

Differentiable Forms in Algebraic Topology

Review: Smooth Manifolds

Sichuan University, Spring 2024

Reminder

Let M be a topological space.

- We say that M is *second countable* when it has a countable basis of open sets.
- A(n open) *neighborhood* of a point $p \in M$ is any open set that contains p .
- An *open cover* of M is a collection $\{U_\alpha\}$ of open sets such that $\bigcup_\alpha U_\alpha = M$.

Remark

In the terminology of Tu's book a neighborhood is always an open neighborhood.

Topological Manifolds

Definition (Locally Euclidean Spaces)

A topological space M is called *locally Euclidean of dimension n* when, for every point p , there is a neighborhood V of p that is homeomorphic to an open subset of \mathbb{R}^n .

Definition (Topological Manifolds)

A *topological manifold of dimension n* is a locally Euclidean of dimension n that is Hausdorff and second countable.

Remark

A topology is 2nd countable if it has a countable basis, i.e., there is countable family (U_α) of open sets such that every open sets is a union of some of the U_α .

Definition (Local Charts)

Let M be locally Euclidean of dimension n .

- 1 A (local) chart near a point $p \in M$ is pair (U, ϕ) where U is a neighborhood of p and $\phi : U \rightarrow \mathbb{R}^n$ is a homeomorphism (from U onto its image).
- 2 The open U is called a *coordinate neighborhood* or *coordinate open set*.
- 3 The map ϕ is called a *coordinate map* or *coordinate system*.
- 4 We say that the chart (U, ϕ) is *centered at* p when $\phi(p) = 0$.

Remark

If $U \rightarrow \mathbb{R}^n$ is homeomorphism onto its image, then $\phi(U)$ must be an open subset of \mathbb{R}^n .

Topological Manifolds

Example

- The Euclidean space \mathbb{R}^n is covered by the single $(\mathbb{R}^n, \text{id}_{\mathbb{R}^n})$, where $\text{id}_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the identity map. Thus, \mathbb{R}^n is a topological manifold of dimension n .
- Every open subset $U \subset \mathbb{R}^n$ is a topological manifold as well, with the single chart (U, id_U) .

Remark

Second countability and Hausdorff condition are “hereditary conditions”, i.e., they are satisfied by subsets.

Example

Any open subset U of a topological manifold M is automatically a topological manifold: if (V, ϕ) is a chart for M , then $(V \cap U, \phi|_{V \cap U})$ is a chart for U .

Compatible Charts

Facts

Let (U, ϕ) and (V, ψ) be two charts of a topological manifold.

① $\phi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of \mathbb{R}^n .

② ϕ and ψ restricts to homeomorphisms,

$$\phi|_{U \cap V} : U \cap V \rightarrow \phi(U \cap V), \quad \psi|_{U \cap V} : U \cap V \rightarrow \psi(U \cap V).$$

③ The compositions $(\psi|_{U \cap V}) \circ (\phi|_{U \cap V})^{-1}$ and $(\phi|_{U \cap V}) \circ (\psi|_{U \cap V})^{-1}$ are denoted by $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$.

④ The maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are inverses of each other.

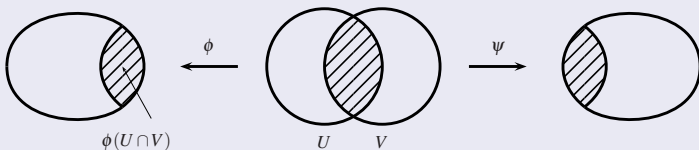
Compatible Charts

Definition (Transition Maps)

The maps

$$\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V) \quad \text{and} \quad \phi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \phi(U \cap V)$$

are called the *transition maps* of the charts (U, ϕ) and (V, ψ) .



Compatible Charts

Definition (C^∞ -Compatible Charts)

We say that two charts (U, ϕ) and (V, ψ) are C^∞ -compatible when the transition maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^∞ -maps.

Remark

As $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are inverses of each other, the above condition means that $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^∞ -diffeomorphisms.

Compatible Charts

Definition (Atlas)

A C^∞ -*atlas*, or simply an *atlas*, on a locally Euclidean space M is a collection $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞ -compatible charts that cover M , i.e., $M = \cup_\alpha U_\alpha$.

Remarks

- 1 The pairwise C^∞ -compatibility means that, for all α, β , the transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are C^∞ -maps.
- 2 This implies that every transition map $\phi_\beta \circ \phi_\alpha^{-1}$ is a C^∞ -diffeomorphism, since its inverse is the transition map $\phi_\alpha \circ \phi_\beta^{-1}$, and hence is C^∞ .

Compatible Charts

Example (Tu2011, Example 5.7; C^∞ -atlas on the circle)

We realize the circle \mathbb{S}^1 a subset of the complex plane,

$$\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\} = \{e^{it}; t \in [0, 2\pi]\}.$$

Let U_1 and U_2 be the open subsets,

$$U_1 = \{e^{it}; t \in (-\pi, \pi)\} = \mathbb{S}^1 \setminus \{-1\},$$

$$U_2 = \{e^{it}; t \in (0, 2\pi)\} = \mathbb{S}^1 \setminus \{1\}.$$

Define $\phi_1 : U_1 \rightarrow (-\pi, \pi)$ and $\phi_2 : U_2 \rightarrow (0, 2\pi)$ as the inverses of the maps $\psi_1 : (-\pi, \pi) \rightarrow U_1$ and $\psi_2 : (0, 2\pi) \rightarrow U_2$ given by

$$\psi_j(t) = e^{it}.$$

Then $\{(U_1, \phi_1), (U_2, \phi_2)\}$ is a C^∞ atlas for \mathbb{S}^1 .

Definition (Smooth Manifold)

A *smooth manifold*, or C^∞ *manifold*, (of dimension n) is a topological manifold (of dimension n) that is equipped with a C^∞ atlas.

Remarks

- A 1-dimensional manifold is called a *curve*.
- A 2-dimensional manifolds is called a *surface*.

Remarks

- 1 Two C^∞ -atlases on a given topological manifold may define the same ring of C^∞ -functions (see Section 6).
- 2 We would like to say that we have the same C^∞ -manifold structure when this happens.
- 3 To deal with this issue it is convenient to use the notion of *maximal atlas*.

Definition (Maximal Atlas)

An atlas \mathcal{M} of a locally Euclidean space is said to be *maximal* when it is not contained in another atlas, i.e., if \mathcal{A} is an atlas containing \mathcal{M} , then it must agree with \mathcal{M} .

Proposition (Tu2011, Proposition 5.8)

Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be a C^∞ -atlas on a locally Euclidean space.

- (i) There is a unique maximal C^∞ -atlas \mathcal{M} that contains \mathcal{A} .
- (ii) \mathcal{M} consists of all local charts (V, ψ) that are C^∞ -compatible with all the charts (U_α, ϕ_α) .

Definition (Smooth Structure)

- A *smooth structure*, or C^∞ -*structure*, on a topological manifold is given by the datum of a maximal C^∞ -atlas.
- A C^∞ -manifold is a topological manifold equipped with a C^∞ -structure (i.e., a maximal C^∞ -atlas).

Remark

The two definitions of C^∞ -manifolds are equivalent.

- A C^∞ -atlas \mathcal{A} on a topological manifold M is contained in a unique maximal C^∞ -atlas \mathcal{M} .
- It thus defines a unique C^∞ -structure on M (given by the maximal atlas \mathcal{M}).

Remark

Two C^∞ -manifolds agree if and only if they agree as sets and have the same topology and C^∞ -structure (i.e., maximal C^∞ -atlas).

Fact

Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta)\}$ be C^∞ -atlases on a topological manifold M . TFAE:

- (i) \mathcal{A} and \mathcal{B} define the same C^∞ -structure on M .
- (ii) \mathcal{A} and \mathcal{B} are contained in the same maximal C^∞ -atlas.
- (iii) The charts of \mathcal{A} and \mathcal{B} are pairwise C^∞ -compatible, i.e., for all α, β the charts (U_α, ϕ_α) and (V_β, ψ_β) are C^∞ -compatible.

Remarks

- In practice we may forget about maximal atlases.
- In order to verify that a topological space M is a C^∞ -manifold we only need to check that
 - (a) M is Hausdorff and second countable.
 - (b) M has a C^∞ -atlas.

Remarks

- ① In what follows, by a “manifold” it will be always meant a “smooth manifold”.
- ② By a *chart* (U, ϕ) about p in a (smooth) manifold M , we shall mean a chart in the maximal C^∞ atlas of M such that $p \in U$.

Notation

(r^1, \dots, r^n) are the standard coordinates in \mathbb{R}^n ,

Definition (Local Coordinates)

- If (U, ϕ) is a chart of a (smooth) manifold, we let $x^i = r^i \circ \phi$ be the i -th coordinate of ϕ .
- The functions x^1, \dots, x^n are called *local coordinates on U* .

Remarks

- ① If $p \in U$, then $(x^1(p), \dots, x^n(p))$ is a point in \mathbb{R}^n .
- ② We often omit p from the notation, so that, depending on context, (x^1, \dots, x^n) may denote local coordinates (functions) or a point in \mathbb{R}^n .

Example (Tu2011, Example 5.11; Euclidean Spaces)

The Euclidean space \mathbb{R}^n is a smooth manifold with single chart $(\mathbb{R}^n, r^1, \dots, r^n)$, where r^1, \dots, r^n are the standard coordinates in \mathbb{R}^n .

Examples of Manifolds

Example (Vector Spaces)

Let E be a (real) vector space of dimension n . Any basis (e_1, \dots, e_n) of E defines a chart (E, ϕ) , where $\phi : E \rightarrow \mathbb{R}^n$ is defined by

$$\phi(r^1 e_1 + \dots + r^n e_n) = (r^1, \dots, r^n), \quad r^i \in \mathbb{R}.$$

This is a linear isomorphism with inverse,

$$\phi^{-1}(r^1, \dots, r^n) = r^1 e_1 + \dots + r^n e_n.$$

Therefore, E is a smooth manifold with single chart (E, ϕ) .

Remarks

- 1 The topology of E is such that the open subsets are of the form $\phi^{-1}(U)$, where U ranges over open subsets of \mathbb{R}^n .
- 2 The topology and smooth structure of E do not depend on the choice of the basis e_1, \dots, e_n .

Examples of Manifolds

Example (Tu2011, Example 5.12; Open subset of a manifold)

An open subset V of a smooth manifold M is a smooth manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is a C^∞ -atlas for M , then $\{(U_\alpha \cap V, \phi_\alpha|_{V \cap U_\alpha})\}$ is a C^∞ -atlas for V .

Examples of Manifolds

Example (Tu2011, Example 5.13; Manifolds of dimension 0)

Let M be a 0-dimensional manifold. Then

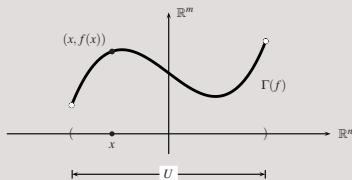
- For every point $p \in M$, the singleton $\{p\}$ is homeomorphic to $\mathbb{R}^0 = \{0\}$, and hence is open. Therefore, M is discrete.
- Second countability then implies that M is countable.
- The charts $(\{p\}, p \rightarrow 0)$, $p \in M$, form a C^∞ -atlas.

Examples of Manifolds

Example (Tu2011, Example 5.14; Graph of a smooth function)

Let $f : U \rightarrow \mathbb{R}^m$ a C^∞ function, where U is an open subset. The graph of f is

$$\begin{aligned}\Gamma(f) &= \{(x, f(x)); x \in U\} \\ &= \{(x, y) \in U \times \mathbb{R}^m; y = f(x)\}.\end{aligned}$$



This is a smooth manifold with single chart $(\Gamma(f), \phi)$, where $\phi : \Gamma(f) \rightarrow U$ is defined by

$$\phi(x, f(x)) = x, \quad x \in U.$$

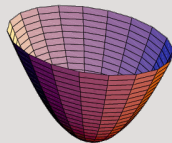
Here $\phi^{-1} : U \rightarrow \Gamma(f)$ is just $x \rightarrow (x, f(x))$.

Examples of Manifolds

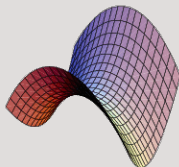
Examples

The following surfaces are graphs of smooth functions, and hence are C^∞ -manifolds:

- Elliptic paraboloid: $z = x^2 + y^2$.



- Hyperbolic paraboloid: $z = y^2 - x^2$.



Examples of Manifolds

Example (Spheres; Tu2011, Example 5.16 and Problem 5.3)

The *unit sphere* of \mathbb{R}^{n+1} is

$$\mathbb{S}^n = \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1}; (x^1)^2 + \dots + (x^{n+1})^2 = 1\}.$$

This is a smooth manifold of dimension n . An atlas is $\{(U_i^\pm, \phi_i^\pm)\}_{i=1}^{n+1}$, where

$$U_i^\pm = \{(x^1, \dots, x^{n+1}) \in \mathbb{S}^n; \pm x^i > 0\},$$

and $\phi_i^\pm : U_i^\pm \rightarrow \mathbb{B}^n$ is defined by

$$\phi_i^\pm(x^1, \dots, x^{n+1}) = (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1}).$$

Here \mathbb{B}^n is the unit ball of \mathbb{R}^n . The inverse map of ϕ_i^\pm is

$$\begin{aligned} (\phi_i^\pm)^{-1}(u^1, \dots, u^n) = \\ (u^1, \dots, u^{i-1}, \pm \sqrt{1 - (u^1)^2 - \dots - (u^n)^2}, u^i, \dots, u^n). \end{aligned}$$

Examples of Manifolds

Remarks

- 1 The above smooth structure on \mathbb{S}^n is called its *standard smooth structure*.
- 2 For $n = 1$ it agrees with the previous smooth structure.
- 3 It can be shown that \mathbb{S}^7 admits exactly 28 distinct smooth structures.

Remarks

- 1 It is known that any topological manifold of dimension ≤ 3 admits at most one smooth structure.
- 2 It can be also shown that (compact) topological manifold of dimension ≥ 5 admits at most finitely many smooth structures.
- 3 In dimension 4 the situation remains unsettled.

Examples of Manifolds

Definition

Let M and N be locally Euclidean spaces of respective dimensions m and n . If (U, ϕ) is a chart for M and (V, ψ) is a chart for N , then the map $\phi \times \psi : U \times V \rightarrow \mathbb{R}^{m+n}$ is defined by

$$(\phi \times \psi)(x, y) = (\phi(x), \psi(y)) \in \mathbb{R}^{m+n}, \quad x \in U, y \in V.$$

Remark

$\phi \times \psi$ is a homeomorphism from $U \times V$ onto the open subset $\phi(U) \times \psi(V) \subset \mathbb{R}^{m+n}$.

Fact (Corollary A.21 and Proposition A.22)

If M and N are both Hausdorff second countable topological spaces, then the product $M \times N$ is again Hausdorff and second countable.

Examples of Manifolds

Proposition (Tu2011, Proposition 5.18 & Example 5.17)

Suppose that M and N are smooth manifolds of respective dimensions m and n . Let $\{(U_\alpha, \phi_\alpha)\}$ be a C^∞ -atlas for M and $\{(V_\beta, \psi_\beta)\}$ a C^∞ -atlas for N . Then

- 1 The collection $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ is a C^∞ atlas for $M \times N$.
- 2 The product $M \times N$ is a smooth manifold of dimension $m + n$.

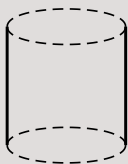
Remark

The smooth structure of $M \times N$ does not depend on the choices of the atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$.

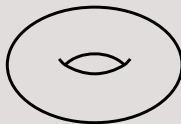
Examples of Manifolds

Example

The infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$ and the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ are both smooth manifolds of dimension 2, since they are product of 1-dimensional smooth manifolds.



Infinite cylinder.



Torus.

Examples of Manifolds

Remark

More generally, if M_1, \dots, M_k are smooth manifolds, then their $M_1 \times \dots \times M_k$ is a smooth manifold of dimension $\dim M_1 + \dots + \dim M_k$.

Example

The n -torus $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ (n times) is a smooth manifold of dimension n .