Differential Forms in Algebraic Topology
Review: Differential 1-Forms

Sichuan University, Spring 2024

Cotangent Space and Differential 1-Forms

Definition (Cotangent Space)

Let M be a smooth manifold.

- The *cotangent space* of M at p, denoted T_p^*M or $T_p^*(M)$, is the dual of the tangent space TpM. That is,

$$T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R}).$$

- An element of T_p^*M is called a *covector* at p.

Remark

In other words, a covector at p is just a linear map $\omega : T_pM \to \mathbb{R}$.

Cotangent Space and Differential 1-Forms

Definition (Differential 1-Forms)

A *differential 1-form* (or a *1-form* or a *covector field*) is the assignment to each $p \in M$ of a covector $\omega_p \in T_p^* M$.

Remark

If ω is a 1-form and X is a vector field on M, then we denote by $\omega(X)$ the function on M defined by

$$\omega(X)(p) = \omega_p(X_p), \qquad p \in M.$$

Cotangent Space and Differential 1-Forms

Definition (Differential of a Function)

Let $f \in C^\infty$. Its *differential* is the 1-form df on M defined by

$$(df)_p(X_p) = X_p f, \qquad X_p \in T_pM, \quad p \in M.$$

Proposition (Proposition 17.2)

If $f : M \to \mathbb{R}$ is a C^∞ function and $p \in M$, then

$$f_{*,p}(X_p) = (df)_p(X_p) \frac{d}{dt}\bigg|_{f(p)} \qquad \forall X_p \in T_pM.$$

Thus, under the identification $T_{f(p)}\mathbb{R} \simeq \mathbb{R}$, we have

$$(df)_p = f_{*,p}.$$

Cotangent Space and Differential 1-Forms

Example

Let $(U, x^1, \ldots, x^n)$ be a local chart near $p \in M$. Then

$$(dx^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \frac{\partial x^i}{\partial x^j}(p) = \delta^i_j.$$

Therefore, we obtain:

Proposition (Proposition 17.3)

Let $(U, x^1, \ldots, x^n)$ be a local chart near $p \in M$. Then $\{(dx^1)_p, \ldots, (dx^n)_p\}$ is the basis of T_p^*M which is dual to the basis $\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\}$ of T_pM.

Cotangent Space and Differential 1-Forms

Consequences

- Every covector $\omega_p \in T_p^*M$, can be uniquely written as
$$\omega_p = \sum a_i (dx^i)_p, \qquad \text{with } a^i = \omega_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right).$$
- Every 1-form on U can be uniquely written as
$$\omega = \sum a_i dx^i, \qquad \text{with } a^i = \omega(\partial/\partial x^i).$$
- If $f \in C^\infty(M)$, then on U we have
$$df = \sum \frac{\partial f}{\partial x^i} dx^i, \qquad \text{since } (df)(\partial/\partial x^i) = \frac{\partial f}{\partial x^i}.$$

The Cotangent Bundle

Definition

- The *cotangent bundle* is
$$T^*M = \bigsqcup_{p \in M} T_p^*M = \left\{(p,\omega);\ p \in M,\ \omega \in T_p^*M\right\}.$$
- The *canonical map* $\pi : T^*M \to M$ is given by
$$\pi\big((p,\omega)\big) = p, \qquad p \in M, \quad \omega \in T_p^*M.$$

Remark

If U is an open set of M, then $T^*U = \sqcup_{p \in U} T_p^*M$.

The Cotangent Bundle

Facts

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M. Set $V = \phi(U)$.

- Every covector $\omega_p \in T_p^* M$, can be uniquely written as

$$\omega_p = \sum a_i (dx^i)_p, \qquad \text{with } a^i = \omega_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right).$$

- We thus get a natural bijection $\tilde{\phi} : T^* U \to V \times \mathbb{R}^n$ such that, for all $p \in M$ and $\omega \in T_p^* M$, we have

$$\tilde{\phi}(p, \omega) = \left(x^1(p), \ldots, x^n(p), \omega(\partial/\partial x^1\big|_p), \ldots, \omega(\partial/\partial x^n\big|_p) \right).$$

Remark

In the same way as with the construction of the tangent bundle TM, the maps $\tilde{\phi}$ allow us to define a topology and a smooth structure on $T^* M$.

The Cotangent Bundle

Definition

Let (U, ϕ) be a chart for M and set $V = \phi(U)$. We endow T^*U with the topology such that

$$W \subset T^*U \text{ is open} \iff \tilde{\phi}(W) \text{ is open in } V \times \mathbb{R}^n.$$

Proposition

Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of M.

- *Define*
$$\mathscr{B} = \bigcup_\alpha \{W;\ W \text{ is an open in } T^*U_\alpha\}.$$
Then $\mathscr{B}$ is the basis for a unique topology on T^*M.
- *The collection $\{(T^*U_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas on T^*M, and hence T^*M is a smooth manifold.*
- *$T^*M \xrightarrow{\pi} M$ is a smooth vector bundle over M.*

Characterization of C^∞ 1-Forms

Remark

A 1-form on M is a section of the tangent bundle T^*M.

Definition

- We say that 1-form is C^∞ when it is C^∞ as a section of T^*M.
- We denote by $\Omega^1(M)$ the space of smooth 1-forms on M.

Remark

In other words, $\Omega^1(M)$ is the space of smooth sections of T^*M. In particular, this is a module over the ring $C^\infty(M)$.

Example

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M. Set $V = \phi(U)$.

- Let $\tilde{\phi} : T^*U \to V \times \mathbb{R}$ be the corresponding chart of T^*M. For all $p \in U$, we have

$$\begin{aligned}\tilde{\phi} \circ dx^i(p) = \tilde{\phi}(p, (dx^i)_p) &= (\phi(p), dx^i(\partial/\partial x^1), \ldots, dx^i(\partial/\partial x^n)) \\ &= (\phi(p), \delta^i_1, \ldots, \delta^i_n) \\ &= (\phi(p), e^i),\end{aligned}$$

  where $(e^1, \ldots, e^n)$ be the canonical basis of $\mathbb{R}^n$.

- Thus,
$$\tilde{\phi} \circ dx^i \circ \phi^{-1}(q) = (q, e^i) \qquad \forall q \in V.$$
  In particular, $\tilde{\phi} \circ dx^i \circ \phi^{-1}$ is a smooth map from V to $V \times \mathbb{R}^n$.

- It follows that dx^i is a smooth map from U to T^*U, and hence this is a smooth 1-form.

Characterization of C^∞ 1-Forms

Consequence

If $(U, x^1, \ldots, x^n)$ is a chart for M, then $\{dx^1, \ldots, dx^n\}$ is a C^∞-frame of T^*U over U.

Reminder (Proposition 12.2)

Let $\{s_1, \ldots, s_r\}$ be a C^∞ frame of a vector bundle E over U. A section $s = \sum c^i s_i$ of E over U is smooth if and only if $c^1, \ldots, c^r$ are smooth functions on U.

We immediately obtain:

Lemma (Lemma 17.5)

Let $(U, x^1, \ldots, x^n)$ be a chart for M. A 1-form $\omega = \sum a_i dx^i$ on U is smooth if and only if the coefficients $a_1, \ldots, a_n$ are smooth functions on U.

Characterization of C^∞ 1-Forms

In the same way as in Section 14, from the previous lemma we obtain:

Proposition (Proposition 17.6)

Let ω be a 1-form on M. Then TFAE:

1. *ω is a smooth 1-form.*
2. *M has an atlas such that, for every chart $(U, x^1, \ldots, x^n)$ of this atlas, we may write $\omega = \sum a_i dx^i$, where the coefficients a^i are smooth functions on U.*
3. *For every chart $(U, x^1, \ldots, x^n)$ of M, we may write $\omega = \sum a_i dx^i$, where the coefficients a^i are smooth functions on U.*

Corollary (Corollary 17.7)

If $f \in C^\infty(M)$, then its differential df is a smooth 1-form.

Characterization of C^∞ 1-Forms

Proposition (Proposition 17.8)

Let ω be a 1-form on M. If X is a vector field on M and f is a function on M, then

$$\omega(fX) = f\omega(X).$$

Proposition (Proposition 17.9)

Let ω be a 1-form on M. Then TFAE:

1. *ω is a smooth 1-form.*
2. *For every smooth vector field X on M the function $\omega(X)$ is smooth on M.*

Characterization of C^∞ 1-Forms

Reminder (see Section 14)

The space $\mathscr{X}(M)$ of smooth vector fields on M is a module over the ring $C^\infty(M)$.

Corollary

Every smooth 1-form ω on M defines a $C^\infty(M)$-module homomorphism,

$$\omega : \mathscr{X}(M) \longrightarrow C^\infty(M), \qquad X \longrightarrow \omega(X).$$

Pullbacks of 1-Forms

Reminder (Pullback by a linear map)

Let $f : V \to W$ be a linear maps between vector spaces.

- By duality we have a linear map,

$$f^* : W^* \longrightarrow V^*, \qquad \varphi \longrightarrow \varphi \circ f.$$

- If $\varphi \in W^*$, we call $f^*\varphi = \varphi \circ f$ the pullback of φ by f.

Consequence

Let $F : N \to M$ be a smooth map and let $p \in N$.

- The differential $F_{*,p} : T_pN \to T_{F(p)}M$ is a linear map.
- We thus get a pullback map $(F_{*,p})^* : T^*_{F(p)}M \to T^*_pN$.

Pullbacks of 1-Forms

Definition (Pullback of 1-forms)

Let $F : N \to M$ be a smooth map. If ω is a 1-form on M, then its *pullback* $F^*\omega$ is the 1-form on N defined by

$$(F^*\omega)_p = \left(F_{*,p}\right)^* \omega_{F(p)} = \omega_{F(p)} \circ F_{*,p}, \qquad p \in N.$$

Remark

In other words, for all $X_p \in T_pN$, we have

$$(F^*\omega)_p(X_p) = \omega_{F(p)} \circ F_{*,p}(X_p) = \omega_{F(p)}\big(F_{*,p}(X_p)\big).$$

Remark

Smooth functions can be pullbacked as well: if $h \in C^\infty(M)$, then $F^*h = h \circ F$.

Pullbacks of 1-Forms

Proposition (Proposition 17.10)

Let $F : N \to M$ be a smooth map. If $h \in C^\infty(M)$, then

$$F^*(dh) = d(F^*h).$$

Proposition (Proposition 17.11)

Let $F : N \to M$ be a smooth map. If $\omega, \tau \in \Omega^1(M)$ and $g \in C^\infty(M)$, then

$$F^*(\omega + \tau) = F^*\omega + F^*\tau,$$
$$F^*(g\omega) = (F^*g)(F^*\omega).$$

Proposition (Proposition 17.12)

Let $F : N \to M$ be a smooth map. If ω is a smooth 1-form on M, then its pullback $F^*\omega$ is a smooth 1-form as well.

Restriction to an Immersed Submanifold

Facts

Let S be an immersed submanifold in M.

- The inclusion $i : S \to M$ is an immersion, and so its differential $i_{*,p} : T_pS \to T_pM$ is an injection for every $p \in S$.
- This allows us to identify T_pS with a subspace of T_pM.

Definition

If ω is a 1-form on M, its *restriction* to S, denoted $\omega_{|S}$, is the 1-form on S defined by

$$\left(\omega_{|S}\right)_p(v) = \omega_p(v) \qquad \text{for all } p \in S \text{ and } v \in T_pS.$$

Proposition

If $i : S \to M$ is the inclusion of S into M and ω is a 1-form on M, then $\omega_{|S} = i^*\omega$.