Differential Forms in Algebraic Topology
Review: Differential k-Forms

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Reminder (see Section 3)

Let V be a vector space (over $\mathbb{R}$). Set $n = \dim V$.

- A *k-covector* on V is an alternating k-linear map $f : V^k \to \mathbb{R}$,

$$f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = (\operatorname{sgn}\sigma) f(v_1, \ldots, v_n) \qquad \forall \sigma \in S_k.$$

- We denote by $A_k(V)$ the space of *k-covectors* on V.
- We have

$$A_0(V) = \mathbb{R}, \qquad A_1(V) = V^*, \qquad A_k(V) = \{0\}, \quad k \geq n+1.$$

Reminder (Wedge product; see Section 3)

- If $f \in A_k(V)$ and $g \in A_\ell(V)$, the *wedge product* $f \wedge g$ is the $(k+\ell)$-covector in $A_{k+l}(V)$ defined by

$$(f \wedge g)(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \mathrm{sgn}(\sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right).$$

- The wedge product $\wedge : A_k(V) \times A_\ell(V) \to A_{k+\ell}(V)$ is a bilinear map which is anti-commutative and associative, i.e.,

$$f \wedge g = (-1)^{k\ell} g \wedge f, \qquad f \wedge f = 0 \quad (k \text{ odd}),$$
$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Reminder (Wedge products of 1-covectors; see Section 3)

- If $\alpha^1, \dots, \alpha^k$ are 1-covectors, then

$$\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)(v_1, \dots, v_k) = \det\left[\alpha^i(v_j)\right], \quad v_i \in V.$$

- Let $\beta^1, \dots, \beta^k$ be k-covectors such that

$$\beta^i = \sum_j a^i_j \alpha^j, \qquad \text{for some matrix } A = [a^i_j] \in \mathbb{R}^{k\times k}.$$

Then

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A)\alpha^1 \wedge \cdots \wedge \alpha^k.$$

Definition

$\mathscr{I}_{k,n}$ is the set of ascending multi-indices $I = (i_1, \ldots, i_k)$ such that $1 \leq i_1 < \cdots < i_k \leq n$.

Reminder (Bases of k-covectors; see Section 3)

Let $e_1, \ldots, e_n$ be a basis of V and let $\alpha^1, \ldots, \alpha^n$ be the dual basis of $V^* = A_1(V)$. For $I = (i_1, \ldots, i_k) \in \mathscr{I}_{k,n}$ set

$$\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

- If $J = (j_1, \ldots, j_k) \in \mathscr{I}_{k,n}$ and $e_J = (e_{j_1}, \ldots, e_{j_k})$, then

$$\alpha^I(e_J) = \delta^I_J.$$

- The k-covectors α^I, $I \in \mathscr{I}_{k,n}$, form a basis of $A_k(V)$.
- In particular $\dim A_k(V) = \binom{n}{k}$ for $k \leq n$.

Differential Forms

Facts

- Any linear map $F : V \to W$ gives rise to a linear map $F^* : A_k(W) \to A_k(V)$ defined by
$$F^*g(v_1, \dots, v_k) = g(Fv_1, \dots, Fv_k), \qquad g \in A_k(W),\ v_i \in V.$$
- If $F : V \to W$ and $G : W \to Z$ are linear maps, then
$$(G \circ F)^* = F^* \circ G^*.$$

Consequence

The construction $V \to A_k(V)$ is a (contravariant) functor from the category $\mathbf{Vect}_{\mathbb{R}}$ to itself.

Differential Forms

Remark

- There is a another construction $V \to \Lambda^k(V)$ called k*-th exterior power*.
- This is a covariant functor on $\mathbf{Vect}_{\mathbb{R}}$.
- We have $A_k(V) = \Lambda^k(V^*)$, so the space of k-covectors is often denoted $\Lambda^k(V^*)$.

Differential Forms

Definition (Differential k-forms)

Let M be a smooth manifold.

- The space $A_k(T_pM)$ is denoted $\Lambda^k(T_p^*M)$.
- An element of $\Lambda^k(T_p^*M)$ is called a *k-covector* at p.
- A *differential k-form* (or a *k-covector field*) is the assignment for each $p \in M$ of a k-covector $\omega \in \Lambda^k(T_p^*M)$.

Remarks

1. Differential k-forms are also called *differential forms of degree* k, or simply *k-forms*.
2. A differential form of degree $k = \dim M$ is also called a *top form*.

Definition

If ω is a differential k-form and $X_1, \ldots, X_k$ are vector fields on M, we denote by $\omega(X_1, \ldots, X_k)$ the function on M defined by

$$\omega(X_1, \ldots, X_k)(p) = \omega_p\big((X_1)_p, \ldots, (X_k)_p\big), \qquad p \in M.$$

Proposition (Proposition 18.1)

Let ω be a differential k-form. For any vector fields $X_1, \ldots, X_k$ and function h on M, we have

$$\omega(X_1, \ldots, hX_i, \ldots, X_k) = h\omega(X_1, \ldots, X_k).$$

Differential Forms

Example

Let $(U, x^1, \ldots, x^n)$ be a chart for M.

- If $p \in U$, then $\left\{\partial/\partial x^1\big|_p, \ldots, \partial/\partial x^n\big|_p\right\}$ is a basis of T_pM.
- The dual basis of T_p^*M is $\left\{(dx^1)_p, \ldots, (dx^n)_p\right\}$.
- For $I = (i_1, \ldots, i_k) \in \mathscr{I}_{k,n}$ let dx^I be the k-form defined by
$$\left(dx^I\right)_p = \left(dx^{i_1}\right)_p \wedge \cdots \wedge \left(dx^{i_k}\right)_p, \qquad p \in U.$$
By the results of Section 3 (see slide 5) $\left\{(dx^I)_p;\ I \in \mathscr{I}_{k,n}\right\}$ is a basis of $\Lambda^k(T_p^*M)$ for every $p \in U$.

Local Expression for a k-Form

Facts

- Let $p \in U$. As $\left\{(dx^I)_p;\ I \in \mathscr{I}_{k,n}\right\}$ is a basis of $\Lambda^k(T_p^*M)$, every k-covector $\omega_p \in \Lambda^k(T_p^*M)$ can be uniquely written as
$$\omega_p = \sum_{I \in \mathscr{I}_{k,n}} a_I \left(dx^I\right)_p, \qquad a_I \in \mathbb{R}.$$
- Set $\partial_i = \partial/\partial x^i$ and for $I = (i_1, \ldots, i_k) \in \mathscr{I}_{k,n}$ set $\partial_I = (\partial_{i_1}, \ldots, \partial_{i_k})$. By the results of Section 3 (see slide 5):
$$dx^I(\partial_J) = \delta^I_J.$$
It follows that if $\omega_p = \sum_{I \in \mathscr{I}_{k,n}} a_I \left(dx^I\right)_p$, then $a_I = \omega_p(\partial_I)$.
- In particular, every k-form ω on U can be uniquely written as
$$\omega = \sum_{I \in \mathscr{I}_{k,n}} a_I dx^I \qquad \text{with } a_I = \omega(\partial_I).$$

Local Expression for a k-Form

Proposition (Proposition 18.3)

Suppose that $(U, x^1, \ldots, x^n)$ *is a chart for* M, *and let* $f^1, \ldots, f^k$ *be smooth functions on* U. *Then*

$$df^1 \wedge \cdots \wedge df^k = \sum_I \frac{\partial(f^1, \ldots, f^k)}{\partial(x^{i_1}, \ldots, x^{i_k})} dx^I.$$

Remark

In fact, in the same way as in Section 3 (see slide 4), we have

$$\left(df^1 \wedge \cdots \wedge df^k\right)(\partial_I) = \det\left[df^i(\partial_{i_j})\right] = \det\left[\partial f^i / \partial x^{i_j}\right]$$
$$= \frac{\partial(f^1, \ldots, f^k)}{\partial(x^{i_1}, \ldots, x^{i_k})}.$$

Local Expression for a k-Form

Example

Let $(V, y^1, \ldots, y^n)$ be another chart. Then on $U \cap V$ we have

$$dy^J = \sum_I \frac{\partial(y^{j_1}, \ldots, y^{j_k})}{\partial(x^{i_1}, \ldots, x^{i_k})} dx^I.$$

Corollary (Corollary 18.4)

Suppose that $(U, x^1, \ldots, x^n)$ is a chart for M, and let $f, f^1, \ldots, f^n$ be smooth functions on U. Then

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i,$$

$$df^1 \wedge \cdots \wedge df^n = \frac{\partial(f^1, \ldots, f^n)}{\partial(x^1, \ldots, x^n)} dx^1 \wedge \cdots \wedge dx^n.$$

Definition

- The k-*th exterior power of the cotangent bundle* is

$$\Lambda^k(T^*M) = \bigsqcup_{p\in M} \Lambda^k(T_p^*M) = \left\{(p,\omega);\ p \in M,\ \omega \in \Lambda^k(T_p^*M)\right\}.$$

- The *canonical map* $\pi : \Lambda^k(T^*M) \to M$ is given by

$$\pi(p,\omega) = p, \qquad p \in M, \quad \omega \in \Lambda^k(T_p^*M).$$

The Bundle Point of View

Facts

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M. Set $V = \phi(U)$.

- Every k-covector $\omega_p \in \Lambda^k(T_p^* M)$ can be uniquely written as
$$\omega_p = \sum_I a_I (dx^I)_p, \qquad \text{with } a^I = \omega_p(\partial_I).$$
- We thus get a natural bijection $\tilde{\phi} : \Lambda^k(T^* U) \to V \times \mathbb{R}^{\binom{n}{k}}$ such that, for all $p \in M$ and $\omega \in \Lambda^k(T_p^* M)$, we have
$$\tilde{\phi}(p, \omega) = \left((x^i(p)), (\omega(\partial_I)) \right).$$

Remark

In the same way as with the constructions of the tangent bundle TM and the cotangent bundle $T^* M$, the maps $\tilde{\phi}$ allow us to define a topology and a smooth structure on $\Lambda^k(T^* M)$.

Definition

Let (U, ϕ) be a chart for M and set $V = \phi(U)$. We endow $\Lambda^k(T^*U)$ with the topology such that

$$W \subset \Lambda^k(T^*U) \text{ is open } \iff \tilde{\phi}(W) \text{ is open in } V \times \mathbb{R}^{\binom{n}{k}}.$$

Proposition

Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of M.

- *Define*
$$\mathscr{B} = \bigcup_\alpha \left\{ W;\ W \text{ is an open in } \Lambda^k(T^*U_\alpha) \right\}.$$
Then $\mathscr{B}$ is the basis for a unique topology on $\Lambda^k(T^*M)$.
- *The collection $\{(T^*U_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas on $\Lambda^k(T^*M)$, and hence $\Lambda^k(T^*M)$ is a smooth manifold.*
- *$\Lambda^k(T^*M) \xrightarrow{\pi} M$ is a smooth vector bundle over M.*

Remark

A k-form on M is a section of the exterior power $\Lambda^k(T^*M)$.

Definition

- We say that k-form is C^∞ when it is C^∞ as a section of $\Lambda^k(T^*M)$.
- We denote by $\Omega^k(M)$ the space of smooth k-forms on M.

Remarks

1. In other words, $\Omega^k(M)$ is the space of smooth sections of T^*M. In particular, this is a module over the ring $C^\infty(M)$.
2. As $\Lambda^0(T_p^*M) = \mathbb{R}$, a 0-form is just a map from M to $\mathbb{R}$. Thus, a smooth 0-form is just a smooth function on M, i.e., $\Omega^0(M) = C^\infty(M)$.

Example

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M. Set $V = \phi(U)$.

- It can be shown that each k-form dx^I, $I \in \mathscr{I}_{k,n}$ is smooth.
- Thus, $\{dx^I;\ I \in \mathscr{I}_{n,k}\}$ is a smooth frame of $\Lambda^k(T^*M)$ over U.

Reminder (Proposition 12.2)

Let $\{s_1, \ldots, s_r\}$ be a C^∞ frame of a vector bundle E over U. A section $s = \sum c^i s_i$ of E over U is smooth if and only if $c^1, \ldots, c^r$ are smooth functions on U.

We immediately obtain:

Lemma (Lemma 18.6)

Let $(U, x^1, \ldots, x^n)$ be a chart for M. A k-form $\omega = \sum a_I dx^I$ on U is smooth if and only if the coefficients a_I are C^∞ functions on U.

Smooth k-Forms

In the same way as with vector fields and 1-forms by using the previous lemma we obtain:

Proposition (Proposition 18.7; 1st part)

Let ω be a k-form on M. Then TFAE:

1. *ω is a smooth k-form.*
2. *M has an atlas such that, for every chart $(U, x^1, \ldots, x^n)$ of this atlas, we may write $\omega = \sum a_I dx^I$ on U with $a^I \in C^\infty(U)$.*
3. *For every chart $(U, x^1, \ldots, x^n)$ of M, we may write $\omega = \sum a_I dx^I$ on U with $a^I \in C^\infty(U)$.*

Proposition (Proposition 18.7; 2nd part)

Let ω be a k-form on M. Then TFAE:

1. *ω is a smooth k-form.*
2. *For any smooth vector fields $X_1, \ldots, X_k$ on M, the function $\omega(X_1, \ldots, X_k)$ is smooth on M.*

Proposition (Proposition 18.8)

Let τ be a smooth k-form defined on a neighborhood of p. Then there exists a smooth k-form $\tilde{\tau}$ on M which agrees with τ near p.

Pullback of k-Forms

Reminder (see slide 6)

Any linear $F : V \to W$ between vector spaces gives rise to a linear map $F^* : A_k(W) \to A_k(V)$ defined by

$$F^*g(v_1, \ldots, v_k) = g(Fv_1, \ldots, Fv_k), \qquad g \in A_k(W),\ v_i \in V.$$

Definition (Pullback of a k-form)

Let $F : N \to M$ be a smooth map. If ω is a k-form on M, then its *pullback* $F^*\omega$ is the k-form on N defined by

$$\left(F^*\omega\right)_p = \left(F_{*,p}\right)^* \omega_{F(p)}, \qquad p \in N.$$

That is,

$$\left(F^*\omega\right)_p(v_1, \ldots, v_k) = \omega_p\left(F_{*,p}v_1, \ldots, F_{*,p}v_k\right), \qquad v_i \in T_pM.$$

Pullback of k-Forms

Proposition (Proposition 18.9)

Let $F : N \to M$ be a smooth map. If ω and τ are k-forms on M and a is a constant, then

$$F^*(\omega + \tau) = F^*\omega + F^*\tau,$$
$$F^*(a\omega) = aF^*\omega.$$

Remark

We will see later that if ω is a smooth k-form, then its pullback $F^*\omega$ is a smooth as well (see slide 29).

The Wedge Product

Definition

If ω is a k-form and τ is a ℓ-form on M, then their *wedge product* $\omega \wedge \tau$ is the $(k+\ell)$-form on M defined by

$$\left(\omega \wedge \tau\right)_p = \omega_p \wedge \tau_p \in \Lambda^{k+\ell}(T_p^* M), \qquad p \in M.$$

Proposition (Proposition 18.10)

If ω and τ are smooth forms on M, then $\omega \wedge \tau$ is smooth on M.

Corollary

The wedge product induces an anti-commutative associative bilinear map,

$$\wedge : \Omega^k(M) \times \Omega^\ell(M) \longrightarrow \Omega^{k+\ell}(M).$$

The Wedge Product

Reminder (Graded Algebras)

1. An algebra A over a field $\mathbb{K}$ is called *graded* when it can be decomposed as
$$A = \bigoplus_{k=0}^{\infty} A^k,$$
where the A^k are subspaces such that the multiplication maps $A^k \times A^\ell$ to $A^{k+\ell}$.
2. We say that A is *anticommutative* (or *graded commutative*) when
$$ba = (-1)^{k\ell} ab \qquad \text{for all } a \in A^k \text{ and } b \in A^\ell.$$

The Wedge Product

Proposition

Set $n = \dim M$. *We Define*

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

Then $\Omega^*(M)$ *is anticommutative graded algebra under the wedge product.*

Remark

$\Omega^*(M)$ is called the *exterior algebra of differential forms* on M.

Proposition (Proposition 18.11)

Let $F : N \to M$ be a smooth map. If ω and τ are differential forms on M, then

$$F^*(\omega \wedge \tau) = (F^*\omega) \wedge (F^*\tau).$$

This result is used to prove:

Lemma (Local expression for pullback)

Suppose that $F : N \to M$ is a smooth map. Let $(U, x^1, \dots, x^n)$ be a chart for N and $(V, y^1, \dots, y^m)$ a chart for M such that $U \subset F^{-1}(V)$. Set $F^j = y^j \circ F$. For any k-form $\omega = \sum b_J dy^J$ on V, we have

$$F^*\omega = \sum_{I,J} (b_J \circ F) \frac{\partial(F^{j_1}, \dots F^{j_k})}{\partial(x^{i_1}, \dots, x^{i_k})} dx^I \quad \text{on } U.$$

Wedge Product and Pullback

Proof.

- Thanks to Proposition 18.9, on $F^{-1}(V)$ we have

$$F^*\omega = F^*\Big(\sum_J b_J y^J\Big) = \sum_J F^* b_J F^*(dy^J) = \sum_J (b_J \circ F) F^*(dy^J).$$

- It remains to determine $F^*(dy^J)$. By Proposition 18.11,

$$F^*(dy^J) = F^*\big(dy^{j_1} \wedge \cdots \wedge dy^{j_k}\big) = (F^* dy^{j_1}) \wedge \cdots \wedge (F^* dy^{j_k}).$$

- By Proposition 17.10 pullback commutes with the differential:

$$(F^* dy^{j_\ell}) = d\big(F^* y^{j_\ell}\big) = d(y^{j_\ell} \circ F) = dF^{j_\ell}.$$

- Thus, on U we have

$$F^*(dy^J) = dF^{j_1} \wedge \cdots \wedge dF^{j_k} = \sum_I \frac{\partial\big(F^{j_1}, \ldots F^{j_k}\big)}{\partial\big(x^{i_1}, \ldots, x^{i_k}\big)} dx^I.$$

This gives the result. $\square$

By combining the previous lemma with the characterization of smoothness of k-forms (Proposition 18.7) we obtain:

Proposition (Proposition 19.7)

Let $F : N \to M$ be a smooth map. If ω is a smooth k-form on M, then $F^*\omega$ is a smooth form on N.

Remark

- In Tu's book the previous result is proved in Section 19. The main step is to prove the lemma of slide 26.
- However, Tu's proof uses Proposition 19.5 whose statement requires Proposition 19.7 in order to makes sense.
- Therefore, Proposition 19.5 cannot be used to prove Proposition 19.7.
- Tu's arguments are fine if we use Proposition 17.10 instead of Proposition 19.5 (as it is done on slide 27).