho_V = 1$).
- Here $\mathrm{supp}(\rho_U\omega) \subseteq \mathrm{supp}(\rho_U) \cap \mathrm{supp}(\omega) \subseteq U \cap \mathrm{supp}(\omega)$.
- The support of $\rho_U\omega$ is compact and contained in U, i.e., $\rho_U\omega \in \Omega^*_{c,U}(M)$. Likewise, $\rho_V\omega \in \Omega^*_{c,V}(M)$.
- Thus, $\omega = \rho_U\omega + \rho_V\omega$ is in $\Omega^*_{c,U}(M) + \Omega^*_{c,V}(M) = \mathrm{im}(i)$.
- This shows that $i : \Omega_c^*(U) \oplus \Omega_c^*(V) \to \Omega_c^*(M)$ is surjective.
- This gives exactness at $\Omega_c^*(M)$.

□

Mayer-Vietoris Sequence for Ω_c^*

Exactness at $\Omega_c^*(U) \oplus \Omega_c^*(V)$.

- We have to show that $\text{im}(j) = \ker(i)$.
- Let $\omega \in \Omega_c^k(U \cap V)$. We have

$$i \circ j(\omega) = i\left(-(j_U)_*\omega, (j_V)_*\omega\right) = -(i_U)_* \circ (j_U)_*\omega + (i_V)_* \circ (j_V)_*\omega.$$

- Let $\ell : U \cap V \to M$ be the inclusion of $U \cap V$ into M.
- As $\ell = i_U \circ j_U = i_V \circ j_V$, we have

$$(i_U)_* \circ (j_U)_*\omega = (i_U \circ j_U)_*\omega = \ell_*\omega,$$
$$(i_V)_* \circ (j_V)_*\omega = (i_V \circ j_V)_*\omega = \ell_*\omega.$$

- We then see that $i \circ j(\omega) = -\ell_*\omega + \ell_*\omega = 0$.
- This shows that $i \circ j = 0$, i.e., $\text{im}(j) \subseteq \ker(i)$.

□

Mayer-Vietoris Sequence for Ω_c^*

Exactness at $\Omega_c^*(U) \oplus \Omega_c^*(V)$.

- It remains to show that $\ker(i) \subseteq \operatorname{im}(j)$.
- Let $(\omega, \tau) \in \Omega_c^k(U) \oplus \Omega_c^k(V)$ be such that $i(\omega, \tau) = 0$.
- This means that $(i_U)_*\omega + (i_V)_*\tau = 0$, i.e., $(i_U)_*\omega = -(i_V)_*\tau$.
- Set $\sigma = \tau_{|U\cap V}$. By restriction to $U \cap V$ we get

$$-\omega_{|U\cap V} = -((i_U)_*\omega)_{|U\cap V} = ((i_V)_*\tau)_{|U\cap V} = \tau_{|U\cap V} = \sigma.$$

- As $\operatorname{supp}(\omega) = \operatorname{supp}((i_U)_*\omega)$ and $\operatorname{supp}(\tau) = \operatorname{supp}((i_V)_*\tau)$, we then see that $\operatorname{supp}(\omega) = \operatorname{supp}(\tau)$.
- By assumption $\operatorname{supp}(\omega) \subseteq U$ and $\operatorname{supp}(\tau) \subseteq V$.
- Therefore, we see that $\operatorname{supp}(\omega) = \operatorname{supp}(\tau) \subseteq U \cap V$.
- This ensures that ω and τ are in $\Omega^k_{c,U\cap V}(U)$. □

Exactness at $\Omega_c^*(U) \oplus \Omega_c^*(V)$.

- As $\tau \in \Omega^k_{c,U\cap V}(V)$, we have
$$\tau = (j_V)_* \circ (j_V)^* \tau = (j_V)_*(\tau_{|U\cap V}) = (j_V)_* \sigma.$$
- Likewise, as $\omega \in \Omega^k_{c,U\cap V}(U)$, we also have
$$\omega = (j_U)_*(\omega_{|U\cap V}) = -(j_U)_* \sigma.$$
- Thus,
$$(\omega, \tau) = (-(j_U)_* \sigma, (j_V)_* \sigma) = j(\sigma) \in \mathrm{im}(j).$$
- This shows that $\ker(i) \subseteq \mathrm{im}(j)$.
- As $\mathrm{im}(j) \subseteq \ker(i)$, we deduce that $\mathrm{im}(j) = \ker(i)$.
- This proves exactness at $\Omega_c^*(U) \oplus \Omega_c^*(V)$.

The proof is complete. □

Mayer-Vietoris Sequence for Ω_c^*

By applying Zig-Zag Lemma we obtain:

Theorem (Mayer-Vietoris Sequence)

If U and V are open subsets of M such that $M = U \cup V$, then we have a long exact sequence,

$$\cdots \longrightarrow H_c^k(U \cap V) \xrightarrow{j_*} H_c^k(U) \oplus H_c^k(V) \xrightarrow{i_*} H_c^k(M) \xrightarrow{\delta} H_c^{k+1}(U \cap V) \longrightarrow \cdots$$

Remark

1. The map $j_* : H_c^k(U \cap V) \to H_c^k(U) \oplus H_c^k(V)$ is given by

$$j_*[\sigma] = [j(\sigma)] = [(-(j_U)_*\sigma, (j_V)_*\sigma)] = (-[(j_U)_*\sigma], [(j_V)_*\sigma]),$$

where $(j_U)_*\sigma$ and $(j_V)_*\sigma$ are extensions by 0 of σ to U and V.

2. The map $i_* : H_c^k(U) \oplus H_c^k(V) \to H_c^k(M)$ is given by

$$i_*([\omega], [\tau]) = [i(\omega, \tau)] = [(i_U)_*\omega + (i_V)_*\tau],$$

where $(i_U)_*\omega$ and $(i_V)_*\tau$ are extensions by 0 of ω and τ to M.

Mayer-Vietoris Sequence for Ω_c^*

Proposition

Let $\{\rho_U, \rho_V\}$ *be a* C^∞*-partition of unity subordinate to* $\{U, V\}$*. Given any closed form* $\omega \in \Omega_c^k(M)$*, we have*

$$\delta([\omega]) = -[(d\rho_U \wedge \omega)_{|U \cap V}].$$

Mayer-Vietoris Sequence for Ω_c^*

Proof.

- We have $\omega = \rho_U \omega + \rho_V \omega = i(\omega_U, \omega_V)$, where

$$\omega_U := (\rho_U \omega)_{|U}, \qquad \omega_V := (\rho_U \omega)_{|V}.$$

- As $d\omega = 0$ and $d\rho_V = -d\rho_V$, we have

$$d\omega_U = (d\rho_U \wedge \omega)_{|U}, \quad d\omega_V = (d\rho_V \wedge \omega)_{|V} = -(d\rho_U \wedge \omega)_{|V}.$$

- As $\mathrm{supp}(d\rho_U) = \mathrm{supp}(d\rho_V) \subseteq U \cap V$, we see that $d\omega_U$ and $d\omega_V$ are supported in $U \cap V$.
- Therefore, if we set $\sigma := -(d\rho_U \wedge \omega)_{|U \cap V}$, then

$$d\omega_U = (j_U)_*((d\omega_U)_{|U \cap V}) = -(j_U)_*\sigma,$$
$$d\omega_V = (j_V)_*((d\omega_V)_{|U \cap V}) = (j_V)_*\sigma.$$

- Thus,

$$d(\omega_U, \omega_V) = (d\omega_U, d\omega_V) = (-(j_U)_*\sigma, (j_V)_*\sigma) = j(\sigma).$$

- We then have $\delta([\omega]) = [\sigma] = -[(d\rho_U \wedge \omega)_{|U \cap V}]$. □