

Differentiable Forms in Algebraic Topology

Review: Smooth Maps on a Manifold

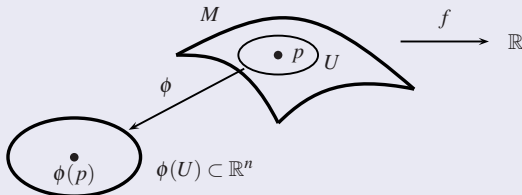
Sichuan University, Spring 2024

Smooth Functions on a Manifold

Definition (Smooth functions)

Let M be a manifold of dimension n .

- A function $f : M \rightarrow \mathbb{R}$ is said to be C^∞ or *smooth at a point* $p \in M$ when there is a chart (U, ϕ) about p in M such that the function $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is C^∞ at $\phi(p)$ (here $\phi(U)$ is an open subset of \mathbb{R}^n).
- We say that f is C^∞ on M when it is C^∞ at every point of M .



Smooth Functions on a Manifold

Remark

- The smoothness condition is independent of the choice of the chart (U, ϕ) .
- If (V, ψ) is another chart about p and $f \circ \phi^{-1}$ is C^∞ , then $f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$ is C^∞ at $\psi(p)$ as well, since the transition map $\phi \circ \psi^{-1}$ is a C^∞ map.

Remark

- If a function $f : M \rightarrow \mathbb{R}$ is C^∞ at p , then it is automatically continuous at p .
- If (U, ϕ) is a chart about p and $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$, then $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p , since ϕ is a continuous map.
- Therefore, any C^∞ -function on M is continuous.

Smooth Functions on a Manifold

Proposition (Tu2011, Proposition 6.3)

Let $f : M \rightarrow \mathbb{R}$ be a function. Then TFAE:

- ① f is C^∞ on M .
- ② There is a C^∞ -atlas $\{(U_\alpha, \phi_\alpha)\}$ on M such that the function $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \supset \phi_\alpha(U_\alpha) \rightarrow \mathbb{R}$ is C^∞ for all α .
- ③ For every chart (U, ϕ) on M , the function $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is C^∞ .

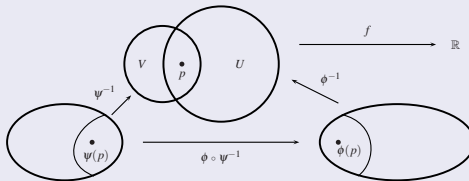
Smooth Maps Between Manifolds

In what follows M is a manifold of dimension m and N is a manifold of dimension n .

Definition (Smooth maps between manifolds)

Let $F : N \rightarrow M$ be a continuous map.

- We say that F is C^∞ or *smooth* at $p \in N$ when there are a chart (U, ϕ) about p in N and a chart (V, ψ) about $F(p)$ on M such that the map $\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^m$ is C^∞ at $\phi(p)$ (here $\phi(F^{-1}(V) \cap U)$ is an open set in \mathbb{R}^n).
- Then map F is C^∞ on N when it is C^∞ at every point $p \in N$.



Smooth Maps Between Manifolds

Remarks

- We assume $F : N \rightarrow M$ to be continuous to ensure that $F^{-1}(V)$ is an open set in N .
- When $M = \mathbb{R}^m$ the continuity assumption can be dropped.

Proposition (Tu2011, Remark 6.6)

A map $F : N \rightarrow \mathbb{R}^m$ is C^∞ at p if and only if there is a chart (U, ϕ) about p in N such that the map $F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^m$ is C^∞ at p (here $\phi(U)$ is an open set in \mathbb{R}^n).

Smooth Maps Between Manifolds

Proposition (Tu2011, Proposition 6.7)

Suppose that $F : N \rightarrow M$ is C^∞ at p . Then, for every chart (U, ϕ) about p in N and every chart (V, ψ) about $F(p)$ in M , the map $\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^m$ is C^∞ at $\phi(p)$.

Proposition (Tu2011, Proposition 6.8)

Let $F : N \rightarrow M$ be a continuous map. TFAE:

- 1 F is a C^∞ map.
- 2 There are C^∞ -atlases (U_α, ϕ_α) for N and (V_β, ψ_β) for M such that the map $\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \rightarrow \mathbb{R}^m$ is C^∞ for every α and β .
- 3 For every chart (U, ϕ) on N and every chart (V, ψ) on M , the map $\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^m$ is C^∞ .

Smooth Maps Between Manifolds

Proposition (Tu2011, Proposition 6.9; Composition of C^∞ maps)

If $F : N \rightarrow M$ and $G : P \rightarrow N$ are C^∞ maps (where P is a manifold), then the composition $F \circ G : P \rightarrow M$ is a C^∞ map.

Diffeomorphisms

Definition

We say that a map $F : N \rightarrow M$ is a *diffeomorphism* when it is a bijective C^∞ map with C^∞ inverse F^{-1} .

Proposition (Tu2011, Proposition 6.10)

If (U, ϕ) is a chart on M , then the coordinate map $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^m$ is a diffeomorphism.

Proposition (Tu2011, Proposition 6.11)

Let U be an open subset of M . If $F : U \rightarrow F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of \mathbb{R}^n , then the pair (U, F) is a chart on M .

Smoothness in Terms of Components

Proposition (Tu2011, Propositions 6.12 & 6.13)

Let $F : N \rightarrow \mathbb{R}^m$ be a map with components $F^1, \dots, F^m : N \rightarrow \mathbb{R}$ (so that $F(p) = (F^1(p), \dots, F^m(p))$). Then TFAE:

- ① F is a C^∞ -map.
- ② For every chart (U, ϕ) on N , the map $F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^m$ is C^∞ .
- ③ All the components $F^1, \dots, F^m : N \rightarrow \mathbb{R}$ are C^∞ maps

Remark

We don't have to assume F to be continuous, since the 2nd and 3rd properties both imply that F is continuous.

Smoothness in Terms of Components

Proposition (Tu2011, Propositions 6.15 & 6.16)

Let $F : N \rightarrow M$ be a continuous map. Then TFAE:

- ① F is a C^∞ map.
- ② For every chart (V, ψ) on M the vector-valued function $\psi \circ F : F^{-1}(V) \rightarrow \mathbb{R}^m$ is C^∞ .
- ③ For every chart $(V, \psi) = (V, y^1, \dots, y^n)$ the component functions $y^i \circ F : F^{-1}(V) \rightarrow \mathbb{R}^m$ are C^∞ .

Remark

We assume F to be continuous to insure that in the 2nd and 3rd properties $F^{-1}(V)$ is an open subset of \mathbb{R}^n .

Examples of Smooth Maps

Example (Tu2011, Example 6.17 + Exercise 6.18)

Let M_1 and M_2 be manifolds.

- 1 The 1st factor projection $\pi_1 : M_1 \times M_2 \rightarrow M_1$, $\pi_1(p_1, p_2) = p_1$ is a C^∞ map. Likewise, the 2nd factor projection $\pi_2 : M_1 \times M_2 \rightarrow M_2$ is a smooth map.
- 2 Given a manifold N , a map $f : N \rightarrow M_1 \times M_2$ is C^∞ if and only if the components $\pi_i \circ f : N \rightarrow M_i$ are C^∞ maps.

Examples of Smooth Maps

Example (Tu2011, Example 6.19)

Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere. If $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a C^∞ function, then the restriction $f|_{\mathbb{S}^n} : \mathbb{S}^n \rightarrow \mathbb{R}$ is a C^∞ function on \mathbb{S}^n .

Partial Derivatives

In what follows M is a manifold of dimension n .

Reminder

If $(U, \phi) = (U, x^1, \dots, x^n)$ a chart on M , then by definition the components x^1, \dots, x^n of ϕ are given by $x^i = r^i \circ \phi : U \rightarrow \mathbb{R}$.

Definition

Let $f : M \rightarrow \mathbb{R}$ be a C^∞ function. For $p \in U$ the *partial derivative of f with respect to x^i at p* is

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial(f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

Remark

The partial derivative $\frac{\partial f}{\partial x^i}(p)$ is also denoted $\frac{\partial}{\partial x^i} \Big|_p f$.

Remark

As $\phi^{-1}(\phi(p)) = p$ the equality $\frac{\partial f}{\partial x^i}(p) = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}(\phi(p))$ can be rewritten as

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1}(\phi(p)) = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

Thus, as functions on $\phi(U)$ we have

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}.$$

In particular, this shows that $\frac{\partial f}{\partial x^i} : U \rightarrow \mathbb{R}$ is C^∞ function on U .

Partial Derivatives

In what follows M is a manifold of dimension m and N is a manifold of dimension n .

Definition (Jacobian matrices and Jacobian determinants)

Let $F : N \rightarrow M$ be a C^∞ map. Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart on N and $(V, \psi) = (V, y^1, \dots, y^m)$ a chart on M such that $F(U) \subset V$. Denote $F^i := y^i \circ F = r^i \circ \psi \circ F : U \rightarrow \mathbb{R}$ the i -th component of F in the chart (V, ψ) .

- 1 The matrix $[\partial F^i / \partial x^j]$ is called the *Jacobian matrix* of F relative to the charts (U, ϕ) and (V, ψ) .
- 2 When $m = n$ the determinant $\det [\partial F^i / \partial x^j]$ is called the *Jacobian determinant* of F relative to the charts.

Remark

The Jacobian determinant is also denoted $\partial(F^1, \dots, F^n) / \partial(x^1, \dots, x^n)$.

Remark

If $N = U$ is an open subset of \mathbb{R}^n and $M = V$ is an open subset of \mathbb{R}^m , and we use the charts (U, r^1, \dots, r^n) and (V, r^1, \dots, r^m) , then the Jacobian matrix $[\partial F^i / \partial r^j]$ is the usual Jacobian matrix from calculus.

Example (Tu2011, Example 6.24; Jacobian matrix of a transition map)

Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ be overlapping charts on N . The transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ is a diffeomorphism between open subsets of \mathbb{R}^n . Given any $p \in U \cap V$, we have

$$\frac{\partial y^i}{\partial x^j}(p) = \frac{(\psi \circ \phi^{-1})^i}{\partial r^j}(\phi(p)).$$

The Inverse Function Theorem

In what follows M and N are manifolds of dimension n .

Definition

We say that a C^∞ map $F : N \rightarrow M$ is *locally invertible*, or is a *local diffeomorphism*, near $p \in N$ if there is an open neighborhood U of p in N such that $F|_U : U \rightarrow F(U)$ is a diffeomorphism.

The Inverse Function Theorem

Theorem (Inverse Function Theorem on \mathbb{R}^n)

Let $F = (F^1, \dots, F^n) : W \rightarrow \mathbb{R}^n$ be a C^∞ -map, where W is an open set in \mathbb{R}^n . Given any $p \in W$, TFAE:

- (i) F is locally invertible near p .
- (ii) The Jacobian determinant $\det[\partial F^i / \partial x^j(p)]$ is non-zero.

The Inverse Function Theorem

Theorem (Tu2011, Theorem 6.26; Inverse Function Theorem for manifolds)

Let $F : N \rightarrow M$ be a C^∞ -map. Given any $p \in N$, TFAE:

- (i) F is locally invertible near p .
- (ii) We have a non-zero Jacobian determinant $\det[\partial F^i / \partial x^j(p)]$.

Remarks

- ❶ In (ii) the Jacobian determinant $\det[\partial F^i / \partial x^j(p)]$ is relative to some chart (U, x^1, \dots, x^n) about p in N and some chart (V, y^1, \dots, y^n) about $F(p)$ in M and we have $F^i = y^i \circ F$.
- ❷ The condition $\det[\partial F^i / \partial x^j(p)] \neq 0$ is independent of the choice of the charts.

The Inverse Function Theorem

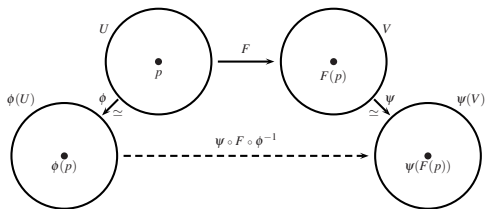


Fig. 6.4. The map F is locally invertible at p because $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$.