Differential Forms in Algebraic Topology: Homotopy Invariance and Poincaré Lemmas

Sichuan University, Spring 2024

References

Main References

- Sections 27 & 29 of Tu2011.
- Section 4 of Bott-Tu.

Smooth Homotopy

Setup

M and N are smooth manifolds.

Definition

Two smooth maps $f, g : M \to N$ are smoothly homotopic if there is a smooth function $F : M \times \mathbb{R} \to N$ (called homotopy) such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x) \quad \text{for all } x \in M.$$

Remark

In other words there is C^∞-family of smooth maps $f_t(x) := F(x, t)$, $t \in \mathbb{R}$, such that $f_0 = f$ and $f_1 = g$.

Smooth Homotopy

Example (Straight-line homotopy)

Suppose that $N = \mathbb{R}^n$.

- Any pair of smooth maps $f, g : M \to \mathbb{R}^n$ are (smoothly) homotopic by means of the straight-line homotopy,

$$F(x,t) = (1-t)f(x) + tg(x), \qquad (x,t) \in M \times \mathbb{R}.$$

- Given any $x \in M$, if $f(x) \neq g(x)$, then, as $t \in \mathbb{R}$ varies, the point $F(x,t)$ ranges over the straight line through $f(x)$ and $g(x)$.

Example

Any smooth map $f : M \to N$ is homotopic to itself by means of the homotopy,

$$F(x,t) = f(x), \qquad (x,t) \in M \times \mathbb{R}.$$

Smooth Homotopy

Definition

If two smooth maps $f, g : M \to N$ are smoothly homotopic, then we write $f \sim g$.

Fact (Tu2011, Exercise 27.2)

Smooth homotopy $\sim$ is an equivalence relation on smooth maps from M to N.

Homotopy Type

Notation (see Tu2011)

$\mathbb{1}_M$ is the identity map of M.

Definition

A (smooth) map $f : M \to N$ is a called a homotopy equivalence if it has a homotopy inverse, i.e., there is a smooth map $g : N \to M$ such that

$$g \circ f \sim \mathbb{1}_M \qquad \text{and} \qquad f \circ g \sim \mathbb{1}_N.$$

Example

Any diffeomorphism $f : M \to N$ is a homotopy equivalence, since

$$f^{-1} \circ f = \mathbb{1}_M \sim \mathbb{1}_M \qquad \text{and} \qquad f \circ f^{-1} = \mathbb{1}_N \sim \mathbb{1}_N.$$

Homotopy Type

Definition

We say that M and N have the same homotopy type whenever there is a homotopy equivalence $f : M \to N$.

Remark

Having the same homotopy type is an equivalence relation for manifolds.

Remark

We will see later that if M and N have the same homotopy type, then any homotopy equivalence $f : M \to N$ gives rise to an isomorphism,

$$f^* : H^*(N) \xrightarrow{\sim} H^*(M).$$

Example

The punctured plane $\mathbb{R}^2 \setminus \{0\}$ and the sphere $\mathbb{S}^1$ have the same homotopy type:

- Let $i : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$ be the inclusion map.
- Define the smooth map $r : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1$ by

$$r(x) = \frac{x}{\|x\|}, \qquad x \neq 0.$$

- We have $r \circ i = \mathbb{1}_{\mathbb{S}^1}$.
- Here $i \circ r(x) = \|x\|^{-1} x \sim \mathbb{1}_{\mathbb{R}^2 \setminus \{0\}}$ by means of the homotopy,

$$F(x,t) = t^2 x + (1-t)^2 \frac{x}{\|x\|}, \qquad (x,t) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}.$$

- Note that if $x \neq 0$, then $F(x,t) \neq 0$ for all $t \in \mathbb{R}$, since $\|F(x,t)\| = (t^2 + (1-t)^2 \|x\|^{-1}) \|x\| > 0$.
- This shows that $i : \mathbb{S}^1 \to \mathbb{R}^2 \setminus \{0\}$ is a homotopy equivalence.

Homotopy Type

Remark

- For any $p \in \mathbb{R}^2$, the punctured plane $\mathbb{R}^2 \setminus \{p\}$ and $\mathbb{S}^1$ have the same homotopy type.
- We just need to replace the maps i and r by

$$i_p(y) = p + y, \qquad r_p(x) = \frac{x-p}{\|x-p\|}, \qquad x \neq p.$$

- We have $r_p \circ i_p = \mathbb{1}_{\mathbb{S}^1}$.
- We also see that $i_p \circ r_p(x) = p + \|x-p\|^{-1}(x-p) \sim \mathbb{1}_{\mathbb{R}^2 \setminus \{p\}}$ by using the homotopy,

$$F(x,t) = p + t^2(x-p) + (1-t)^2 \frac{x-p}{\|x-p\|}, \qquad x \neq p, \ t \in \mathbb{R}.$$

Remark

More generally, if $p \in \mathbb{R}^n$, then $\mathbb{R}^n \setminus \{p\}$ and $\mathbb{S}^{n-1}$ have the same homotopy type for any $n \geq 2$.

Definition

We say that M is contractible if it has the same homotopy type as a point.

Remark

- If $N = \{q\}$ is a singleton, then the unique (smooth) map $f : M \to N$ is the constant map $x \to q$.
- In particular, the unique smooth map $N \to N$ is the identity map $\mathbb{1}_N$.

Facts

- Let $f : M \to N$ have homotopy inverse $g : N \to M$, and set $p = g(q)$.
- Then $f \circ g$ maps N to itself, and hence $f \circ g = \mathbb{1}_N$.
- The map $g \circ f : M \to M$ is the constant map $x \to p$.
- By assumption $g \circ f$ is homotopic to the identity map $\mathbb{1}_M$.

Therefore, we have the following result:

Proposition

The following are equivalent:

1. M *is contractible.*
2. *The identity map* $\mathbb{1}_M$ *is homotopic to a constant map.*

Remarks

1. The 2nd condition means there are $p \in M$ and a smooth map $F : M \times \mathbb{R} \to M$ such that
$$F(x, 1) = x \qquad \text{and} \qquad F(x, 0) = p \quad \text{for all } x \in M.$$
2. This implies that any contractible manifold is path-connected, and hence is connected.

Example

The Euclidean spaces $\mathbb{R}^n$, $n \geq 1$, are contractible:

- Define $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by
$$F(x,t) = tx, \qquad x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$
- As $F(x,1) = x$ and $F(x,0) = 0$ we get a smooth homotopy between the identity map $\mathbb{1}_{\mathbb{R}^n}$ and the zero map.
- It follows that $\mathbb{R}^n$ is contractible.

Deformation Retractions

Setup

S is a submanifold of M with inclusion map $i : S \to M$.

Definition

A retraction from M to S is a smooth map $r : M \to S$ such that $r(x) = x$ for all $x \in S$.

Remark

In other words, a retraction $r : M \to S$ is such that $r \circ i = \mathbb{1}_S$, i.e., this is a left-inverse of the inclusion map $i : S \to M$.

Remark

If there exists a retraction $r : M \to S$, then we say that S is a retract of M.

Deformation Retractions

Definition

We say that S is a deformation retract of M is there is a smooth homotopy $F : M \times \mathbb{R} \rightarrow M$ such that

(i) $F(x, 0) = x$ for all $x \in M$.

(ii) $F(x, 1) \in S$ for all $x \in M$.

(iii) $F(x, t) = x$ for all $x \in S$ and $t \in \mathbb{R}$.

Remarks

- Define $r : M \rightarrow S$ by $r(x) = F(x, 1)$, $x \in M$.
- By (iii) we have $r(x) = F(x, 1) = x$ for all $x \in S$, i.e., r is a retraction from M to S (and hence $r \circ i = \mathbb{1}_S$).
- Moreover, $F(x, t)$ is a smooth homotopy from $F(\cdot, 1) = i \circ r$ and $F(\cdot, 0) = \mathbb{1}_M$, and hence $i \circ r \sim \mathbb{1}_M$.
- Thus, r is a homotopy inverse of the inclusion $i : S \rightarrow M$.

Deformation Retractions

Therefore, we have the following result:

Proposition

If S is a deformation retract of M, then there is a retraction $r : M \to S$ such that

$$r \circ i = \mathbb{1}_S \qquad \text{and} \qquad i \circ r \sim \mathbb{1}_M.$$

In particular, the inclusion map $i : S \to M$ is a homotopy equivalence.

Corollary

If S is a deformation retract of M, then M has the same homotopy type as S.

Example

The singleton $\{0\}$ is a deformation retract of $\mathbb{R}^n$:

- We use the straight-line homotopy,

$$F(x,t) = (t-1)x, \qquad x \in \mathbb{R}^n, \; t \in \mathbb{R}.$$

- We have

$$F(x,0) = x, \qquad F(x,1) = 0, \qquad F(0,t) = 0.$$

- Thus, F is a deformation retraction from $\mathbb{R}^n$ to $\{0\}$.

Deformation Retractions

Example

The circle $\mathbb{S}^1$ is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$:

- We use the homotopy $F : (\mathbb{R}^2 \setminus 0) \times \mathbb{R} \to \mathbb{R}^2 \setminus 0$ given by
$$F(x,t) = \cos^2(\pi t/2)x + \sin^2(\pi t/2)\frac{x}{\|x\|}, \qquad x \neq 0,\ t \in \mathbb{R}.$$
- We have
$$F(x,0) = x \quad \text{and} \quad F(x,1) = \frac{x}{\|x\|} \in \mathbb{S}^1 \qquad \text{for } x \neq 0,$$
$$F(x,t) = (\cos^2(\pi t/2)x + \sin^2(\pi t/2))x = x \qquad \text{for all } x \in \mathbb{S}^1.$$
- Thus F is a deformation retraction from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{S}^1$.

Homotopy Axiom for De Rham Cohomology

Theorem (Homotopy axiom for de Rham cohomology)

If two smooth maps $f_0, f_1 : M \to N$ *are homotopic, then they induce the same map on de Rham cohomology,*

$$f_0^* = f_1^* : H^*(N) \longrightarrow H^*(M).$$

Remark

The proof of the theorem is postponed to the end of these slides.

Corollary

If $f : M \to N$ is a smooth homotopy equivalence, then it descends to an isomorphism,

$$f^* : H^*(N) \xrightarrow{\sim} H^*(M).$$

Proof.

- Let $g : N \to M$ be a homotopy inverse of g, i.e., $g \circ f \sim \mathbb{1}_M$ and $f \circ g \sim \mathbb{1}_N$.
- The fact that $g \circ f \sim \mathbb{1}_M$ ensures that at the level of cohomology, we have
$$f^* \circ g^* = (g \circ f)^* = \mathbb{1}_M^* = \mathrm{id} \qquad \text{on } H^*(M).$$
- Likewise,
$$g^* \circ f^* = (f \circ g)^* = \mathbb{1}_N^* = \mathrm{id} \qquad \text{on } H^*(N).$$
- Thus, $f^* : H^*(N) \to H^*(M)$ and $g^* : H^*(M) \to H^*(N)$ are inverses of each other, and hence are isomorphisms. □

Homotopy Axiom for De Rham Cohomology

Corollary

If a submanifold $S \subseteq M$ is a deformation retract of M, then the inclusion map $i : S \to M$ gives rise to an isomorphism,

$$i^* : H^*(M) \xrightarrow{\sim} H^*(S).$$

Proof.

- If S is a deformation retract of M, then the inclusion map $i : S \to M$ is a homotopy equivalence.
- It then induces an isomorphism on cohomology. □

Remark

- The pullback map $i^* : \Omega^*(M) \to \Omega^*(S)$ agrees with the restriction map $\omega \to \omega_{|S}$.
- Therefore, if S is a deformation retract, then the restriction map induces an isomorphism on cohomology.

Homotopy Axiom for De Rham Cohomology

Remark

If $N = \{q\}$, then $\dim N = 0$, and hence $H^k(N) = 0$ for $k \geq 1$.

Corollary

If M is contractible, then

$$H^k(M) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{for } k \geq 1. \end{cases}$$

Proof.

- As M is contractible, it is connected, and so $H^0(M) = \mathbb{R}$.
- M has the same homotopy type as a singleton $N = \{q\}$.
- Thus $H^k(M) = H^k(N) = 0$ for $k \geq 1$.

□

Poincaré Lemma

As a special case of the previous result we get:

Theorem (Poincaré Lemma)

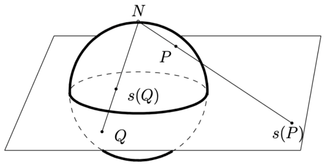
For all $n \geq 1$, we have

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{for } k \geq 1. \end{cases}$$

In particular, for $k \geq 1$, every closed k-form on $\mathbb{R}^n$ is exact.

Lemma

Set $N = (0, \ldots, 0, 1) \in \mathbb{S}^n$.



1. *In Cartesian coordinates, the stereographic projection* $\varphi : \mathbb{S}^n \setminus \{N\} \rightarrow \mathbb{R}^n$ *is given by*

$$\varphi(x) = \frac{1}{1 - x^{n+1}}(x^1, \ldots, x^n), \qquad x = (x^1, \ldots, x^{n+1}) \in \mathbb{S}^n.$$

2. *This is a smooth diffeomorphism with inverse,*

$$\varphi^{-1}(y) = \frac{1}{\|y\|^2 + 1}(2y^1, \ldots, 2y^n, \|y\|^2 - 1), \quad y = (y^1, \ldots, y^n) \in \mathbb{R}^n.$$

De Rham Cohomology of $\mathbb{S}^n$

Proposition

We have

$$H^k(\mathbb{S}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

- As $\mathbb{S}^n$ is a connected manifold of dimension n, we have
$$H^0(\mathbb{S}^n) = \mathbb{R}, \qquad H^k(\mathbb{S}^n) = 0 \quad \text{for } k \geq n+1.$$
- To prove the result for $1 \leq k \leq n$ we proceed by induction.
- We know the result for $n = 1$ already.
- Suppose that the result is known for $n-1$ with $n \geq 2$.
- We have an open covering $\mathbb{S}^n = U \cup V$, where
$$U = \mathbb{S}^n \setminus \{N\}, \quad V = -U = \mathbb{S}^n \setminus \{S\}, \quad U \cap V = \mathbb{S}^n \setminus \{N, S\}.$$
- We thus have a Mayer-Vietoris long exact sequence,

$$\cdots \rightarrow H^{k-1}(U) \oplus H^{k-1}(V) \rightarrow H^{k-1}(U \cap V) \rightarrow H^k(\mathbb{S}^n) \rightarrow H^k(U) \oplus H^k(V) \rightarrow \cdots$$

□

Proof (continued).

- The stereographic projection gives a diffeomorphism $U \simeq \mathbb{R}^n$.
- Thus,
$$H^k(U) = H^k(\mathbb{R}^n) = 0 \qquad \text{for } k \geq 1.$$
- As V is diffeomorphic to U under the involution $x \to -x$, we also have
$$H^k(V) = H^k(U) = 0 \qquad \text{for } k \geq 1.$$
- The stereographic projection of $S \in \mathbb{S}^n$ is the origin $0 \in \mathbb{R}^n$.
- We thus get a diffeomorphism $U \cap V = \mathbb{S}^n \setminus \{N, S\} \simeq \mathbb{R}^n \setminus 0$.
- We know that $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{R}^n \setminus 0$.
- Thus,
$$H^k(U \cap V) = H^k(\mathbb{R}^n \setminus 0) = H^k(\mathbb{S}^{n-1}).$$

□

Proof (continued).

- If $k \geq 2$, then

$$H^{k-1}(U) \oplus H^{k-1}(V) = H^k(U) \oplus H^k(V) = 0,$$
$$H^{k-1}(U \cap V) = H^{k-1}(\mathbb{S}^{n-1}).$$

- The Mayer-Vietoris sequence then yields an exact sequence,

$$0 \longrightarrow H^{k-1}(\mathbb{S}^{n-1}) \longrightarrow H^k(\mathbb{S}^n) \longrightarrow 0.$$

- We then get

$$H^k(\mathbb{S}^n) \simeq H^{k-1}(\mathbb{S}^{n-1}) = \begin{cases} \mathbb{R} & \text{for } k = n, \\ 0 & \text{for } 2 \leq k \leq n-1. \end{cases}$$

□

De Rham Cohomology of $\mathbb{S}^n$

Proof (continued).

- It remains to compute $H^1(\mathbb{S}^n)$.
- As $H^1(U) \oplus H^1(V) = 0$, the Mayer-Vietoris sequence yields an exact sequence,

$$0 \rightarrow H^0(\mathbb{S}^n) \rightarrow H^0(U) \oplus H^0(V) \rightarrow H^0(U \cap V) \rightarrow H^1(\mathbb{S}^n) \rightarrow 0.$$

- Here $H^0(U) = H^0(V) = H^0(U \cap V) = H^0(\mathbb{S}^n) = \mathbb{R}$.
- We thus get an exact sequence,

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow H^1(\mathbb{S}^n) \longrightarrow 0.$$

- Taking the alternating sum of dimensions then gives

$$1 - 2 + 1 - \dim H^1(\mathbb{S}^n) = 0.$$

- That is, $\dim H^1(\mathbb{S}^n) = 0$, and hence $H^1(\mathbb{S}^n) = 0$.

This completes the proof. □

Proof of Homotopy Invariance: Reduction to Two Sections

Setup

- $f, g : M \to N$ are homotopic smooth maps.
- $F : M \times \mathbb{R} \to N$ is a smooth homotopy such that

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x) \quad \text{for all } x \in M.$$

We want to prove:

Theorem

f and g induce the same map on de Rham cohomology,

$$f^* = g^* : H^*(N) \longrightarrow H^*(M).$$

Proof of Homotopy Invariance: Reduction to Two Sections

Definition

The C^∞-maps $i_0 : M \to M \times \mathbb{R}$ and $i_1 : M \to M \times \mathbb{R}$ are given by

$$i_0(x) = (x, 0) \quad \text{and} \quad i_1(x) = (x, 1), \qquad x \in M.$$

Facts

- We have

$$f(x) = F(x, 0) = f \circ i_0(x), \qquad g(x) = F(x, 1) = f \circ i_1(x).$$

- Thus, at the level of cohomology, we get:

$$f^* = (F \circ i_0)^* = i_0^* \circ F^*, \qquad g^* = (F \circ i_1)^* = i_1^* \circ F^*.$$

- Therefore, **in order to show that** $f^* = g^*$ **it is enough to prove that** $i_0^* = i_1^*$.

Proof of Homotopy Invariance: Cochain Homotopy

Setup

- $\mathscr{A} = (A^*, d)$ and $\mathscr{B} = (B^*, d)$ are cochain complexes.
- $\varphi, \psi : A^* \to B^*$ are cochain maps.

Definition

A cochain homotopy from φ to ψ is a degree -1 linear map $K : A^* \to B^{*-1}$ such that

$$\varphi - \psi = d \circ K + K \circ d.$$

Proposition

If there is a cochain homotopy from φ to ψ, then φ and ψ induce the same map on cohomology,

$$\varphi^* = \psi^* : H^*(\mathscr{A}) \longrightarrow H^*(\mathscr{B}).$$

Proof of Homotopy Invariance: Cochain Homotopy

Proof.

- Given any cocycle $a \in Z^k(\mathscr{A})$, we have
$$\varphi^*[a] - \psi^*[a] = [\varphi(a)] - [\psi(a)] = [\varphi(a) - \psi(a)].$$
- As $\varphi - \psi = dK + Kd$ and $da = 0$, we have
$$\varphi(a) - \psi(a) = d(K(a)) + K(da) = d(K(a)).$$
- Thus,
$$\varphi^*[a] - \psi^*[a] = [d(K(a))] = 0.$$

This proves the result. □

Proof of Homotopy Invariance: Homotopy from i_1^* to i_0^*

Setup

- M is a smooth manifold of dimension n.
- $i_0, i_1 : M \to M \times \mathbb{R}$ are the embeddings $x \to (x, 1)$ and $x \to (x, 0)$.
- They give rise to cochain maps $i_0^*, i_1^* : \Omega^*(M \times \mathbb{R}) \to \Omega^*(M)$.

Strategy

- We shall construct a linear map $K : \Omega^*(M \times \mathbb{R}) \to \Omega^{*-1}(M)$ such that
$$i_1^* - i_0^* = d \circ K + K \circ d.$$
- This will exhibit a cochain homotopy from i_1^* to i_0^*.
- It will then follow that i_1^* and i_0^* induce the same map on de Rham cohomology.

Proof of Homotopy Invariance: Homotopy from i_1^* to i_0^*

Facts

- If $(U, x^1, \ldots, x^n)$ are local coordinates for M, then $(U \times \mathbb{R}, x^1, \ldots, x^n, t)$ are local coordinates for $M \times \mathbb{R}$.
- Thus, on $U \times \mathbb{R}$, any $\omega \in \Omega^k(M)$, can be uniquely written as
$$\omega = \sum_I a_I(x,t)dx^I + \sum_J b_J(x,t)dx^J \wedge dt,$$
where I ranges over $\mathscr{I}_{n,k}$ and J ranges over $\mathscr{I}_{n,k-1}$.
- Thus, if we set $\omega_0 := \sum a_I dx^I$ and $\omega_1 := \sum b_J dx^J$, then
$$\omega = \omega_0 + \omega_1 \wedge dt.$$

Proof of Homotopy Invariance: Homotopy from i_1^* to i_0^*

Lemma

There a well-defined linear map $K : \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$ *such that, given any* $\omega \in \Omega^k(M \times \mathbb{R})$*, if* $(U, x^1, \ldots, x^n)$ *are local coordinates for* M *and* $\omega = \sum a_I dx^I + \sum b_J dx^J \wedge dt$ *on* $U \times \mathbb{R}$*, then*

$$K\omega = (-1)^{k-1} \sum_J \left(\int_0^1 b_J(x,t)dt \right) dx^J \quad \text{on } U.$$

Proof.

- The map K is well defined in a local chart.
- We need to show that the definition does not depend on the choice of the local coordinates.
- Namely, if $(U, y^1, \ldots, y^n)$ are local coordinates on U and $\omega = \sum_I c_I dy^I + \sum_J d_J y^J$, then we need to show that, for all $p \in U$, we have

$$\sum_J \left(\int_0^1 b_J(p,t)dt \right) dx^J = \sum_J \left(\int_0^1 d_J(p,t)dt \right) dy^J.$$

- On U we may write

$$y^I = \sum_{I'} \varepsilon^I_{I'} dx^{I'}, \qquad \varepsilon^I_{I'} = \frac{\partial(y^{i_1}, \ldots, y^{i_k})}{\partial(x^{i'_1}, \ldots, x^{i'_k})} \in C^\infty(U),$$

with $I = (i_1, \ldots, i_k)$ and $I' = (i'_1, \ldots, i'_k)$. □

Proof.

- Let $p \in U$. We have
$$\omega(p) = \sum_{I'} a_{I'}(p,t)dx^{I'} + \sum_{J'} b_{J'}(p,t)dx^{J'} \wedge dt.$$
- We also have
$$\begin{aligned}\omega(p) &= \sum_{I} c_I(p,t)dy^I + \sum_J d_J(p,t)dy^J \wedge dt \\ &= \sum_{I,I'} c_I(p,t)\varepsilon^I_{I'}(p)dx^{I'} + \sum_{J,J'} d_J(p,t)\varepsilon^J_{J'}(p)dx^{J'} \wedge dt \\ &= \sum_{I'}\left(\sum_I c_I(p,t)\varepsilon^I_{I'}(p)\right)dx^{I'} \\ &\quad + \sum_{J'}\left(\sum_J d_J(p,t)\varepsilon^J_{J'}(p)\right)dx^{J'} \wedge dt.\end{aligned}$$
- Thus,
$$b_{J'}(p,t) = \sum_J \varepsilon^J_{J'}(p)d_J(p,t).$$

□

Proof.

- Therefore, we have

$$\sum_{J'} \left(\int_0^1 b_{J'}(p,t)dt \right) dx^{J'} = \sum_{J',J} \left(\int_0^1 \varepsilon^J_{J'}(p) d_J(p,t)dt \right) dx^{J'}$$
$$= \sum_{J} \left(\int_0^1 d_J(p,t)dt \right) \sum_{J'} \varepsilon^J_{J'}(p) dx^{J'}.$$

- As $dy^J = \sum_J \varepsilon^J_{J'}(p) dx^{J'}$, we then get

$$\sum_{J'} \left(\int_0^1 b_{J'}(p,t)dt \right) dx^{J'} = \sum_{J} \left(\int_0^1 d_J(p,t)dt \right) dy^J.$$

This completes the proof. □

Proof of Homotopy Invariance: Homotopy from i_1^* to i_0^*

Lemma

For all $\omega \in \Omega^k(M \times \mathbb{R})$, *we have*

$$i_1^*\omega - i_0^*\omega = d(K\omega) + K(d\omega).$$

Proof.

- It's enough to prove the result in local coordinates.
- Let $(U, x^1, \ldots, x^n)$ be local coordinates for M.
- $(U \times \mathbb{R}, x^1, \ldots, x^n, t)$ then are local coordinates for $M \times \mathbb{R}$.
- Thus, we may write $\omega = \sum a_I dx^I + \sum b_J dx^J \wedge dt$ on $U \times \mathbb{R}$.

□

Proof of Homotopy Invariance: Homotopy from i_1^* to i_0^*

Proof – Computation of $i_1^*\omega - i_0^*\omega$.

- In local coordinates, $i_0 : M \to M \times \mathbb{R}$ is just the embedding $(x^1, \ldots, x^n) \to (x^1, \ldots, x^n, 0)$, and hence $(i_0)_*(\partial_{x^i}) = \partial_{x^i}$.
- Thus, if $I = (i_1, \ldots, i_k)$ and $\partial_I = (\partial_{x^{i_1}}, \ldots, \partial_{x^{i_k}})$, then

$$(i_0^*\omega)(\partial_I)(p) = \omega(i_0^*\partial_{x^{i_1}}, \ldots, i_0^*\partial_{x^{i_1}})(i_0(p)) = \omega(\partial_I)(p, 0).$$

- If $\omega = \sum a_I dx^I + \sum b_J dx^J \wedge dt$, then $\omega(\partial_I) = a_I$.
- It then follows that $(i_0^*\omega)(p)$ is equal to

$$\sum (i_0^*\omega)(\partial_I)(p) dx^I = \sum_I \omega(\partial_I)(p, 0) dx^I = \sum_I a_I(p, 0) dx^I.$$

- Likewise, we have

$$(i_1^*\omega)(p) = \sum_I a_I(p, 1) dx^I.$$

□

Proof – Computation of $i_1^*\omega - i_0^*\omega$.

- We then see that $(i_1^*\omega)(p) - (i_0^*\omega)(p)$ is equal to

$$\sum_I a_I(p,1)dx^I - \sum_I a_I(p,0)dx^I = \sum_I \left(a_I(p,1) - a_I(p,0)\right) dx^I.$$

- Note that

$$a_I(p,1) - a_I(p,0) = \int_0^1 \partial_t a_I(x,t)dt.$$

- Thus,

$$(i_1^*\omega)(p) - (i_0^*\omega)(p) = \sum_I \left(\int_0^1 \partial_t a_I(x,t)dt \right) dx^I.$$

□

Proof – Computation of $d(K\omega)$.

- By definition, on U we have

$$K\omega = (-1)^{k-1}\sum_J \left(\int_0^1 b_J(x,t)dt\right)dx^J.$$

- Thus, on U we have

$$\begin{aligned} K\omega &= (-1)^{k-1}\sum_i\sum_J \partial_{x^i}\left(\int_0^1 b_J(x,t)dt\right)dx^i\wedge dx^J \\ &= (-1)^{k-1}\sum_{i,J}\left(\int_0^1 \partial_{x^i} b_J(x,t)dt\right)dx^i\wedge dx^J. \end{aligned}$$

□

Proof – Computation of $K(d\omega)$.

- As $\omega = \sum a_I dx^I + \sum b_J dx^J \wedge dt$ on U, we have

$$\begin{aligned} d\omega = & \sum_{i,I} \partial_{x^i} a_I dx^i \wedge dx^I + \sum_I \partial_t a_I dt \wedge dx^I \\ & + \sum_{i,J} \partial_{x^i} b_J dx^i \wedge dx^J \wedge dt \\ = & \sum_{i,I} \partial_{x^i} a_I dx^i \wedge dx^I + (-1)^k \sum_I \partial_t a_I dx^I \wedge dt \\ & + \sum_{i,J} \partial_{x^i} b_J dx^i \wedge dx^J \wedge dt. \end{aligned}$$

□

Proof – Computation of $K(d\omega)$.

- Thus, taking into account that $d\omega$ has degree $k+1$, we get

$$\begin{aligned} K(d\omega)(p) = & \sum_I \left(\int_0^1 \partial_t a_I(p,t)dt \right) dx^I \\ & + (-1)^k \sum_{i,J} \left(\int_0^1 \partial_{x^i} b_J(p,t)dt \right) dx^i \wedge dx^J \\ = & i_1^*\omega(p) - i_0^*\omega(p) - d(K\omega)(p). \end{aligned}$$

- This shows that

$$i_1^*\omega - i_0^*\omega = K(d\omega) + d(K\omega).$$

The proof is complete. □

Poincaré Lemma for Compact Cohomology

Setup

M is a smooth manifold of dimension n.

For the compactly supported de Rham cohomology we are going to show the following result:

Proposition (see Bott-Tu)

We have

$$H_c^k(M \times \mathbb{R}) \simeq H_c^{k-1}(M).$$

Corollary (Poincaré Lemma for Compact Cohomology)

We have

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

- We proceed by induction on n.
- We know the result for $n = 1$.
- Assume the result is true for $n - 1$ with $n \geq 2$.
- The previous proposition then gives

$$H_c^n(\mathbb{R}^n) = H_c^{n-1}(\mathbb{R}^{n-1}) = \mathbb{R},$$
$$H_c^k(\mathbb{R}^n) = H_c^{k-1}(\mathbb{R}^{n-1}) = 0, \qquad k \neq n.$$

This completes the proof. □

Notation (Shifted cochain complex)

If $\mathscr{A} = (A^*, d)$ is a cochain complex, then $\mathscr{A}[-1]$ is the cochain complex such that

- The space of k-cochains is A^{k-1}.
- The differential in degree k is $d : A^{k-1} \rightarrow A^k$.

Remark

We then have

$$H^k(\mathscr{A}[-1]) = H^{k-1}(\mathscr{A}).$$

Poincaré Lemma for Compact Cohomology

Setup

M is a smooth manifold of dimension n.

Reminder

- If $(U, x^1, \ldots, x^n)$ are local coordinates for M, then $(U \times \mathbb{R}, x^1, \ldots, x^n, t)$ are local coordinates for $M \times \mathbb{R}$.
- Thus, on $U \times \mathbb{R}$ any form $k \in \Omega^k(M \times \mathbb{R})$ takes the form,

$$\omega = \sum_I a_I(x,t)dx^I + \sum_J b_J(x,t)dx^J \wedge dt,$$

where I ranges over $\mathscr{I}_{n,k}$ and J ranges over $\mathscr{I}_{n,k-1}$.

Poincaré Lemma for Compact Cohomology

Lemma

1. *There is a well-defined lin. map* $\pi : \Omega_c^k(M \times \mathbb{R}) \rightarrow \Omega_c^{k-1}(M)$ *such that, given any* $\omega \in \Omega_c^k(M \times \mathbb{R})$, *if* $(U, x^1, \ldots, x^n)$ *are local coordinates for* M *and* $\omega = \sum a_I dx^I + \sum b_J dx^J \wedge dt$ *on* $U \times \mathbb{R}$, *then*
$$\pi(\omega) = \sum_J \left(\int_{-\infty}^{\infty} b_J(x, t) dt \right) dx^J \quad \text{on } U.$$
2. π *commutes with* d.
3. *We thus get a cochain map,*
$$\pi : \Omega_c^*(M \times \mathbb{R}) \longrightarrow \Omega_c^*(M)[-1].$$

Poincaré Lemma for Compact Cohomology

Lemma

Let $\rho(t) \in C_c^\infty(\mathbb{R})$ *be such that* $\int_{-\infty}^{\infty} \rho(t)dt = 1$.

1. *There is a well-defined lin. map* $\varepsilon : \Omega^{k-1}(M) \to \Omega^k(M \times \mathbb{R})$ *such that, given any* $\omega \in \Omega^k(M)$, *if* $(U, x^1, \ldots, x^n)$ *are local coordinates for* M *and* $\omega = \sum b_J dx^J$ *on* U, *then*
$$\varepsilon(\omega) = \sum_J b_J(x)\rho(t)dx^J \wedge dt \quad \text{on } U \times \mathbb{R}.$$
2. ε *commutes with* d.
3. *It maps* $\Omega_c^{k-1}(M)$ *to* $\Omega_c^k(M \times \mathbb{R})$.
4. *We thus get a cochain map,*
$$\varepsilon : \Omega_c^*(M)[-1] \longrightarrow \Omega_c^*(M \times \mathbb{R}).$$

Poincaré Lemma for Compact Cohomology

Fact

$\pi \circ \varepsilon = \mathrm{id}$ on $\Omega_c^{k-1}(M)$.

Lemma (see Bott-Tu)

There is a cochain homotopy $K : \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M \times \mathbb{R})$ *such that*

$$\mathrm{id} - \varepsilon \circ \pi = dK + Kd \qquad \text{on } \Omega_c^k(M \times \mathbb{R}).$$

Remark

Given any $\omega \in \Omega_c^k(M \times \mathbb{R})$, if $(U, x^1, \ldots, x^n)$ are local coordinates for M and $\omega = \sum a_I dx^I + \sum b_J dx^J \wedge dt$ on $U \times \mathbb{R}$ on $U \times \mathbb{R}$, then

$$K(\omega) = (-1)^k \sum_J \left(\int_{-\infty}^t \tilde{b}_J(x, s) ds \right) dx^J \quad \text{on } U \times \mathbb{R},$$

where we have set $\tilde{b}_J(x,t) := b_J(x,t) - \rho(t) \int_{-\infty}^{\infty} b_J(x,s) ds$.

Poincaré Lemma for Compact Cohomology

This leads to the following result:

Proposition (see Bott-Tu)

1. *The cochain maps* $\pi : \Omega^*_c(M \times \mathbb{R}) \rightarrow \Omega^*_c(M)[-1]$ *and* $\varepsilon : \Omega^*_c(M)[-1] \rightarrow \Omega^*_c(M \times \mathbb{R})$ *are quasi-inverses of each other.*
2. *Therefore, on cohomology they induce isomorphisms,*

$$H^k_c(M \times \mathbb{R}) \simeq H^k(\Omega^*_c(M)[-1]) = H^{k-1}_c(M).$$

Poincaré Lemma for Compact Cohomology

Proof.

- As $\pi \circ \varepsilon = \mathrm{id}$, on $H^k(\Omega_c^*(M)[-1]) = H_c^{k-1}(M)$ we have

$$\pi^* \circ \varepsilon^* = (\pi \circ \varepsilon)^* = \mathrm{id}\,.$$

- By the previous lemma $\varepsilon \circ \pi$ is chain homotopic to the identity map on $\Omega_c^*(M \times \mathbb{R})$.
- Thus, it induces the identity map on cohomology, i.e.,

$$\varepsilon^* \circ \pi^* = (\varepsilon \circ \pi^*) = \mathrm{id}\,.$$

- This shows that π^* and ε^* are inverses of each other on cohomology.

This proves the result. □