Differential Forms in Algebraic Topology: Cochain Complexes and Cohomology

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References

Main References

- Section 25 of Tu2011.
- Section 1 of Bott-Tu.

Cochain Complexes

Definition

- A cochain complex $\mathscr{C}$ is given by vector spaces C^k, $k \in \mathbb{Z}$, and linear maps $d_k : C^k \to C^{k+1}$,

$$\cdots C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots,$$

  such that

$$d_k \circ d_{k+1} = 0.$$

- The collection of the linear maps $(d_k)_{k\in\mathbb{Z}}$ is called the differential of the cochain complex $\mathscr{C}$.

Example

Given any (smooth) manifold M the exterior algebra $\Omega^*(M)$ of (smooth) differential forms along with the exterior derivative d give rise to a cochain complex,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots .$$

This cochain complex is called the de Rham complex of M.

Exact Sequences

Definition

A sequence of linear maps,

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

is said to be exact at B if $\operatorname{im} f = \ker g$.

Definition

An exact sequence is a sequence of linear maps,

$$0 \longrightarrow A^0 \xrightarrow{f_0} A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A^n$$

which is exact at A^i for $i = 1, \ldots, n-1$. That is,

$$\operatorname{im} f^{i-1} = \ker f^i \qquad \text{for } i = 1, \ldots, n-1.$$

Exact Sequences

Definition

A short exact sequence is a 5-term exact sequence, i.e., an exact sequence of the form,

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Remark

This means that

$$0 = \ker f, \qquad \operatorname{im} f = \ker g, \qquad \operatorname{im} g = \ker(0) = C.$$

That is, the sequence is exact at B, the map f is injective, and the linear map g is surjective.

Exact Sequences

Remark

If $A = 0$, then a sequence $0 \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if

$$0 = \operatorname{im} f = \ker g.$$

That is, g is injective.

Remark

If $C = 0$, then a sequence $A \xrightarrow{f} B \xrightarrow{g} 0$ is exact if and only if

$$\operatorname{im} f = \ker g = B.$$

That is, f is surjective.

Exact Sequences

Proposition (Proposition 25.2)

Let $A \xrightarrow{f} B \xrightarrow{g} C$ *be an exact sequence. Then:*

(i) *The map* f *is surjective if and only if* $g = 0$*.*

(ii) *The map* g *is injective if and only if* $f = 0$*.*

Remark

If $f : A \to B$ is a linear map, then its cokernel is $\operatorname{coker} f := B/\operatorname{im} f$.

Proposition (4-term exact sequences; Proposition 25.3)

1. *A sequence* $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ *is exact if and only if* f *is an isomorprhism.*
2. *If the sequence* $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *is exact, then*

$$C \simeq \operatorname{coker} f.$$

Cohomology of cochain complexes

Setup

$\mathscr{C}$ is a cochain complex,

$$\cdots C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots .$$

In particular, $d_{k-1} \circ d_k = 0$.

Fact

$d_{k-1} \circ d_k = 0 \iff \operatorname{im} d_{k-1} \subseteq \ker d_k$.

Cohomology of cochain complexes

Definition

The degree k cohomology space of $\mathscr{C}$ is

$$H^k(\mathscr{C}) = \frac{\ker d_k}{\operatorname{im} d_{k-1}}.$$

Remark

$$H^k(\mathscr{C}) = 0 \iff \frac{\ker d_k}{\operatorname{im} d_{k-1}} = 0 \iff \operatorname{im} d_{k-1} = \ker d_k.$$

Thus, $H^k(\mathscr{C}) = 0$ if and only if the cochain complex is exact at C^k.

Cohomology of cochain complexes

Definition

- The elements of C^k are called cochains of degree k (or k-cochains).
- A k-cochain c such that $d_k c = 0$ is a called a k-cocycle.
- A k cochain for which there is $c' \in C^{k-1}$ such that $c = d_{k-1} c'$ is called a k-coboundary.

Definition

- The space of k-cocycles is denoted $Z^k(\mathscr{C})$.
- The space of k-coboundaries is denoted $B^k(\mathscr{C})$.

Remark

We have

$$Z^k(\mathscr{C}) = \ker d_k, \qquad B^k(\mathscr{C}) = \operatorname{im} d_{k-1}.$$

Thus,

$$H^k(\mathscr{C}) = \frac{\ker d_k}{\operatorname{im} d_{k-1}} = Z_k(\mathscr{C})/B^k(\mathscr{C}).$$

Definition

If $c \in Z^k(\mathscr{C})$ is a k-cocycle, then its class in $H^k(\mathscr{C})$ is denoted $[c]$ and is called its cohomology class.

Remark

The subscript in d_k is often omitted, and so we write $d \circ d = 0$ instead of $d_k \circ d_{k+1} = 0$.

Cohomology of cochain complexes

Example

Let M be a smooth manifold with de Rham complex,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots .$$

- The space of k-cocycles is

$$Z^k(M) = \left\{\omega \in \Omega^k(M);\ d\omega = 0\right\}$$
$$= \{\text{closed } k\text{-forms}\} .$$

- The space of k-coboundaries is

$$B^k(M) = \left\{\omega \in \Omega^k(M); \exists \eta \in \Omega^{k-1}(M) \text{ s.t. } \omega = d\eta\right\}$$
$$= \{\text{exact } k\text{-forms}\} .$$

Cochain Maps

Definition

Let $\mathscr{A} = (A^*, d)$ and $\mathscr{B} = (B^*, d')$ be cochain complexes. A cochain map $\varphi : \mathscr{A} \to \mathscr{B}$ is given by a collection of linear maps $\varphi_k : A^k \to B^k$ such that

$$d'_k \circ \varphi_k = \varphi_{k+1} \circ d.$$

Remarks

1. In other words, we have a commutative diagram,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{k-1} & \xrightarrow{d} & A^k & \xrightarrow{d} & A^{k+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varphi_{k-1}} & & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \varphi_{k+1}} & & \\ \cdots & \longrightarrow & B^{k-1} & \xrightarrow{d'} & B^k & \xrightarrow{d'} & B^{k+1} & \longrightarrow & \cdots . \end{array}$$

2. We will often omit the subscript in φ_k.

Cochain Maps

Fact

If $\varphi : \mathscr{A} \rightarrow \mathscr{B}$ is a cochain map, then

$$\varphi(Z^k(\mathscr{A})) \subseteq Z^k(\mathscr{B}), \qquad \varphi(B^k(\mathscr{A})) \subseteq B^k(\mathscr{B})$$

As a consequence, we get:

Proposition

If $\varphi : \mathscr{A} \rightarrow \mathscr{B}$ is a cochain map, then, for every k, it descends to a unique linear map,

$$\varphi^* : H^k(\mathscr{A}) \longrightarrow H^k(\mathscr{B}),$$

such that

$$\varphi([c]) = [\varphi(c)] \qquad \forall c \in Z^k(\mathscr{A}).$$

Cochain Maps

Proposition

If $\varphi : \mathscr{A} \to \mathscr{B}$ *and* $\psi : \mathscr{B} \to \mathscr{C}$ *are cochain maps, then* $\psi \circ \varphi : \mathscr{A} \to \mathscr{B}$ *is cochain map, and we have*

$$(\psi \circ \varphi)^* = \psi^* \circ \varphi^*.$$

Consequence

We have a category of cochain complexes, where:

- The objects are cochain complexes.
- The morphisms are cochain maps.

Cochain Maps

Example

Let M and N be smooth manifolds with de Rham complexes,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots,$$
$$0 \longrightarrow \Omega^0(N) \xrightarrow{d} \Omega^1(N) \xrightarrow{d} \Omega^2(N) \xrightarrow{d} \cdots,$$

If $F : N \to M$ be a smooth map, then:

- By pullback we get a linear maps $F^* : \Omega^k(M) \to \Omega^k(N)$.
- The pullback maps commute with the exterior derivative, i.e., $F^* \circ d = d \circ F^*$.
- We thus get a cochain map $F^* : \Omega^*(M) \to \Omega^*(N)$.
- In particular, this descends to linear maps,

$$F^* : H^k(M) \longrightarrow H^k(N).$$

Cochain Maps

Remark

The previous example shows that the assignment $M \to (\Omega^*(M), d)$ is a functor from the category of smooth manifolds to the category of cochain complexes.

Cochain Maps

Example

Let M be a smooth manifold with de Rham complex,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots .$$

Let X be a smooth vector field on M.

- The Lie derivative yieds linear maps $\mathscr{L}_X : \Omega^k(M) \to \Omega^k(M)$.
- The Lie derivative commute with the exterior derivative, i.e., $\mathscr{L}_X \circ d = d \circ \mathscr{L}_X$.
- We thus get a cochain map $\mathscr{L}_X : \Omega^*(M) \to \Omega^*(M)$.
- In particular, this descends to linear maps,

$$\mathscr{L}_X : H^k(M) \longrightarrow H^k(M).$$

Cochain Maps

Definition

Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a cochain map

1. We say that φ is a quasi-isomorphism if the induced maps $\varphi^* : H^k(\mathscr{A}) \to H^k(\mathscr{B})$ are isomorphisms.
2. A quasi-inverse for φ is a cochain map $\psi : \mathscr{B} \to \mathscr{A}$ such that

$$\psi^* \circ \varphi^* = \mathrm{id}_{H^k(\mathscr{A})}, \qquad \varphi^* \circ \psi^* = \mathrm{id}_{H^k(\mathscr{B})} .$$

Example

If $F : N \to M$ is a diffeomorphism between C^∞-manifolds, then:

1. The pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ is a quasi-isomorphism.
2. A quasi-inverse is $(F^{-1})^* : \Omega^*(N) \to \Omega^*(M)$.
3. We thus get isomorphisms,

$$H^k(M) \simeq H^k(N).$$

The Connecting Homomorphism

Definition

A sequence of cochain complexes $0 \to \mathscr{A} \xrightarrow{i} \mathscr{B} \xrightarrow{j} \mathscr{C} \to 0$ is called short-exact if:

- i and j are cochain maps.
- Each sequence $0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \to 0$ is a short exact sequence of vector spaces.

Remark

We shall omit the subscripts in i_k and j_k.

The Connecting Homomorphism

Lemma

If $0 \to \mathscr{A} \xrightarrow{i} \mathscr{B} \xrightarrow{j} \mathscr{C} \to 0$ *is a short-exact sequence of cochain complexes, then there is a well-defined linear map,*

$$\delta : H^k(\mathscr{C}) \longrightarrow H^{k+1}(\mathscr{A}),$$

which is defined as follows: if $c \in Z^k(\mathscr{A})$ *and* $b \in B^k$ *are such that* $c = j(b)$, *then*

$$\delta[c] = [a],$$

where $a \in Z^{k+1}(\mathscr{A})$ *is such that* $i(a) = db$.

Definition

The linear map $\delta : H^*(\mathscr{C}) \to H^{*+1}(\mathscr{A})$ is called the connecting map (or connecting homomorphism).

The Connecting Homomorphism

Sketch of Proof.

- Let $c \in Z^k(\mathscr{C})$. Then there is $b \in B^k$ and $a \in Z^{k+1}(\mathscr{A})$ such that $c = j(b)$ and $db = i(a)$ and the cohomology class $[a]$ depends only on c.
- We thus get a linear map $\delta : Z^k(\mathscr{C}) \to H^{k+1}(\mathscr{A})$, $c \to [a]$.
- If $c \in B^k(\mathscr{C})$, then $\delta c = 0$.
- It then follows that $\delta : Z^k(\mathscr{C}) \to H^{k+1}(\mathscr{A})$ descends to a linear map,

$$\delta : H^k(\mathscr{C}) \longrightarrow H^{k+1}(\mathscr{A}),$$

□

The Connecting Homomorphism

Remark

The construction of the connecting map can be summarized as the diagram,

$$\begin{array}{ccc} a & \hookrightarrow & db & & \\ & & \Big\uparrow{\scriptstyle d} & & \\ & & b & \twoheadrightarrow & c. \end{array}$$

where the arrows $\hookrightarrow$ and $\twoheadrightarrow$ are used to emphasize that i and j are injective and surjective, respectively.

The Zig-Zag Lemma

Theorem (Zig-Zag Lemma; Theorem 25.6)

Any short-exact sequence $0 \to \mathscr{A} \xrightarrow{i} \mathscr{B} \xrightarrow{j} \mathscr{C} \to 0$ *of cochain complexes given rise to a long exact sequence,*

$$\cdots \xrightarrow{\delta} H^k(\mathscr{A}) \xrightarrow{i^*} H^k(\mathscr{B}) \xrightarrow{j^*} H^k(\mathscr{C}) \xrightarrow{\delta} H^{k+1}(\mathscr{A}) \xrightarrow{i^*} \cdots,$$

where:

- i^* *and* j^* *are the maps induced on cohomology by the cochain maps* i *and* j.
- δ *is the connecting map.*

The Zig-Zag Lemma

Remark

The proof requires showing the following:

- Exactness at $H^{k+1}(\mathscr{A})$, i.e., $\operatorname{im} \delta = \ker i^*$.
- Exactness at $H^k(\mathscr{B})$, i.e., $\operatorname{im} i^* = \ker j^*$.
- Exactness at $H^k(\mathscr{C})$, i.e., $\operatorname{im} j^* = \ker \delta$.

The Zig-Zag Lemma

Corollary

Let $0 \to \mathscr{A} \xrightarrow{i} \mathscr{B} \xrightarrow{j} \mathscr{C} \to 0$ *be a short-exact sequence of cochain complexes.*

1. *If* $H^k(\mathscr{C}) = 0$*, then* $i : A^* \to B^*$ *is a quasi-isomorrphism, and hence* $H^k(\mathscr{A}) \simeq H^k(\mathscr{B})$*.*
2. *If* $H^k(\mathscr{A}) = 0$*, then* $j : B^* \to C^*$ *is a quasi-isomorrphism, and hence* $H^k(\mathscr{B}) \simeq H^k(\mathscr{C})$*.*
3. *If* $H^k(\mathscr{B}) = 0$*, then* $\delta : H^k(\mathscr{C}) \to H^{k+1}(\mathscr{A})$ *is an isomorphism.*

The Zig-Zag Lemma

Remark (Degree shifting)

- We have a shifted cochain complex $\mathscr{A}[1]$, where
  - The space of k-cochains is $A^k[1] := A^{k+1}$.
  - The differential in degree k is $d_{k+1} : A^{k+1} \to A^{k+2}$.
- We then have $H^k(\mathscr{A}[1]) = H^{k+1}(\mathscr{A})$.
- Therefore, the 3rd part of the corollary yields a cohomology space isomorphism,

$$H^k(\mathscr{C}) \simeq H^k(\mathscr{A}[1]).$$

The Zig-Zag Lemma

Corollary (The Snake Lemma; see Tu2011, Exercise 25.4)

Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^1 & \longrightarrow & B^1 & \longrightarrow & C^1 & \longrightarrow & 0 \\ & & \alpha\uparrow & & \beta\uparrow & & \gamma\uparrow & & \\ 0 & \longrightarrow & A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. Then we have an exact sequence,

$$0 \longrightarrow \ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \longrightarrow \operatorname{coker}(\beta) \longrightarrow \operatorname{coker}(\gamma) \longrightarrow 0.$$

where δ *is the connecting map.*