# Commutative Algebra
# Chapter 6: Chain Conditions

Sichuan University, Fall 2023

**Reminder**

- An order relation on a set $\Sigma$ is a reflexive and transitive relation $\leq$ on $\Sigma$ such that

$$x \leq y \text{ and } y \leq x \implies x = y.$$

- A set equipped with an order relation is called *partially ordered*.

**Example**

Let $\Sigma$ be the set of submodules of a module $M$ over a ring $A$. Then $(\Sigma, \subseteq)$ and $(\Sigma, \supseteq)$ are both partially ordered sets.

# Chain Conditions

**Proposition (Proposition 6.1)**

*Let $(\Sigma, \leq)$ be a partially ordered set. TFAE:*

(i) *Every increasing sequence $x_1 \leq x_2 \leq \cdots \leq x_j \leq \cdots$ is stationary (i.e., there is $n$ such that $x_j = x_n$ for all $j \geq n$).*

(ii) *Every non-empty subset of $\Sigma$ has a maximal element.*

**Proof.**

- Let $S \subseteq \Sigma$ have no maximal element, i.e., for every $x \in S$ there is $y \in S$ such that $x \leq y$ and $y \neq x$.
- Starting with any $x_0 \in S$ we can construct by induction a sequence $(x_n)_{n \geq 0} \subseteq S$ such that $x_n \leq x_{n+1}$ and $x_n \neq x_{n+1}$, and hence (i) fails.
- By contraposition (i) implies (ii).
- Suppose that (ii) holds, let $x_1 \leq x_2 \leq \cdots$ be an increasing sequence.
- There is $n \geq 1$ such that $x_n$ is maximal in the set $\{x_j; j \geq 0\}$.
- If $j \geq n$, then $x_n \leq x_j$, and so $x_j = x_n$, since $x_n$ is maximal.
- Thus, the sequence is stationary, and hence (ii) implies (i).

The proof is complete. □

**Definition**

Let $M$ be a module over a ring $A$ and denote by $\Sigma$ be the set of its submodules.

- For the partially ordered set $(\Sigma, \subseteq)$ the condition (i) is called the *ascending chain condition* (a.c.c.) and the condition (ii) is called the *maximal condition*.
- For the partially ordered set $(\Sigma, \supseteq)$ the condition (i) is called the *descending chain condition* (d.c.c.) and the condition (ii) is called the *minimal condition*.

**Definition**

Let $M$ be a module over $A$.

1. We say that $M$ is *Noetherian* (after Emmy Noether) if $M$ satisfies a.c.c. or the maximal condition.
2. We say that $M$ is *Artinian* (after Emil Artin) if $M$ satisfies d.c.c. or the minimal condition.

**Remark**

Let $\mathfrak{a}$ be an ideal of $A$ such that $\mathfrak{a}M = 0$. Then:

- $M$ is an $A/\mathfrak{a}$-module.
- $M$ is Noetherian (resp., Artinian) over $A$ if and only if it is Noetherian (resp., Artinian) over $A/\mathfrak{a}$.

**Example**

Any finite group is both Noetherian and Artinian (as a module over $\mathbb{Z}$).

**Proof of the Noetherian Property.**

Denote by $|G|$ the order of $G$ (i.e., its number of elements).

- Let $G_1 \subseteq G_2 \subseteq$ be an ascending chain of subgroups.
- Then $|G_1| \leq |G_2| \leq \cdots \leq |G|$ is a bounded non-decreasing sequence of integers, and hence $\exists n$ s.t. $|G_j| = |G_n|$ for $j \geq n$.
- As $G_j \supseteq G_n$, it follows that $G_j = G_n$ for $j \geq n$.
- Thus, the chain is stationary, and hence $G$ is a Noetherian $\mathbb{Z}$-module.

$\square$

**Proof of the Artinian Property.**

- Let $G_1 \supseteq G_2 \supseteq$ be an descending chain of subgroups.
- Then $|G_1| \geq |G_2| \geq \cdots \geq 0$ is a non-increasing sequence of non-negative integers, and so $\exists n$ s.t. $|G_j| = |G_n|$ for $j \geq n$.
- As $G_j \subseteq G_n$, it follows that $G_j = G_n$ for $j \geq n$.
- Thus, the chain is stationary, and hence $G$ is an Artinian $\mathbb{Z}$-module.

□

**Example (see also Proposition 6.10)**

Let $V$ be a vector space over a field $k$. Then:

1. If $\dim V < \infty$, then $V$ is both Noetherian and Artinian.
2. If $\dim V = \infty$, then $V$ is neither Noetherian nor Artinian.

**Proof ($\dim V = \infty$ Case).**

- If $\dim V = \infty$, then $V$ contains a linearly independent sequence $(v_j)_{j\geq 1}$.
- Set $V_j = \text{Span}\{x_k; k \leq j\}$. Then $V_1 \subsetneq V_2 \subsetneq \cdots$ is a non-stationary ascending chain, and hence $V$ is not Noetherian.
- Set $W_j = \text{Span}\{x_k; k \geq j\}$. Then $V = W_1 \supsetneq W_2 \supsetneq \cdots$ is a non-stationary descending chain, and hence $V$ is not Artinian.

□

**Proof ($\dim V < \infty$ Case).**

- Suppose that $\dim V < \infty$.
- Let $V_1 \subseteq V_2 \subseteq \cdots$ be an ascending chain of vector subspaces.
- Here $0 \leq \dim V_1 \leq \dim V_2 \leq \cdots \leq \dim V$, and hence $(\dim V_j)_{j \geq 0}$ is a bounded non-decreasing sequence of integers.
- Thus, $\exists n$ s.t. $\dim V_j = \dim V_n$ for $j \geq n$.
- As $V_j \supseteq V_n$, it follows that $V_j = V_n$ for $j \geq n$, i.e., we have a stationary chain, and hence $V$ is Noetherian.
- Let $V_1 \supseteq V_2 \supseteq \cdots$ be a descending chain of vector subspaces.
- Here $\dim V \geq \dim V_1 \geq \dim V_2 \geq \cdots \geq 0$, and hence $(\dim V_j)_{j \geq 0}$ is a bounded non-increasing sequence of integers.
- Thus, $\exists n$ s.t. $\dim V_j = \dim V_n$ for $j \geq n$.
- As $V_j \subseteq V_n$, it follows that $V_j = V_n$ for $j \geq n$, i.e., we have a stationary chain, and hence $V$ is Artinian.

□

**Example**

The ring $\mathbb{Z}$ (as a module over itself) is Noetherian, but is not Artinian.

**Proof that $\mathbb{Z}$ is not Artinian.**

- Every ideal of $\mathbb{Z}$ is principal, i.e., of the form $(m)$ with $m \in \mathbb{Z}$.
- We have $(m_1) \subseteq (m_2)$ if and only if $m_2$ divides $m_1$.
- Let $m \in \mathbb{Z} \setminus 0$. Then $(m) \supsetneq (m^2) \supsetneq (m^3) \supsetneq \cdots$ is a non-stationary descending chain of ideals.
- Thus, $\mathbb{Z}$ is not Artinian.

$\square$

**Proof that $\mathbb{Z}$ is Noetherian.**

- Let $(m_1) \subseteq (m_2) \subseteq \cdots$ be an ascending chain of ideals (with $m_j \geq 0$).
- Suppose the chain is non-stationary, i.e., $(m_j) \subsetneq (m_{j+1})$ for all $j \geq 1$.
- Thus $m_{j+1}$ divides $m_j$ and $m_{j+1} \neq m_j$, so $m_j \geq 2m_{j+1}$, i.e., $m_{j+1} \leq 2^{-1}m_j$.
- By induction $m_j \leq 2^{-j+1}m_1 < 1$ for $j$ large (impossible).
- Thus, the chain is stationary, and hence $\mathbb{Z}$ is Noetherian. □

**Example**

If $k$ is a field, then, in the same way as with $\mathbb{Z}$, it can be shown that the polynomial ring $k[x]$ (seen as a module over itself) is Noetherian, but not Artinian.

**Example**

If $k$ is a field, then the polynomial ring $A = k[x_1, x_2, \ldots]$ with an infinite number of variables is neither Noetherian nor Artinian.

**Proof.**

- If $\mathfrak{a}_j = (x_1, \ldots, x_j)$, $j \geq 1$, then $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots$ is a non-stationary ascending chain of ideals.
- Thus, $A$ is not Noetherian.
- $(x_1) \supsetneq (x_1^2) \supsetneq \cdots$ is a non-stationary descending chain.
- Thus, $A$ is not Artinian.

□

**Proposition (Proposition 6.2)**

*Let $M$ be a module over $A$. TFAE:*

(i) *$M$ is Noetherian.*

(ii) *Every submodule of $M$ is finitely generated.*

**Proof of (i)⇒(ii).**

- Assume $M$ is a Noetherian, and let $N \subseteq M$ be a submodule.
- Let $\Sigma$ be the set of all finitely generated submodules of $N$,
- Here $\Sigma \neq \emptyset$, since $0 \in \Sigma$, it admits a maximal element $N_0$.
- Suppose that $N_0 \subsetneq N$. Let $x \in N \setminus N_0$. Then $N_0' = N_0 + Ax$ is finitely generated and $N_0' \supsetneq N_0$, and hence $N_0$ is not maximal (contradiction).
- Thus, $N = N_0$, and hence $N$ is finitely generated. □

**Proof of (ii)⇒(i).**

- Suppose that every submodule is finitely generated.
- Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending sequence of submodules.
- $N = \cup_{j=1}^{\infty} M_j$ is a submodule, and hence is finitely generated by $x_1, \ldots, x_r$.
- For each $i$ there is $n_i$ such that $x_i \in M_{n_i}$. Set $n = \max(n_1, \ldots, n_r)$.
- If $j \geq n$, then $x_i \in M_{n_i} \subseteq M_n \subseteq M_j$ for all $i$, and hence $N \subseteq M_n \subseteq M_j$.
- As $M_n \subseteq M_j \subseteq N$, we deduce that $M_n = M_j = N$ for all $j \geq n$.
- Thus, the chain is stationary, and hence $M$ is Noetherian.

The proof is complete. □

**Proposition (Proposition 6.3)**

*Let* $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ *be an exact sequence of* $A$*-modules.*

(i) $M$ *is Noetherian if and only if* $M'$ *and* $M''$ *are both Noetherian.*

(ii) $M$ *is Artinian if and only if* $M'$ *and* $M''$ *are both Artinian.*

**Remarks**

- As $\alpha$ is injective, if $N \subseteq M'$, then $\alpha^{-1}(\alpha(N)) = N$. Thus,
$$N_1 = N_2 \Longleftrightarrow \alpha(N_1) = \alpha(N_2)$$
- As $\beta$ is surjective, if $N \subseteq M''$, then $\beta(\beta^{-1}(N)) = N$. Thus,
$$N_1 = N_2 \Longleftrightarrow \beta^{-1}(N_1) = \beta^{-1}(N_2)$$

## Proof of Proposition 6.3(i).

- Assume $M$ is Noetherian.
- Let $M'_1 \subseteq M'_2 \subseteq \cdots$ be an ascending chain of submod. of $M'$.
- $\alpha(M'_1) \subseteq \alpha(M'_2) \subseteq \cdots$ is an ascending chain of submod. of $M$.
- As $M$ is Noetherian, there is $n$ such that if $j \geq n$, then $\alpha(M'_j) = \alpha(M'_n)$ for $j \geq n$, and hence $M'_j = M'_n$.
- Thus, we have a stationary chain, and hence $M'$ is Noetherian.
- Let $M''_1 \subseteq M''_2 \subseteq \cdots$ be an ascending chain of submod. of $M''$.
- $\beta^{-1}(M''_1) \subseteq \beta^{-1}(M''_2) \subseteq \cdots$ is an ascending chain of submodules of $M$.
- As $M$ is Noetherian, there is $n$ such that if $j \geq n$, then $\beta^{-1}(M''_j) = \beta^{-1}(M''_n)$ for $j \geq n$, and hence $M''_j = M''_n$.
- Thus, we have a stationary chain, and hence $M''$ is Noetherian.

□

**Proof of Proposition 6.3(i); Continued.**

- Assume $M'$ and $M''$ are Noetherian, and let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending chain of submod. of $M$.
- $\alpha^{-1}(M_1) \subseteq \alpha^{-1}(M_2) \subseteq \cdots$ is an ascending chain of submod. of $M'$, and hence $\alpha^{-1}(M_j) = \alpha^{-1}(M_{n'})$ for $j \geq n'$.
- $\beta(M_1) \subseteq \beta(M_2) \subseteq \cdots$ is an ascending chain of submodules of $M''$, and hence $\beta(M_j) = \beta(M_{n''})$ for $j \geq n''$.
- Set $n = \max(n', n'')$, and let $x \in M_j$, $j \geq n$.
- As $\beta(x) \in \beta(M_j) = \beta(M_n)$, $\exists y \in M_n$ s.t. $\beta(x) = \beta(y)$.
- Thus, $x - y \in \ker \beta = \operatorname{ran} \alpha$, and hence $\exists z \in M'$ s.t. $\alpha(z) = x - y \in M_j$, and hence $z \in \alpha^{-1}(M_j) = \alpha^{-1}(M_n)$.
- Thus, $x = y + \alpha(z) \in M_n + M_n = M_n$, and hence $M_j \subseteq M_n$.
- Thus, $M_j = M_n$ for $j \geq n$, i.e., the chain is stationary, and hence $M$ is Noetherian.

The proof is complete. □

**Corollary (Corollary 6.4)**

*If $M_1, \ldots, M_n$ are Noetherian (resp., Artinian) modules over $A$, then so is their direct sum $\bigoplus_{i=1}^{n} M_i$.*

**Proof.**

- Assume that the modules $M_i$ are Noetherian. We proceed by induction on $n$.
- For $n = 1$ the result is immediate.
- Suppose that $\bigoplus_{i=1}^{n-1} M_i$ is Noetherian. We have an exact sequence,
$$0 \longrightarrow M_n \longrightarrow \bigoplus_{i=1}^{n} M_i \longrightarrow \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0.$$
- As $M_n$ and $\bigoplus_{i=1}^{n-1} M_i$ are Noetherian, $\bigoplus_{i=1}^{n} M_i$ is Noetherian by Proposition 6.3.
- This proves the Noetherian case. The Artinian case is proved similarly. □

# Chain Conditions

## Definition

A ring $A$ is called *Noetherian* (resp., *Artinian*) when it is Noetherian (resp., Artinian) as a module over itself, i.e., it satisfies a.c.c. (resp., d.c.c.) on *ideals*.

## Examples

1. Any field $k$ is both Noetherian and Artinian (the only ideals are $0$ and $k$).
2. The ring $\mathbb{Z}$ is Noetherian, but not Artinian (see slide 11).
3. Any principal ideal domain is Noetherian (this follows from Proposition 6.2).

**Example**

Let $A = k[x_1, x_2, \ldots]$ be the polynomial ring with infinite variables over a field $k$.

- $A$ is not Noetherian (see slide 13).
- It is an integral domain, and so its fraction field $k = \mathrm{Frac}(A)$ is Noetherian.
- This shows that a subring of a Noetherian ring need not be Noetherian.

**Example**

Let $X$ be a compact Hausforff space. Then the ring $C(X)$ of continuous functions on $X$ is not Noetherian.

### Proposition (Proposition 6.5)

*Let $A$ be a Noetherian (resp., Artinian) ring and $M$ a finitely generated $A$-module. Then $M$ is Noetherian (resp., Artinian).*

### Proof (Noetherian Case).

- Assume $A$ is Noetherian, and let $x_1, \ldots, x_n$ generate $M$.
- Define $\phi : A^n \to M$, $\phi(a_1, \ldots, a_n) = a_1x_1 + \cdots + a_nx_n$.
- We then have an exact sequence,

$$0 \longrightarrow \ker \phi \longrightarrow A^n \longrightarrow M \longrightarrow 0.$$

- By Corollary 6.4 $A^n$ is Noetherian.
- Thus, $M$ is Noetherian by Proposition 6.3.

The proof is complete. □

**Reminder**

A ring $A$ is Noetherian (resp., Artinian) if and only if any ascending (resp., descending) chain of ideals is stationary.

**Proposition (Proposition 6.6)**

*Let $A$ be a Noetherian (resp., Artinian) ring and $\mathfrak{a}$ an ideal of $A$. Then $A/\mathfrak{a}$ is a Noetherian (resp., Artinian) ring.*

**Proof of Proposition 6.6.**

- We have an exact sequence of $A$-modules,

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0.$$

- If $A$ is Noetherian, then $A/\mathfrak{a}$ is a Noetherian $A$-module by Proposition 6.3.
- If $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$ is an ascending chain of ideals of $A/\mathfrak{a}$, then this is an ascending chain of $A$-submodules of $A/\mathfrak{a}$.
- As $A/\mathfrak{a}$ is a Noetherian $A$-module, this chain is stationary, and hence $A/\mathfrak{a}$ is Noetherian as an $A/\mathfrak{a}$-module.
- Likewise, if $A$ is Artinian, then the ring $A/\mathfrak{a}$ is Artinian as well.

The proof is complete. □

# Chain Conditions

## Definition

Let $M$ be a module over $A$.

- A *chain* of submodules of $M$ is a strictly descending finite sequence of the form,

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n = 0.$$

- We call $n$ the *length* of the chain.
- A *composition series* is a maximal chain, i.e., we cannot insert any module between $M_i$ and $M_{i+1}$.

## Remarks

- The composition series condition is equivalent to requiring each module $M_i/M_{i+1}$ to be *simple*, i.e., it has no submodules but 0 and itself.
- If $M = A$ and the $M_i$ are ideals, then this is equivalent to requiring $M_i/M_{i+1}$ to be a field.

# Chain Conditions

**Proposition (Proposition 6.7)**

*Suppose that $M$ has a composition series of length $n$. Then:*

(i) *Every composition series has length $n$.*

(ii) *Every chain in $M$ can be extended to a composition series.*

**Definition**

- If $M$ has a composition series, then we denote by $\ell(M)$ the length of any composition series.
- Otherwise we set $\ell(M) = \infty$.
- We call $\ell(M)$ the *length* of $M$.

**Remark**

We say that $M$ *is of finite length* it admits a composition series.

**Proposition (Proposition 6.8)**

*Let $M$ be a module over $A$. TFAE:*

(i) *$M$ is of finite length.*

(ii) *$M$ is both Noetherian and Artinian.*

**Proof of (i)⇒(ii).**

- If $M$ is of finite length, then by Proposition 6.7 all chains are bounded.
- Thus, every ascending or descending chain is stationary, and hence $M$ is both Noetherian and Artinian. □

**Proof of (ii)⇒(i).**

- Suppose that $M$ is Noetherian and Artinian.
- If $N \subseteq M$ is a non-zero sub-module, then the set of proper submodules of $N$ is non-empty.
- As $M$ is Noetherian, this set admits a maximal element.
- That is, there is submodule $N' \subsetneq N$ such that $N/N'$ is simple.
- Thus, starting with $M_0 = M$ by induction we can construct a chain $M_0 \supseteq M_1 \supseteq \cdots$ such that either $M_j = 0$ or $M_j/M_{j+1}$ is simple.
- As $M$ is Artinian, this chain must be stationary, and so there is $n$ such that $M_n = 0$ for all $j \geq n$.
- Taking the smallest such $n$ then yields a composition series $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$, and hence $M$ is of finite length.

The proof is complete. □

**Remark (Jordan-Hölder Theorem; see Carlson + Gaillard)**

Let $M$ be an $A$-module of finite length $n$. If $(M_i)_{0 \leq i \leq n}$ and $(M'_i)_{0 \leq i \leq n}$ are two composition series for $M$. Then the quotients $M_{i-1}/M_i$ and $M'_{i-1}/M'_i$ are isomorphic for $i = 1, \ldots, n$,

**Proposition (Proposition 6.9)**

*The length is an additive function on finite-length $A$-modules. That is, if $0 \to M' \to M \to M'' \to 0$ is any exact sequence of finite-length $A$-modules, then*

$$\ell(M) = \ell(M') + \ell(M'').$$

**Proof of Proposition 6.9 (Sketch).**

- Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence.
- Let $M'_0 \supsetneq \cdots \supsetneq M'_{n'}$ be a comp. series for $M'$.
- Let $M''_0 \supsetneq \cdots \supsetneq M''_{n''}$ be a comp. series for $M''$.
- Note that
$$\beta^{-1}(M''_{n''}) = \beta^{-1}(0) = \ker\beta = \operatorname{ran}\alpha = \alpha(M') = \alpha(M'_0).$$
- Then
$$M = \beta^{-1}(M''_0) \supsetneq \cdots \supsetneq \beta^{-1}(M''_{n''}) = \alpha(M'_0) \supsetneq \cdots \supsetneq \alpha(M_{n'}) = 0$$
is a composition series for $M$.
- Thus, $M$ of finite length $\ell(M) = n' + n'' = \ell(M') + \ell(M'')$.

This gives the result. □

**Proposition (Proposition 6.10)**

*Let $V$ be a vector space over a field $k$. TFAE:*

(i) *$V$ has finite dimension.*

(ii) *$V$ has finite length.*

(iii) *$V$ is Noetherian.*

(iv) *$V$ is Artinian.*

*Furthermore, if these conditions hold, then $\ell(V) = \dim V$.*

**Proof of Proposition 6.10.**

- Assume $\dim V = n$ and let $\{v_1, \ldots, v_n\}$ be a basis for $V$.
- Set $V_j = \text{Span}\{v_{j+1}, \ldots, v_n\}$, $0 \leq j \leq n-1$, and $V_n = \{0\}$.
- Then $V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_n = 0$ is a composition series for $V$, and hence $V$ is of finite length $n = \dim V$. Thus (i)$\Rightarrow$(ii).
- It is immediate that (ii)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iv).
- If $\dim V = \infty$, then $V$ is not Noetherian or Artinian (see Slide 9). By contraposition (iii)$\Rightarrow$(i) and (iv)$\Rightarrow$(i).

The proof is complete. □

**Corollary (Corollary 6.11)**

*Let $A$ be a ring such that there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ so that $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$. Then $A$ is Noetherian if and only if it is Artinian.*

Proof.

- Set $\mathfrak{a}_0 = M$ and $\mathfrak{a}_i = \mathfrak{m}_1 \cdots \mathfrak{m}_i$, $i = 1, \ldots, n$.
- We thus get a chain of ideals,

$$M = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_n = 0.$$

- Here $V_i := \mathfrak{a}_{i-1}/\mathfrak{a}_i$ is a vector space over the field $A/\mathfrak{m}_i$.
- Thus, by Proposition 6.10 we have

$$(V_i \text{ Noetherian } A/\mathfrak{m}_i\text{-module}) \Longleftrightarrow (V_i \text{ Artinian } A/\mathfrak{m}_i\text{-module}).$$

- As $\mathfrak{m}_i V_i = (\mathfrak{m}_i \mathfrak{a}_{i-1})/\mathfrak{a}_i = \mathfrak{a}_i = 0$, we see that

$$(V_i \text{ Noetherian } A/\mathfrak{m}_i\text{-module}) \Longleftrightarrow (V_i \text{ Notherian } A\text{-module}),$$
$$(V_i \text{ Artinian } A/\mathfrak{m}_i\text{-module}) \Longleftrightarrow (V_i \text{ Artinian } A\text{-module}).$$

- Thus,

$$(V_i \text{ Noetherian } A\text{-module}) \Longleftrightarrow (V_i \text{ Artinian } A\text{-module}).$$

□

**Proof.**

- We have exact sequences of $A$-modules,

$$0 \longrightarrow \mathfrak{a}_i \longrightarrow \mathfrak{a}_{i-1} \longrightarrow V_i \longrightarrow 0.$$

- Thus, by Proposition 6.3 we get

$$(\mathfrak{a}_{i-1} \text{ Noetherian}) \Longleftrightarrow (\mathfrak{a}_i \text{ and } V_i \text{ Noetherian})\,.$$

- By induction we then obtain

$$\begin{aligned}(A = \mathfrak{a}_0 \text{ Noetherian}) &\Longleftrightarrow (\mathfrak{a}_1 \text{ and } V_1 \text{ Noetherian})\\ &\Longleftrightarrow (\mathfrak{a}_2 \text{ and } V_1, V_2 \text{ Noetherian})\\ &\Longleftrightarrow \qquad\qquad \cdots\\ &\Longleftrightarrow (\mathfrak{a}_n \text{ and } V_1, \ldots, V_n \text{ Noetherian})\end{aligned}$$

- As $\mathfrak{a}_n = 0$ we see that

$$(A \text{ Noetherian}) \Longleftrightarrow (V_1, \ldots, V_n \text{ Noetherian})\,.$$

□

**Proof.**

- Likewise,

$$(A \text{ Artinian}) \iff (V_1, \ldots, V_n \text{ Artinian}).$$

- As $V_i$ is Noetherian if and only if it is Artinian, we get

$$(A \text{ is a Noetherian ring}) \iff (A \text{ is an Artin ring}).$$

The proof is complete. $\square$