Differentiable Forms in Algebraic Topology
Review: Vector Bundles

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Vector Bundles

Definition

A *vector bundle of rank* r over a manifold M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ such that:

(i) For every $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a vector space of dimension r.

(ii) For each $p \in M$ there is an open neighborhood U of p in M and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ (called *trivialization of* E *over* U) such that

- $\pi \circ \phi^{-1}(q, \xi^1, \dots, \xi^r) = q$ for all $q \in U$ and $(\xi^1, \dots, \xi^r) \in \mathbb{R}^r$.
- For each $q \in U$, the restriction of ϕ to E_q is a vector space isomorphism from E_q onto $\{q\} \times \mathbb{R}^r$.

Vector Bundles

Remarks

- We sometimes write a vector bundle as $E \xrightarrow{\pi} M$.
- We may also think of a vector bundle as a triple (E, M, π). In this picture E is called the *total space*, M is called the *base space*, and π is called the *projection*.

Remark

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle and S a regular submanifold in M. Then $\pi^{-1}(S) \xrightarrow{\pi} S$ is a smooth vector bundle over S denoted $E_{|S}$ and called the *restriction of E to S*.

Example

- A *trivial vector bundle* is of the form $E = M \times \mathbb{R}^r$.
- In this case the projection $\pi : M \times \mathbb{R}^r \to M$ is just the projection onto the first factor.

Example

- The tangent bundle TM is a vector bundle of rank n.
- If $(U, x^1, \dots, x^n)$ is a chart, then a trivialization of TM over U is the map $\psi : TU \to U \times \mathbb{R}^n$ given by

$$\psi\left(\sum v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = (p, v^1, \dots, v^n), \qquad p \in U,\ v^i \in \mathbb{R}.$$

In particular, $(\phi \times \mathbb{1}_{\mathbb{R}^n}) \circ \psi = \tilde{\phi}$.

Remark

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle. Suppose that $(U, \psi) = (U, x^1, \ldots, x^n)$ is a chart for M and we have a local trivialization,

$$\phi : E_{|U} \longrightarrow U \times \mathbb{R}^r, \qquad \phi(\xi) = \big(\pi(\xi), c^1(\xi), \ldots, c^r(\xi)\big).$$

Then $(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi : E_{|U} \to \psi(U) \times \mathbb{R}^r$ is a diffeomorphism, and we have

$$\begin{aligned}(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi &= (\psi \times \mathbb{1}_{\mathbb{R}^r})\big(\pi, c^1, \ldots, c^r\big) \\ &= (x^1 \circ \pi, \ldots, x^n \circ \pi, c^1, \ldots, c^r).\end{aligned}$$

In particular, $(\pi^{-1}(U), (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi)$ is a chart for E. We call $x^1, \ldots, x^n$ the *base coordinates* and $c^1, \ldots, c^n$ the *fiber coordinates*

Definition (Bundle Maps)

Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be smooth vector bundles. A *bundle map* from E to F is given by a pair of smooth maps $(f, \tilde{f})$, $f : M \to N$, $\tilde{f} : E \to F$ such that:

(i) $\pi_F \circ \tilde{f} = f \circ \pi_E$, i.e., we have a commutative diagram,

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow{f} & N'. \end{array}$$

(ii) For every $p \in M$, the map $\tilde{f}$ restricts to a linear map $\tilde{f} : E_p \to F_{f(p)}$.

Example

Any smooth map $f : M \to N$ gives rise to a bundle map $(f, \tilde{f})$ from TM to TN with $\tilde{f} = f_*$. Namely,

$$\tilde{f}(v) = f_{*,p}(v) \qquad p \in M, \; v \in T_pM.$$

Remarks

- The smooth vector bundles define a category where the objects are smooth vector bundles and the morphisms are bundle maps.
- From this point of view, the tangent bundle construction defines a functor from the category of smooth manifolds to the category of smooth vector bundles.

Remark

- We may also consider the category of vector bundles over a fixed manifold M.
- In this case the morphisms are bundle maps $(f, \tilde{f})$ with $f = \mathbb{1}_M$.

Smooth Sections

Definition (Section of a Vector Bundle)

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle.

- A *section* of E is any map $s : M \to E$ such that $\pi \circ s = \mathbb{1}_M$, i.e., $s(p) \in E_p$ for all $p \in M$,
- A *smooth section* is a section which is smooth as a map from M to E.

Remarks

- The set of smooth sections of E is denoted $\Gamma(E)$ or $\Gamma(M, E)$.
- If U is an open subset of M, we denote by $\Gamma(U, E)$ the set of smooth sections of $E_{|U}$.
- Sections of $E_{|U}$ are called *local sections*, whereas sections defined on the entire manifold M are called *global sections*.

Example: Vector Fields

Definition (Vector Field)

- A *vector field* is a section of the tangent bundle TM.
- A *smooth vector field* is a smooth section of TM.

Remark

In other words, a vector field $X : M \to TM$ assigns to each $p \in M$ a tangent vector $X_p \in T_pM$.

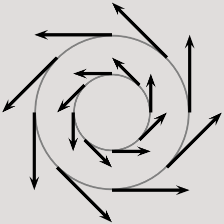
Example: Vector Fields

Example

On $\mathbb{R}^2$

$$X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \langle -y, x\rangle$$

is a smooth vector field on $\mathbb{R}^2$.



Smooth Sections – Module Structure

Proposition (Proposition 12.9)

Let E be a vector bundle over M. Then its set of smooth sections $\Gamma(E)$ is a module over the ring $C^\infty(M)$ with respect to the addition and scalar multiplication given by

$$(s_1 + s_2)(p) = s_1(p) + s_2(p), \quad s_i \in \Gamma(E), \quad p \in M,$$
$$(fs)(p) = f(p)s(p), \quad f \in C^\infty(M),\ s \in \Gamma(E),\ p \in M.$$

Remarks

- Here $s_1(p) + s_2(p)$ and $f(p)s(p)$ make sense as elements of the fiber E_p, since E_p is a vector space.
- If U is an open set, then $\Gamma(U, E)$ is a module over $C^\infty(U)$.

Smooth Frames

Definition (Frames of Vector Bundles)

Let E be a smooth vector bundle of rank r over M.

- A *frame* of E over an open $U \subseteq M$ is given by sections $s_1, \ldots, s_r$ such that $\{s_1(p), \ldots, s_r(p)\}$ is a basis of the fiber E_p for every $p \in U$.
- We say that the frame $\{s_1, \ldots, s_r\}$ is *smooth* when the sections $s_1, \ldots, s_r$ are smooth.

Remarks

- A frame of the tangent bundle is called a *tangent frame*, or simply a *frame*.
- For instance, $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is a smooth tangent frame over $\mathbb{R}^2$.

Example

Let $e_1, \ldots, e_r$ be the canonical basis of $\mathbb{R}^r$. For $i = 1, \ldots, r$, define $\tilde{e}_i : M \to M \times \mathbb{R}^r$ by

$$\tilde{e}_i(p) = (p, e_i), \qquad p \in M.$$

- Each map $\tilde{e}_i$ is a smooth section of the trivial bundle $M \times \mathbb{R}^r$.
- If $p \in M$, then $\{\tilde{e}_1(p), \ldots, \tilde{e}_r(p)\}$ is a basis of $\{p\} \times \mathbb{R}^r$.

Therefore, $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ is a smooth frame of $M \times \mathbb{R}^r$ over M.

Smooth Frames

Example (Frame of a trivialization)

Suppose E is a smooth vector bundle of rank r over M. Let $\phi : E_{|U} \to U \times \mathbb{R}^r$ be a trivialization over an open $U \subseteq M$.

- From the previous example $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ is a smooth frame of $U \times \mathbb{R}^r$ over U.
- As ϕ is smooth, $t_i = \phi^{-1} \circ \tilde{e}_i$ is a smooth map from U to $E_{|U}$.
- If $p \in U$, then $t_i(p) = \phi(\tilde{e}_i(p)) = \phi(p, e_i) \in E_p$, so t_i is a smooth section of E.
- The trivialization ϕ induces a linear isomorphism from E_p to $\{p\} \times \mathbb{R}^r$. It pullbacks the basis $\{\tilde{e}_i(p), \ldots, \tilde{e}_r(p)\}$ of $\{p\} \times \mathbb{R}^r$ to $\{t_1(p), \ldots, t_r(p)\}$, so the latter is a basis of E_p.
- Therefore, $\{t_1, \ldots, t_r\}$ is a smooth frame of E over U. It is called the *frame of the trivialization* (U, ϕ).

Smooth Frames

Facts

Let s be a section of E over U. If $p \in U$, then $s(p) \in E_p$ and $\{t_1(p), \ldots, t_r(p)\}$ is a basis of E_p. Thus, we may write

$$s(p) = \sum b^i(p) t_i(p), \qquad b^i(p) \in \mathbb{R}.$$

- If the coefficients $b_i(p)$ depends smoothly on p, then s is smooth.
- Conversely, suppose that s is a smooth section.
  - This implies that $\phi \circ s : U \to U \times \mathbb{R}^r$ is a smooth map.
  - If $p \in U$, then $\phi \circ s(p) = \phi\left[\sum b^i(p) t_i(p)\right] = \sum b^i(p)\phi[t_i(p)]$.
  - As $\phi[t_i(p)] = \phi[\phi^{-1}(\tilde{e}_i(p))] = \tilde{e}_i(p) = (p, e_i)$, we get

    $$\phi \circ s(p) = \sum b^i(p)(p, e_i) = (p, b^1(p), \ldots, b^r(p)).$$
  - As $\phi \circ s$ is a smooth map, the components $b^1(p), \ldots, b^r(p)$ must be smooth functions.

Smooth Frames

From the previous slide we obtain:

Lemma (Lemma 12.11)

Let $\phi : E_{|U} \to U \times \mathbb{R}^r$ be a trivialization of E over an open $U \subseteq M$ with frame $\{t_1, \ldots, t_n\}$. A section $s = \sum b^i t_i$ of E over U is smooth if and only if $b^1, \ldots, b^r$ are smooth functions.

More generally, we have:

Proposition (Proposition 12.12; see Tu's book)

Let $\{s_1, \ldots, s_r\}$ be a smooth frame of E over an open $U \subseteq M$. A section $s = \sum c^i s_i$ of E over U is smooth if and only if $c^1, \ldots, c^r$ are smooth functions.

Corollary

If $\{s_1, \ldots, s_r\}$ is a smooth frame of E over an open $U \subseteq M$, then this is a $C^\infty(U)$-basis of the $C^\infty(U)$-module $\Gamma(U, E)$.

Smooth Frames

Remark

Let $\{s_1, \ldots, s_r\}$ be a smooth frame of E over an open $U \subseteq M$. Define $\sigma : U \times \mathbb{R}^n \to E_{|U}$ by

$$\sigma(p, \xi^1, \ldots, \xi^r) = \sum \xi^i s_i(p), \qquad p \in U, \ \xi^i \in \mathbb{R}.$$

- The map σ is a smooth bijection that induces a linear isomorphism from $\{p\} \times \mathbb{R}^r$ onto E_p.
- It can be shown that the inverse map $\phi = \sigma^{-1} : E_{|U} \to U \times \mathbb{R}^r$ is smooth, and so this is a trivialization of E over U.
- The frame of (ϕ, U) is $\{s_1, \ldots, s_r\}$, since

$$\phi^{-1}(\tilde{e}_i(p)) = \sigma(p, e_i) = s_i(p).$$

It follows that we have a one-to-one correspondance between trivializations and smooth frames.

Example

Let $(U, x^1, \ldots, x^n)$ be a local chart for M.

- We know that $(U, x^1, \ldots, x^n)$ gives rise to the trivialization $\psi : TU \to U \times \mathbb{R}^n$ given by

$$\psi(v) = (p, v^1, \ldots, v^n) \quad \text{if } v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p \in T_pM, \; p \in U.$$

- In particular, as $\psi(\frac{\partial}{\partial x^i}\big|_p) = (p, e_i) = \tilde{e}_i(p)$, we have

$$t_i(p) = \psi^{-1}(\tilde{e}_i(p)) = \frac{\partial}{\partial x^i}\bigg|_p.$$

  Thus, $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ is the frame of the trivialization (U, ψ). In particular, this is a smooth tangent frame over U.