Differentiable Forms in Topology
Review: The Tangent Space

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Directional Derivative

Remarks

- A vector at a point p in $\mathbb{R}^3$ can be visualized as an arrow emanating from p.
- It can also be represented as a column,

$$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}.$$

Definition (Tangent Space)

- The tangent space $T_p(\mathbb{R}^n)$ at $p \in \mathbb{R}^n$ is the vector space of all arrows emanating from p.
- Elements of $T_p(\mathbb{R}^n)$ are called tangent vectors (or simply vectors).

Remarks

1. We identify $T_p(\mathbb{R}^n)$ with the space of n-columns, and hence tangent vectors are identified with n-columns.
2. We sometime write $T_p\mathbb{R}^n$ for $T_p(\mathbb{R}^n)$.

Directional Derivative

Convention

- We shall denote a point in $\mathbb{R}^n$ as $p = (p^1, \ldots, p^n)$ and a tangent vector in $T_p(\mathbb{R}^n)$ as
$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad \text{or} \quad v = \langle v^1, \ldots, v^n \rangle.$$
- We usually denote the canonical basis of $\mathbb{R}^n$ or $T_p(\mathbb{R}^n)$ by $e_1, \ldots, e_n$, so that $v = \sum v^j e_j$.

Directional Derivative

Definition

Let f be a C^∞-function on a neighbrohood of $p = (p^1, \ldots, p^n)$, and let $v = \langle v^1, \ldots, v^n \rangle$ be a tangent vector. The directional derivative of f in the direction of v at p is defined to be

$$D_v f = \frac{d}{dt}\bigg|_{t=0} f(p + tv).$$

Remarks

1. In other words $D_v f = \frac{d}{dt}\big|_{t=0} f(c(t))$, where $c(t) = p + tv$.
2. In the notation $D_v f$ it is implicitly understood that we evaluate at p, since v is a tangent vector at p. Thus, $D_v f$ is a number, not a function.

Directional Derivative

Fact

By using the Chain Rule we find that

$$D_v f = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p).$$

Definition

We write

$$D_v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p$$

for the map that assigns to any C^∞-function f near p its directional derivative $D_v f$.

Remark

As we shall see, the assignment $v \to D_v$ provides us with an alternative description of tangent vectors at p.

Observation

Two C^∞ functions that agree on a neighborhood of p have the same directional derivatives at p. Therefore, it is natural to declare such functions to be equivalent.

Germs of Functions

Definition

We define a relation on C^∞-functions near p at follows:

- S is a set of pairs (f, U), where U is a neighborhood of p and f is a C^∞-function on U.
- The relation on S is given by

$$(f, U) \sim (g, V) \iff f = g \text{ near } p.$$

Fact

This defines an equivalence relation.

Definition

1. The equivalence class of (f, U) is called the germ of f at p.
2. The set of all germs at p is denoted by $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞.

Example

The functions

$$f(x) = \frac{1}{1-x}, \qquad x \neq 1,$$
$$g(x) = 1 + x + x^2 + \cdots, \qquad |x| < 1,$$

have the same germ at any point of the interval $(-1, 1)$.

Fact

Let p be a point in $\mathbb{R}^n$.

1. The addition, scalar multiplication and multiplication of functions induces corresponding operations on the set of germs C_p^∞ (see Problem 2.2).
2. This turns C_p^∞ into an algebra over $\mathbb{R}$.

Facts

Let p be a point in $\mathbb{R}^n$ and v a tangent vector at p.

1. The directional derivative gives rise to a map,
$$D_v : C_p^\infty \longrightarrow \mathbb{R}.$$
2. This map is $\mathbb{R}$-linear and satisfies Leibniz's Rule:
$$D_v(fg) = f(p)D_v(g) + (D_v f)g(p).$$

Derivations at a Point

Definition

1. Any linear map $D : C_p^\infty \to \mathbb{R}$ that satisfies Leibniz's Rule is called a derivation at p (or a point-derivation of C_p^∞).
2. The set of all derivations at p is denoted $\mathscr{D}_p(\mathbb{R}^n)$.

Fact

$\mathscr{D}_p(\mathbb{R}^n)$ is a vector space over $\mathbb{R}$.

Theorem (Theorem 2.2)

Let $\phi : T_p(\mathbb{R}^n) \to \mathscr{D}_p(\mathbb{R}^n)$ be the map defined by

$$\phi(v) = D_v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p, \qquad v = \langle v^1, \ldots, v^n \rangle \in T_p(\mathbb{R}^n).$$

Then ϕ is a linear isomorphism.

Derivations at a Point

Consequence

- This isomorphism allows us to identify tangent vectors at p with derivation at p.
- Under this identification,

$$\text{Canonical basis } e_1, \dots, e_n \quad \longleftrightarrow \quad \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p,$$
$$v = \langle v^1, \dots, v^n \rangle = \sum v^i e_i \quad \longleftrightarrow \quad v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

Remarks

1. From now on we will write a tangent vector as $v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p$.
2. Although not as geometric as the realization as arrows, the description of the tangent space in terms of derivations is more suitable for the generalization to manifolds.

The Tangent Space at a Point

Facts

- Let $\mathscr{F}_p(M)$ consist of pairs (U, f), where U is an open neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function.
- On $\mathscr{F}_p(M)$ we define an equivalence relation by

  $$(U, f) \sim (V, g) \iff f = g \text{ near } p.$$

  Thus, $f \sim g$ means there is an open $W \subset U \cap V$ such that $p \in W$ and $f = g$ on W.

Definition

- The equivalence class of (U, f) is called the *germ of f at p*.
- The quotient $\mathscr{F}_p(M)/\sim$ is denoted $C_p^\infty(M)$; this is the set of germs of C^∞ functions at p.

The Tangent Space at a Point

Facts

- $C_p^\infty(M)$ is a vector space with respect to the scalar multiplication and the addition given by

$$\lambda \cdot (\text{germ at } p \text{ of } f) = \text{germ at } p \text{ of } \lambda f, \quad \lambda \in \mathbb{R},$$
$$(\text{germ at } p \text{ of } f) + (\text{germ at } p \text{ of } g) = \text{germ at } p \text{ of } f + g.$$

- $C_p^\infty(M)$ is also an algebra with respect to the multiplication given by

$$(\text{germ at } p \text{ of } f) \cdot (\text{germ at } p \text{ of } g) = \text{germ at } p \text{ of } fg.$$

The Tangent Space at a Point

Remark

Let U be an open set in M containing p.

- As $\mathscr{F}_p(U) \subset \mathscr{F}_p(M)$ we get an inclusion,
$$C_p^\infty(U) \subset C_p^\infty(M).$$
- As $(V, f) \in \mathscr{F}_p(M)$ and $(V \cap U, f_{|V \cap U})$ are equivalent, we actually have an equality. That is,
$$C_p^\infty(U) = C_p^\infty(M).$$

The Tangent Space at a Point

Definition

A *derivation at* p is any linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Remarks

1. By abuse of notation, we use the same letter f or g to denote a function and its germ at p.
2. The set of all derivations at p is a subspace of the space of linear maps $C_p^\infty \to \mathbb{R}$.

Definition

- The *tangent space of* M *at* p, denoted $T_p(M)$ or T_pM, is the vector space of all derivations at p.
- An element of $T_p(M)$ is now called a *tangent vector at* p.

Example

For $M = \mathbb{R}^n$ we recover the description of the tangent space $T_p(\mathbb{R}^n)$ in terms of derivations.

Example (see Remark 8.2)

- Let U be an open in M containing p. As $C_p^\infty(U) = C_p^\infty(M)$, we see that
$$T_p(U) = T_p(M).$$
- In particular, if $M = \mathbb{R}^n$, then
$$T_p(U) = T_p(\mathbb{R}^n) \simeq \mathbb{R}^n.$$

The Tangent Space at a Point

Example

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart about p in M. Denote by $(r^1, \ldots, r^n)$ the standard coordinates in $\mathbb{R}^n$ (so that $x^i = r^i \circ \phi$).

- By definition, if f is C^∞ at p, then
$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} \left(f \circ \phi^{-1}\right) \in \mathbb{R}.$$
- We have $\frac{\partial}{\partial x^i}\big|_p (fg) = (\frac{\partial}{\partial x^i}\big|_p f)g(p) + f(p)\frac{\partial}{\partial x^i}\big|_p g$.
- If $f = g$ near p, then $\frac{\partial}{\partial x^i}\big|_p f = \frac{\partial}{\partial x^i}\big|_p g$.
- Thus, $\frac{\partial}{\partial x^i}\big|_p$ induces a map,
$$\frac{\partial}{\partial x^i}\bigg|_p : C_p^\infty \longrightarrow \mathbb{R}.$$
  We obtain a derivation at p, i.e., a tangent vector at p.

The Tangent Space at a Point

Remarks

- We sometimes write $\frac{\partial}{\partial x^i}$ instead $\left.\frac{\partial}{\partial x^i}\right|_p$ when it is understood that derivatives are evaluated at the point p.
- When M has dimension 1 and t is a local coordinate, we write $\frac{d}{dt}$ instead of $\frac{\partial}{\partial t}$.
- We will see later that $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$ is a basis of $T_p(M)$.

The Differential of a Map

Facts

Let $F : M \to N$ be a C^∞ map, where M and N are manifolds.

- Given $X \in T_p(M)$ define $F_*(X) : C^\infty_{F(p)}(N) \to \mathbb{R}$ by
$$F_*(X)f = X(f \circ F), \qquad f \in C^\infty_{F(p)}(N).$$
- $F_*(X)$ is a linear map.
- As X is a derivation at p, we have
$$\begin{aligned} F_*(X)(fg) &= X\left[(f \circ F)(g \circ F)\right] \\ &= X\left[(f \circ F)\right](g \circ F)(p) + (f \circ F)(p)X\left[(g \circ F)\right] \\ &= \left[F_*(X)f\right] g\left(F(p)\right) + f\left(F(p)\right)\left[F_*(X)g\right]. \end{aligned}$$
  That is, $F_*(X)$ is a derivation at $F(p)$, i.e., $F_*(X) \in T_{F(p)}(N)$.
- We thus get a map $F_* : T_p(M) \to T_{F(p)}(N)$, $X \to F_*(X)$.

The Differential of a Map

Fact

The map $F_* : T_p(M) \to T_{F(p)}(N)$ is linear, since we have

$$F_*(\lambda X)f = \lambda X(f \circ F) = \lambda F_*(X)f,$$
$$F_*(X+Y)f = X(f \circ F) + Y(f \circ F) = F_*(X)f + F_*(Y)f.$$

Definition

The linear map $F_* : T_p(M) \to T_{F(p)}(N)$ is called the *differential of* F *at* p.

Remarks

1. To emphasize the dependence on the point p we sometimes write $F_{*,p}$ for F_*.
2. There are various notations for the differential. For instance, it also denoted d_pF, $dF(p)$, D_pF or even $F'(p)$.

The Differential of a Map

Example

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ map. Denote by $(x^1, \ldots, x^n)$ the coordinates on $\mathbb{R}^n$ and by $(y^1, \ldots, y^m)$ the coordinates on $\mathbb{R}^m$. Set $F = (F^1, \ldots, F^m)$.

- Let $p \in \mathbb{R}^n$. Then $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of $T_p(\mathbb{R}^n)$.
- Likewise, $\left\{ \frac{\partial}{\partial y^1}\Big|_{F(p)}, \ldots, \frac{\partial}{\partial y^m}\Big|_{F(p)} \right\}$ is a basis of $T_{F(p)}(\mathbb{R}^m)$.
- Given any $f \in C^\infty_{F(p)}(\mathbb{R}^m)$, we have

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p \right) f = \frac{\partial}{\partial x^j}\bigg|_p (f \circ F) = \sum_{i=1}^m \frac{\partial F^i}{\partial x^j}(p) \frac{\partial}{\partial y^i}\bigg|_{F(p)} f.$$

- This means that $F_*\left(\frac{\partial}{\partial x^j}\Big|_p \right) = \sum_{i=1}^m \frac{\partial F^i}{\partial x^j}(p) \frac{\partial}{\partial y^i}\Big|_{F(p)}$.
- In other words, the matrix of F_* relative to the bases $\left\{ \frac{\partial}{\partial x^j}\Big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}$ is precisely the Jacobian matrix $\left[\frac{\partial F^i}{\partial x^j}(p) \right]$.

Fact

Let $F : N \to M$ and $G : M \to P$ be C^∞ maps. Given any $p \in N$ the differentials $F_{*,p}$ and $G_{*,F(p)}$ are linear maps,

$$T_p(N) \xrightarrow{F_{*,p}} T_{F(p)}(M) \xrightarrow{G_{*,F(p)}} T_{G(F(p))}(P).$$

Theorem (Theorem 8.5; Chain Rule)

If $F : N \to M$ and $G : M \to P$ are C^∞ maps, then, for every $p \in N$, we have

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}.$$

Remark

Let $\mathbb{1}_M : M \to M$ be the identity, and let $p \in M$. Given $X \in T_p(M)$ and $f \in C_p^\infty$, we have

$$(\mathbb{1}_M)_*(X)f = X(f \circ \mathbb{1}_M) = Xf.$$

Thus, $(\mathbb{1}_M)_*(X) = X$, and so the differential $(\mathbb{1}_M)_*$ is the identity map $\mathbb{1}_{T_p(M)} : T_p(M) \longrightarrow T_p(M)$.

Corollary (Corollary 8.6)

If $F : N \to M$ is a diffeomorphism, then, for every $p \in N$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Corollary (Corollary 8.7; Invariance of Dimension)

If an open $U \subset \mathbb{R}^n$ is diffeomorphic to an open $V \subset \mathbb{R}^m$, then we must have $n = m$.

Bases for the Tangent Space at a Point

Facts

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart about p in M. Denote by $(r^1, \ldots, r^n)$ the coordinates in $\mathbb{R}^n$. Then:

- The map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.
- The differential $F_{*,p}$ is an isomorphism from $T_p(U) = T_p(M)$ to $T_{\phi(p)}(\phi(U)) = T_{\phi(p)}\mathbb{R}^n$.
- $\{\frac{\partial}{\partial r^1}\big|_{\phi(p)}, \ldots, \frac{\partial}{\partial r^n}\big|_{\phi(p)}\}$ is a basis of $T_{\phi(p)}(\mathbb{R}^n)$.
- By definition of ϕ_* and $\frac{\partial}{\partial x^i}\big|_p$, if $f \in C^\infty_{\phi(p)}(\mathbb{R}^n)$, then

$$\phi_*\left(\frac{\partial}{\partial x^i}\bigg|_p\right) f = \frac{\partial}{\partial x^i}\bigg|_p (f \circ \phi) = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} \left[(f \circ \phi) \circ \phi^{-1}\right] = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} f.$$

Thus,

$$\phi_*\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)}.$$

Bases for the Tangent Space at a Point

Facts

To sum up:

- The differential $\phi_{*,p} : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism.
- It maps the family $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ in T_pM to the basis $\left\{ \frac{\partial}{\partial r^1}\Big|_{\phi(p)}, \ldots, \frac{\partial}{\partial r^n}\Big|_{\phi(p)} \right\}$ of $T_{\phi(p)}\mathbb{R}^n$.

We deduce from this the following result:

Proposition (Proposition 8.9)

If $(U, x^1, \ldots, x^n)$ is a chart about p in M, then $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of T_pM. In particular, T_pM has dimension n.

Corollary (Invariance of Dimension)

If M and N are diffeomorphic manifolds, then $\dim M = \dim N$.

Bases for the Tangent Space at a Point

Proposition (Proposition 8.10)

Let $(U, x^1, \ldots, x^n)$ *and* $(V, y^1, \ldots, y^n)$ *be charts around* p *in* M. *Then, on* $U \cap V$ *we have*

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}.$$

Bases for the Tangent Space at a Point

Remarks

1. There are alternative definitions of the tangent space T_pM.
2. What is important to keep in mind is the following:
    - The tangent space T_pM is a vector space that has basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$, where $(x^1, \ldots, x^n)$ are local coordinates.
    - We keep the same vector space upon changing local coordinates.

Local Expression for the Differential

Proposition (Proposition 8.11)

Let $F : N \to M$ *be a* C^∞ *map and* $p \in N$. *Let* $(U, x^1, \ldots, x^n)$ *be a chart around* p *in* N *and* $(V, y^1, \ldots, y^m)$ *a chart around* $F(p)$ *in* M. *Then, relative to the bases* $\left\{ \frac{\partial}{\partial x^j}\Big|_p \right\}$ *and* $\left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}$ *of* T_pN *and* $T_{F(p)}M$, *the differential* $F_{*,p} : T_pN \to T_{F(p)}M$ *has matrix*

$$\left[\frac{\partial F^i}{\partial x^j}(p) \right]_{\substack{1 \le i \le m \\ 1 \le j \le n}} \qquad \text{where } F^i = y^i \circ F.$$

Remark (Remark 8.12)

- The inverse function theorem for manifolds (Theorem 6.26) asserts that F is locally invertible at p if and only if $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] \neq 0$.
- Therefore, we obtain the following coordinate-free description of this result:

Theorem (Inverse Function Theorem)

Let $F : N \to M$ be a C^∞ map and $p \in N$. TFAE:

1. *F is locally invertible at p.*
2. *The differential $F_* : T_pN \to T_{F(p)}M$ is an isomorphism.*