Differentiable Forms in Algebraic Topology
Review: Projective Spaces

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The Quotient Topology

Reminder

An equivalence relation on a set S is given by a subset $\mathscr{R} \subset S \times S$ with the following properties:

- Reflexivity: $(x, x) \in \mathscr{R}$ for all $x \in S$.
- Symmetry: $(x, y) \in \mathscr{R} \Leftrightarrow (y, x) \in \mathscr{R}$.
- Transitivity: $(x, y) \in \mathscr{R}$ and $(y, z) \in \mathscr{R} \Rightarrow (x, z) \in \mathscr{R}$.

When $(x, y) \in \mathscr{R}$ we say that x and y are *equivalent* and write $x \sim y$.

The set $\mathscr{R}$ is called the *graph* of the equivalence relation.

The Quotient Topology

Definition

Let $\sim$ be an equivalence relation on S.

- The *class* of $x \in S$, denoted $[x]$, is the subset of S consisting of all $y \in S$ that are equivalent to x.
- The set of equivalence classes is denoted $S/\!\sim$ and is called the *quotient* of S by $\sim$.
- The map $\pi : S \rightarrow S/\!\sim$, $x \rightarrow [x]$ is called the *natural projection map* (or *canonical projection*)

Remarks

1. The equivalence classes form of partition of S.
2. The canonical projection $\pi : S \rightarrow S/\!\sim$ is always onto.

The Quotient Topology

Fact

Suppose that S is a topological space. Let $\mathcal{T}$ be the collection of subsets $U \subset S/\sim$ such that $\pi^{-1}(U)$ is an open in S.

- $\mathcal{T}$ is closed under unions and finite intersections: if $U_\alpha \in \mathcal{T}$ and $V_i \in \mathcal{T}$, then

$$\pi^{-1}(\bigcup U_\alpha) = \bigcup \pi^{-1}(U_\alpha) \quad \text{and} \quad \pi^{-1}(V_1 \cap V_2) = \pi^{-1}(V_1) \cap \pi^{-1}(V_2)$$

  are again contained in $\mathcal{T}$.
- Therefore $\mathcal{T}$ defines a topology on $S/\sim$.

Definition

- The topology $\mathcal{T}$ is called the *quotient topology*.
- Equipped with this topology $S/\sim$ is called the *quotient space* of S by $\sim$.

The Quotient Topology

Remarks

1. A subset $U \subset S/\sim$ is open if and only if $\pi^{-1}(U)$ is an open in S.
2. This implies that the projection map $\pi : S \to S/\sim$ is always continuous.
3. The quotient topology is actually the strongest topology on $S/\sim$ for which the map $\pi : S \to S/\sim$ is continuous.

Continuity of a Map on a Quotient

Fact

Let $f : S \to Y$ be a map that is constant on each equivalence class, i.e.,

$$x \sim y \Rightarrow f(x) = f(y).$$

Then f descends to a map $\overline{f} : S/\!\sim \to Y$ such that

$$\overline{f}([x]) = f(x), \qquad x \in S.$$

Remarks

1. The definition of $\overline{f}$ means that if c is an equivalence class in $S/\!\sim$, then $\overline{f}(c) = f(x)$ for any $x \in c$.
2. The equality $\overline{f}([x]) = f(x)$ for all $x \in S$ means that $\overline{f} \circ \pi = f$. That is, we have a commutative diagram,

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/\!\sim \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Continuity of a Map on a Quotient

Proposition (Tu2011, Proposition 7.1)

The induced map $\bar{f} : S/\sim \to Y$ is continuous if and only if the original map $f : S \to Y$ is continuous.

Corollary

A map $g : S/\sim \to Y$ is continuous if and only if the composition $g \circ \pi : S \to Y$ is continuous.

A Necessary Condition for a Hausdorff Quotient

Facts

- If X is a Hausdorff topological space, then every singleton $\{x\}$, $x \in X$, is a closed set in X.
- If the quotient space $S/\sim$ is Hausdorff, then every singleton $\{[x]\}$, $x \in S$, is closed in $S/\sim$. This means that the preimage $\pi^{-1}(\{[x]\}) = [x]$ is closed in S.

Proposition (Tu2011, Proposition 7.4)

If the quotient space $S/\sim$ is Hausdorff, then all the equivalence classes $[x]$, $x \in S$, are closed sets in S.

Consequence

If there is an equivalence class that is not a closed set, then the quotient space $S/\sim$ is not Hausdorff.

Open Equivalence Relations

Reminder

A map $f : X \to Y$ is open when the image of any open set in X is an open set in Y.

Definition

We say that an equivalence relation $\sim$ on a topological space S is open when the projection $\pi : S \to S/\sim$ is an open map.

Remark

- If $A \subset S$, then $\pi(A)$ is open in $S/\sim$ if and only if $\pi^{-1}(\pi(A)) = \cup_{x \in A}[x]$ is an open set in S.
- Thus, the equivalence relation $\sim$ is open if and only if, for every open U in S, the set $\cup_{x \in U}[x]$ is open in S.

Open Equivalence Relations

Reminder

If $\sim$ is an equivalence relation, then its graph is

$$\mathscr{R} = \{(x, y) \in S \times S;\ x \sim y\} \subset S \times S.$$

Theorem (Tu2011, Theorem 7.7)

Suppose that $\sim$ *is an open equivalence relation on a topological space* S*. Then the quotient space* $S/\sim$ *is Hausdorff if and only if the graph* $\mathscr{R}$ *of* $\sim$ *is closed in* $S \times S$*.*

Open Equivalence Relations

Proposition (Tu2011, Proposition 7.9)

Suppose that $\sim$ is an open equivalence relation on S. If $\{U_\alpha\}$ is a basis for the topology of S, then $\{\pi(U_\alpha)\}$ is a basis for the quotient topology on $S/\sim$.

Corollary (Tu2011, Corollary 7.10)

If $\sim$ is an open equivalence relation on S, and S is second countable, then the quotient space $S/\sim$ is second countable.

Real Projective Space

Remarks

1. Intuitively speaking the real projective space $\mathbb{R}P^n$ is the set of lines in $\mathbb{R}^{n+1}$ through the origin.
2. Two non-zero vectors $x, y \in \mathbb{R}^{n+1} \setminus 0$ are the same line through the origin if and only if there is $t \neq 0$ such that $y = tx$.

Fact

1. We define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus 0$ by
$$x \sim y \Longleftrightarrow y = tx \text{ for some } t \neq 0.$$
2. The equivalence classes consist precisely of the lines through the origin (with the origin deleted).

Real Projective Space

Definition

The *real projective space* $\mathbb{R}P^n$ is the quotient space $(\mathbb{R}^{n+1} \setminus 0)/\sim$.

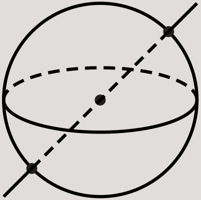
Remarks

1. We denote by $[a^0, \ldots, a^n]$ the class of $(a^0, \ldots, a^n) \in \mathbb{R}^{n+1}/\sim$.
2. We call $[a^0, \ldots, a^n]$ *homogeneous coordinates* on $\mathbb{R}P^n$.
3. We also let $\pi : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{R}P^n$ be the canonical projection.

Real Projective Space

Remark

1. Every line in $\mathbb{R}^{n+1}$ through the origin meets the unit sphere $\mathbb{S}^{n+1}$ at a pair of antipodal points.
2. Conversely, there is a unique line through the origin and two antipodal points of $\mathbb{S}^{n+1}$



Facts

- On $\mathbb{S}^n \subset \mathbb{R}^{n+1} \setminus 0$ we define an equivalence relation by
$$x \sim y \iff x = \pm y.$$
- The restriction of the canonical projection $\pi_{|\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{R}P^n$ induces a continuous map $\overline{\pi} : \mathbb{S}^n/\sim \to \mathbb{R}P^n$.
- The continuous map $f : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$, $x \to \frac{x}{\|x\|}$ induces a continuous map $\overline{f} : \mathbb{R}P^n \to \mathbb{S}^n/\sim$.
- The maps $\overline{\pi} : \mathbb{S}^n/\sim \to \mathbb{R}P^n$ and $\overline{f} : \mathbb{R}P^n \to \mathbb{S}^n/\sim$ are inverses of each other.

Proposition (Tu2011, Exercise 7.11)

The real projective space $\mathbb{R}P^n$ is homeomorphic to the quotient space $\mathbb{S}^n/\sim$.

Real Projective Space

Example (Real projective line $\mathbb{R}P^1$; see also Example 7.12)

- If we regard the unit circle $\mathbb{S}^1$ as a subset of $\mathbb{C}$, then the map $\mathbb{S}^1 \to \mathbb{S}^1$, $z \to z^2$ induces a continuous map $\mathbb{S}^1/\!\sim \to \mathbb{S}^1$.
- This is a continuous bijection between compact spaces, and hence this is a homeomorphism (by Corollary A.36).
- Here $\mathbb{S}^1/\!\sim$ is compact, since this is the image of $\mathbb{S}^1$ by the canonical projection map $\mathbb{S}^1 \to \mathbb{S}^1/\!\sim$, which is continuous.
- We thus have a sequence of homeomorphisms,

$$\mathbb{R}P^1 \simeq \mathbb{S}^1/\!\sim \simeq \mathbb{S}^1.$$

Real Projective Space

Proposition (Tu2011, Proposition 7.14)

The equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus 0$ is an open equivalence relation.

Corollary (Tu2011, Corollary 7.15)

The real projective space $\mathbb{R}P^n$ is second countable.

Corollary (Tu2011, Corollary 7.16)

The real projective space $\mathbb{R}P^n$ is Hausdorff.

The Standard Differentiable Structure of $\mathbb{R}P^n$

Facts

- For $i = 0, \ldots, n$, define
$$U_i = \left\{[a^0, \ldots, a^n] \in \mathbb{R}P^n;\ a^i \neq 0\right\}.$$
- As the property $a^i \neq 0$ remains unchanged when we replace $(a^0, \ldots, a^n)$ by $(ta^0, \ldots, ta^n)$ with $t \neq 0$, we see that U_i is well defined.
- We have $\pi^{-1}(U_i) = \tilde{U}_i$, where
$$\tilde{U}_i = \left\{(a^0, \ldots, a^n) \in \mathbb{R}^{n+1} \setminus 0;\ a^i \neq 0\right\}.$$
- As $\tilde{U}_i$ is an open set in $\mathbb{R}^{n+1} \setminus 0$, this shows that U_i is an open set in $\mathbb{R}P^n$.

Facts

- Define $\tilde{\phi}_i : \tilde{U}_i \to \mathbb{R}^n$ by

$$\tilde{\phi}_i(a^0, \ldots, a^n) = \left(\frac{a^0}{a^i}, \ldots, \frac{a^{i-1}}{a^i}, \frac{a^{i+1}}{a^i}, \ldots, \frac{a^n}{a^i} \right).$$

- As $\tilde{\phi}_i(ta^0, \ldots, ta^n) = \tilde{\phi}_i(a^0, \ldots, a^n)$ for all $t \neq 0$, the map $\tilde{\phi}_i$ induces a map $\phi_i : U_i \to \mathbb{R}^n$ such that

$$\begin{aligned} \phi_i\left([a^0, \ldots, a^n]\right) &= \tilde{\phi}_i(a^0, \ldots, a^n), \\ &= \left(\frac{a^0}{a^i}, \ldots, \frac{a^{i-1}}{a^i}, \frac{a^{i+1}}{a^i}, \ldots, \frac{a^n}{a^i} \right). \end{aligned}$$

- As $\tilde{\phi}_i : \tilde{U}_i \to \mathbb{R}^n$ is a continuous map, the induced map $\phi_i : U_i \to \mathbb{R}^n$ is continuous as well.

The Standard Differentiable Structure of $\mathbb{R}P^n$

Facts

- The map $\phi_i : U_i \to \mathbb{R}^n$ is a bijection with inverse $\psi_i : \mathbb{R}^n \to U_j$, where

$$\psi_i(x^1, \ldots, x^n) = [x^1, \ldots, x^i, 1, x^{i+1}, \ldots, x^n].$$

- The inverse map $\psi_i = \phi_i^{-1}$ is continuous, since $\psi_i = \pi \circ \tilde{\psi}_i$, where $\tilde{\psi}_i : \mathbb{R}^n \to \tilde{U}_i$ is the continuous map given by

$$\tilde{\psi}_i(x^1, \ldots, x^n) = (x^0, \ldots, x^i, 1, x^{i+1}, \ldots, x^n).$$

- Thus, the map $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism.

The Standard Differentiable Structure of $\mathbb{R}P^n$

Facts

- We have

$$\phi_0(U_0 \cap U_1) = \left\{ \left(\frac{a^1}{a^0}, \ldots, \frac{a^n}{a^0} \right); a^j \in \mathbb{R},\ a^0 \neq 0,\ a^1 \neq 0 \right\}$$
$$= \left\{ (x^1, \ldots, x^n) \in \mathbb{R}^n;\ x^1 \neq 0 \right\}.$$

- The transition map $\phi_1 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_1) \to \mathbb{R}^n$ is given by

$$\phi_1 \circ \phi_0^{-1}(x^1, \ldots, x^n) = \phi_1 \left([1, x^1, \ldots, x^n] \right),$$
$$= \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right).$$

  In particular, this is a C^∞ map.

- It can be similarly shown that all the other transition maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \mathbb{R}^n$ are C^∞ maps.

The Standard Differentiable Structure of $\mathbb{R}P^n$

Conclusion

The collection $\{(U_i, \phi_i)\}_{i=0}^n$ is a C^∞ atlas for $\mathbb{R}P^n$, and so $\mathbb{R}P^n$ is a smooth manifold.

Definition

The differentiable structure defined by the atlas $\{(U_i, \phi_i)\}_{i=0}^n$ is called the *standard differentiable structure* of $\mathbb{R}P^n$.

Complex Projective Space

Facts

We also define complex projective spaces.

- On $\mathbb{C}^{n+1} \setminus 0$ consider the equivalence relation

$$x \sim y \iff \exists \lambda \in \mathbb{C} \setminus 0 \text{ such that } x = \lambda y.$$

  In other words $x \sim y$ if and only if x and y lie on the same complex line through the origin.

- The equivalence classes are the complex lines through the origin (minus the origin).
- The *complex projective space* $\mathbb{C}P^n$ is the quotient space $(\mathbb{C}^{n+1} \setminus 0)/\sim$.
- The class of $a = (a^0, \ldots, a^n)$ is denoted $[a^0, \ldots, a^n]$. We call $[a^0, \ldots, a^n]$ *homogeneous coordinates*.
- The space $\mathbb{C}P^n$ is Hausdorff and 2nd countable.

Differentiable Structure on $\mathbb{C}P^n$

Facts

- For $i = 1, \ldots, n$, define
$$U_i = \left\{[a^0, \ldots, a^n];\ (a^0, \ldots, a^n) \in \mathbb{C}^{n+1} \setminus 0,\ a^i \neq 0\right\}.$$
This is an open set in $\mathbb{C}P^n$.
- Define $\phi_i : U_i \to \mathbb{C}^n$ by
$$\phi_i\left([a^0, \ldots, a^n]\right) = \left(\frac{a^0}{a^i}, \ldots, \frac{a^{i-1}}{a^i}, \frac{a^{i+1}}{a^i}, \ldots, \frac{a^n}{a^i}\right).$$
This is a homeomorphism from U_i on $\mathbb{C}^n$. It has inverse
$$\psi_i(z^1, \ldots, z^n) = \left[z^1, \ldots, z^i, 1, z^{i+1}, \ldots, z^n\right].$$
- The transition maps $\phi_i \circ \phi_j^{-1}$ are C^∞ maps (they even are holomorphic maps).
- Thus, $\{(U_i, \phi_i)\}_{i=1}^n$ is a C^∞ atlas for $\mathbb{C}P^n$, and so the complex projective space $\mathbb{C}P^n$ is a manifold.