Differential Forms in Algebraic Topology
Review: Lie Bracket, Lie Derivative, and Interior Multiplication

Sichuan University, Spring 2024

Reminder

Let X be a smooth vector field on a smooth manifold M. Then X defines a derivation on $C^\infty(M) = \Omega^0(M)$,

$$X : C^\infty(M) \longrightarrow C^\infty(M), \qquad f \longrightarrow Xf.$$

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Can we extend this derivation to a derivation on all $\Omega^*(M)$?

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Solution

The solution is provided by the *Lie derivative* (see Tu2011, §20).

Lie Bracket of Vector Fields

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$$X(Yf)(p) = X_p(Yf).$$

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- It follows we get a linear map,

$$C_p^\infty \ni f \longrightarrow X_p(Yf) \in \mathbb{R}$$

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Definition

If X and Y are smooth vector fields on M, then their *Lie bracket at a point* $p \in M$ is the linear map $[X, Y]_p : C_p^\infty \to \mathbb{R}$ defined by

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf), \qquad f \in C_p^\infty.$$

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Lemma

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Definition

If X and Y are smooth vector fields on M, then their *Lie bracket* is the vector field,

$$[X, Y] : M \longrightarrow TM, \qquad p \longrightarrow [X, Y]_p.$$

Proposition (Proposition 14.10)

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If $f \in C^\infty(M)$ and $p \in M$, then

$$\begin{aligned}\big([X, Y]f\big)(p) = [X, Y]_p(f) &= X_p(Yf) - Y_p(Xf) \\ &= X(Yf)(p) - Y(Xf)(p).\end{aligned}$$

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Thus,

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Thus,

$$[X, Y]f = X(Yf) - Y(Xf) \in C^\infty(M).$$

Therefore, if we regard X, Y and $[X, Y]$ are derivations on $C^\infty(M)$, then

$$[X, Y] = X \circ Y - Y \circ X.$$

Definition (Lie algebras)

A *Lie algebra over a field* $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ together with an alternating bilinear map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathbb{K}$ satisfying Jacobi's identity,

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \quad \text{for all } X,Y,Z\in\mathfrak{g}.$$

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Remark

In general, a Lie algebra $(\mathfrak{g},[\cdot,\cdot])$ need not be an algebra, since the bracket $[\cdot,\cdot]$ may fail to be associative.

Example

Any vector space V over $\mathbb{K}$ is a Lie algebra with respect to the zero bracket $[x, y] = 0$. Such a Lie algebra is called an *Abelian Lie algebra*.

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Any algebra A over $\mathbb{K}$ is a Lie algebra with respect to the bracket,

$$[x, y] = xy - yx, \qquad x, y \in A.$$

Proposition (see Exercise 14.11)

The space $\mathscr{X}(M)$ of smooth vector fields on M is a Lie algebra over $\mathbb{R}$ with respect to the Lie bracket of vector fields.

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Remark

Let A be an algebra over $\mathbb{K}$. Denote by $\mathrm{Der}(A)$ the space of derivations of A.

- If D_1 and D_2 are derivations of A, then

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$$

is again a derivation of A.

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  is again a derivation of A.
- $(\mathrm{Der}(A), [\cdot, \cdot])$ is a Lie algebra.

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2. For any smooth vector fields $X_1, \ldots, X_k$ on M, the function $\omega(X_1, \ldots, X_k)$ is smooth on M.

The Lie Derivative

Definition (see also Theorem 20.12)

Let X be a smooth vector field on M and $\omega \in \Omega^k(M)$. The Lie derivative $\mathscr{L}_X\omega$ is the unique smooth k-form on M such that, for any smooth vector fields $Y_1, \ldots, Y_k$ on M, we have

$$(\mathscr{L}_X\omega)(Y_1, \ldots, Y_k) = X\big(\omega(Y_1, \ldots, Y_k)\big) - \sum_{i=1}^{k} \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_k).$$

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Remark

In Tu2011 the Lie derivative is defined in terms of the flow of the vector field X. This leads to the same formula as above.

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(i) *The Lie derivative $\mathscr{L}_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation, i.e, it's a linear map such that*

$$\mathscr{L}_X(\omega \wedge \tau) = (\mathscr{L}_X \omega) \wedge \tau + \omega \wedge (\mathscr{L}_X \tau) \quad \forall \omega, \tau \in \Omega^*(M).$$

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(ii) *It commutes with exterior differentiation, i.e.,*

$$d\mathscr{L}_X = \mathscr{L}_X d.$$

Interior Multiplication

Definition (Interior multiplication)

Let V be a vector space. If β is a k-covector on V and $v \in V$, then the *interior multiplication* or *contraction* of β with v is the $(k-1)$-covector $\imath_v \beta$ defined as follows:

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- If $k = 1$, then $\imath_v\beta = \beta(v)$.
- If $k = 0$, then $\imath_v\beta = 0$.

Proposition (Proposition 20.7)

Let $\alpha^1, \ldots, \alpha^k$ *be* 1*-covectors (i.e., elements of* V^**). Then*

$$\imath_v(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k,$$

where $\widehat{\cdot}$ *means omission.*

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1. $\imath_v \circ \imath_v = 0$.
2. *If $\beta \in A_k(V)$ and $\gamma \in A_\ell(V)$, then*
$$\imath_v(\beta \wedge \gamma) = (\imath_v \beta) \wedge \gamma + (-1)^k \beta \wedge (\imath_v \gamma).$$

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$$\imath_v(\beta \wedge \gamma) = (\imath_v \beta) \wedge \gamma + (-1)^k \beta \wedge (\imath_v \gamma).$$

In other words, $\imath_v$ is an antiderivation of degree -1 whose square is zero.

Interior Multiplication

Definition

Let M be a smooth manifold. If X is a vector field and ω is a k-form on M, then the interior product $\imath_X \omega$ is defined by

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Remark

- If $k \geq 2$, then, for any vector fields $X_1, \ldots, X_{k-1}$ on M, we have
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Proposition

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Sketch of Proof.

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- If $k \geq 2$, then for any smooth vector fields $X_1, \ldots, X_{k-1}$ on M we have

$$\imath_X\omega(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}) \in C^\infty(M).$$

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The proof follows by induction. □

Interior Multiplication

Corollary

If X is a smooth vector field on M, the interior product with X defines a degree -1 anti-derivation $\imath_X : \Omega^*(M) \to \Omega^{*-1}(M)$ such that $\imath_X \circ \imath_X = 0$.

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Theorem (Theorem 20.10; Cartan Homotopy Formula)

If X a smooth vector field on M, then

$$\mathscr{L}_X = d\imath_X + \imath_X d.$$

Reminder

Let X be a smooth vector field on M and $\omega \in \Omega^k(M)$. For any smooth vector fields $Y_1, \ldots, Y_k$ on M, we have

$$(\mathscr{L}_X\omega)(Y_1, \ldots, Y_k) = X\big(\omega(Y_1, \ldots, Y_k)\big) - \sum_{i=1}^{k} \omega\big(Y_1, \ldots, [X, Y_i], \ldots, Y_k\big).$$

Global Formulas for the Exterior Derivative

Reminder

Let X be a smooth vector field on M and $\omega \in \Omega^k(M)$. For any smooth vector fields $Y_1, \ldots, Y_k$ on M, we have

$$\begin{aligned}(\mathscr{L}_X\omega)(Y_1, \ldots, Y_k) =& X\big(\omega(Y_1, \ldots, Y_k)\big) \\ & - \sum_{i=1}^{k} \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_k).\end{aligned}$$

Proposition (Proposition 20.13)

Let $\omega \in \Omega^1(M)$. Then, for any smooth vector fields X and Y on M, we have

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega([X, Y]).$$

By using Cartan homotopy formula and the previous formulas we get:

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Theorem (Theorem 20.14; Global formula for the exterior derivative)

Let $\omega \in \Omega^k(M)$, $k \geq 1$. Then, for any smooth vector fields $Y_0, \ldots, Y_k$ on M, we have

$$d\omega(Y_0, \ldots, Y_k) = \sum_{i=1}^{k} (-1)^i Y_i\big(\omega(Y_0, \ldots, \widehat{Y}_i, \ldots, Y_k)\big) + \sum_{1 \leq i < j \leq k} (-1)^{i+j} \omega\big([Y_i, Y_j], Y_0, \ldots, \widehat{Y}_i, \ldots, \widehat{Y}_j, \ldots, Y_k\big).$$