Commentative Algebra
Chapter 3: Rings and Modules of Fractions

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The Field of Fractions of an Integral Domain

Reminder

We say that a ring A is an *integral domain* when it has no non-zero divisors, i.e.,

$$xy = 0 \iff x = 0 \text{ or } y = 0.$$

Fact

In the same way we construct the rational field $\mathbb{Q}$ from the ring of integers $\mathbb{Z}$, with any integral domain A we can associate its field of fractions $\mathrm{Frac}(A)$.

The Field of Fractions of an Integral Domain

Facts

Let A is an integral domain. Set $S = A \setminus \{0\}$. On $A \times S$ define a relation $\equiv$ by

$$(a, s) \equiv (b, t) \iff at = bs.$$

- This relation is reflexive and symmetric,

$$(a, s) \equiv (a, s), \qquad (a, s) \equiv (b, t) \iff (b, t) \equiv (a, s).$$

- To check transitivity, suppose that $(a, s) \equiv (b, t)$ and $(b, t) \equiv (c, u)$, i.e., $at = bs$ and $bu = ct$. Then

$$t(au - cs) = (at)u - (ct)s = (bs)u - (bu)s = 0.$$

- As $t \neq 0$ and A is an integral domain, this implies that $au = cs$, i.e., $(a, s) \equiv (c, u)$.
- Therefore, the relation $\equiv$ is an equivalence relation on $A \times S$.

The Field of Fractions of an Integral Domain

Definition

1. The class of (a, s) is denoted by a/s.
2. The set of equivalence classes is denoted by $\operatorname{Frac}(A)$.

Proposition

1. $\operatorname{Frac}(A)$ *is a field with respect to the addition and multiplication given by*

$$(a/s) + (b/t) = (at + bs)/st, \qquad (a/s) \cdot (b/t) = ab/st.$$

2. *The map* $A \ni a \to a/1 \in \operatorname{Frac}(A)$ *is an injective ring homomorphism, and hence embeds* A *as a subring into* $\operatorname{Frac}(A)$*.*

Definition

The field $\operatorname{Frac}(A)$ is called the *field of fractions* of A.

Examples

1. If $A = \mathbb{Z}$, then $\mathrm{Frac}(A) = \mathbb{Q}$.
2. If A is a polynomial ring $k[x]$, k field, then $\mathrm{Frac}(A)$ is the field of rational functions over k.
3. If A is the ring of holomorphic functions on an open $\Omega \subseteq \mathbb{C}$, then $\mathrm{Frac}(A)$ is the field of meromorphic functions on Ω.

Remark

- The construction of the field $\mathrm{Frac}(A)$ uses the fact that A is an integral domain.
- It still can be adapted for arbitrary rings.

In what follows we let A be a ring.

Definition

A subset S of A is called *multiplicatively closed* when

$$1 \in S \quad \text{and} \quad x, y \in S \Longrightarrow xy \in S.$$

Example

The ring A is an integral domain if and only if $A \setminus \{0\}$ is multiplicatively closed.

Facts

Let S be a multiplicatively closed subset of A. On $A \times S$ define a relation $\equiv$ by

$$(a, s) \equiv (b, t) \iff \exists u \in S \text{ such that } (at - bs)u = 0.$$

- This relation is reflexive and symmetric.
- To check transitivity, suppose that $(a, s) \equiv (b, t)$ and $(b, t) \equiv (c, u)$, i.e., there are $v, w \in S$ such that

$$(at - bs)v = (bu - ct)w = 0.$$

- Then $(au - cs)tvw$ is equal to

$$\big[(at)v\big]uw - \big[(ct)w\big]sv = \big[(bs)v\big]uw - \big[(bu)w\big]sv = 0.$$

- As S is multiplicatively closed, $tvw \in S$, and so $(a, s) \equiv (c, u)$.
- Thus, we have an equivalence relation on $A \times S$.

Rings of Fractions

Definition

1. The class of (a, s) is denoted by a/s.
2. The set of equivalence classes is denoted by $S^{-1}A$.

Proposition

1. *$S^{-1}A$ is a ring with respect to the addition and multiplication given by*

$$(a/s) + (b/t) = (at + bs)/st, \qquad (a/s) \cdot (b/t) = ab/st.$$

2. *The map $f : A \to S^{-1}A$, $a \to a/1$ is a ring homomorphism.*

Remarks

1. The ring homomorphism $f : A \to S^{-1}A$ is not injective in general.
2. If A is an integral domain and $S = A \setminus \{0\}$, then $S^{-1}A$ is the field of fractions $\mathrm{Frac}(A)$.

Definition

The ring $S^{-1}A$ is called the *ring of fractions* of A with respect to S.

Proposition (Universal Property of $S^{-1}A$; Proposition 3.1)

Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for all $s \in S$. Then there is a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$.

Fact

The ring $S^{-1}A$ and the homomorphism $f : A \to S^{-1}A$ satisfy the following properties:

(i) $f(s)$ is a unit in $S^{-1}A$ for all $s \in S$.

(ii) If $f(a) = 0$, then $as = 0$ for some $s \in S$.

(iii) Every element of $S^{-1}A$ is of the form $f(a)f(s)^{-1}$ with $a \in A$ and $s \in S$.

Corollary (Corollary 3.2)

Let B be a ring and $g : A \to B$ a ring homomorphism satisfying the properties (i)–(iii) above. Then there is a unique ring isomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$.

Example

- The set $S = \{1, 0\}$ is multiplicatively closed.
- In this case $S^{-1}A$ is the zero ring, since $(a, s) \equiv (0, 0)$ for all $a \in A$ and $s \in S$.
- In fact, we have

$$S^{-1}A \text{ is the zero ring} \iff 0 \in S.$$

Example

Let $\mathfrak{a}$ be an ideal in A, and set

$$S = 1 + \mathfrak{a} = \{1 + x;\ x \in \mathfrak{a}\} = \{x \in A; x = 1 \bmod \mathfrak{a}\}.$$

Then S is multiplicatively closed.

Examples of Rings of Fractions

Example

Let $f \in A$ and set $S = \{f^n;\ n \geq 0\}$.

- The subset S is multiplicatively closed.
- We write A_f for $S^{-1}A$ in this case.
- If $A = \mathbb{Z}$ and $f = q \in \mathbb{Z}$, then A_f consists of rational numbers of the form mq^{-n} with $m \in \mathbb{Z}$ and $n \geq 0$.

Examples of Rings of Fractions

Reminder

- An ideal $\mathfrak{p}$ of A is called a *prime ideal* when

$$xy \in \mathfrak{p} \iff x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}.$$

- Any *maximal ideal* is prime.
- A *local ring* is a ring that has a unique maximal ideal.

Example

Let $\mathfrak{p}$ be a prime ideal, and set $S = A \setminus \mathfrak{p}$. We have

$$\mathfrak{p} \text{ is prime} \iff S \text{ is multiplicatively closed}.$$

We denote by $A_{\mathfrak{p}}$ the ring $S^{-1}A$ in this case.

Facts

Let $\mathfrak{m}$ be the subset of $A_{\mathfrak{p}}$ consisting of elements of the form a/s with $a \in \mathfrak{p}$ and $s \in S$.

- $\mathfrak{m}$ is an ideal of $A_{\mathfrak{p}}$.
- If $b/t \notin \mathfrak{m}$, then $b \notin \mathfrak{p}$, i.e, $b \in S$, and so b/t is a unit in $A_{\mathfrak{p}}$ (with inverse t/b).
- Thus, if $\mathfrak{a}$ is an ideal such that $\mathfrak{a} \not\subseteq \mathfrak{m}$, then $\mathfrak{a}$ contains a unit, and hence $\mathfrak{a} = A_{\mathfrak{p}}$.
- It follows that $\mathfrak{m}$ is a maximal ideal of A and is the only such ideal. Thus, $A_{\mathfrak{p}}$ is a *local ring*.

Definition

The ring $A_{\mathfrak{p}}$ is called the *localization* of A at $\mathfrak{p}$.

Examples of Rings of Fractions

Example

$A = \mathbb{Z}$ and $\mathfrak{p} = (p)$, where p is a prime number. Then $\mathbb{Z}_{\mathfrak{p}}$ consists of all rational numbers of the form m/n where n is prime to p.

Modules of Fractions

The construction of $S^{-1}A$ can be further extended to A-modules.

Facts

Let S be a multiplicatively closed subset of A and M an A-module. On $M \times S$ we define a relation $\equiv$ by

$$(m, s) \equiv (m', s') \iff \exists t \in S \text{ such that } t(s'm - sm') = 0.$$

As before, this is an equivalence relation.

Definition

1. The equivalence class of (m, s) is denoted m/s.
2. The set of equivalence classes is denoted $S^{-1}M$.

Modules of Fractions

Proposition

$S^{-1}M$ is an $S^{-1}A$-module with respect to the addition and scalar multiplication given by

$$(m/s) + (m'/s') = (s'm + sm')/ss', \qquad (a/s) \cdot (m/t) = am/st.$$

Definition

$S^{-1}M$ is called the *module of fractions* of M with respect to S.

Modules of Fractions

Fact

- If $u : M \to N$ is an A-module homomorphism, then we get an $S^{-1}A$-module homomorphism,
$$S^{-1}u : S^{-1}M \longrightarrow S^{-1}N, \qquad m/s \longrightarrow u(m)/s.$$
- Thus, the operation S^{-1} is a functor from the category of A-modules to the category of $S^{-1}A$-modules.

Proposition (Proposition 3.3)

The functor S^{-1} is exact, i.e., if $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M, then $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact at $S^{-1}M$.

Remark

Let M' be a sub-module of M.

- Applying the previous result to $0 \to M' \hookrightarrow M$ produces an injective $S^{-1}A$-module homomorphism $S^{-1}M' \to S^{-1}M$.
- This allows us to identify $S^{-1}M'$ with a sub-module of $S^{-1}M$.

Corollary (Corollary 3.4)

Let N and P be sub-modules of M. Then:

1. $S^{-1}(N+P) = S^{-1}(N) + S^{-1}(P)$.
2. $S^{-1}(N \cap P) = S^{-1}(N) \cap S^{-1}(P)$.
3. *The $S^{-1}A$-modules $S^{-1}(M/N)$ and $S^{-1}M/S^{-1}N$ are isomorphic.*

Remark

By exactness the exact sequence $0 \to N \hookrightarrow M \to M/N \to 0$ gives an exact sequence $0 \to S^{-1}N \hookrightarrow S^{-1}M \to S^{-1}(M/N) \to 0$.

Modules of Fractions

Proposition (Proposition 3.5)

We a canonical A-module isomorphism,

$$S^{-1}A \otimes_A M \simeq S^{-1}M, \qquad (a/s) \otimes m \longrightarrow am/s.$$

Remarks

1. As $(a/s) \times m \to am/s$ is A-bilinear, by the universal property of the tensor product there is a unique A-module homomorphism $f : S^{-1}A \otimes_A M \to S^{-1}M$ such that
$$f\big((a/s) \otimes m\big) = am/s.$$
2. The A-module map $g : S^{-1}M \to S^{-1}A \otimes_A M$, $m/s \to (1/s) \otimes m$ is an inverse of f, since
$$f \circ g(m/s) = f((1/s) \otimes m) = 1m/s = m/s,$$
$$g \circ f\big((a/s) \otimes m\big) = g(am/s) = (1/s) \otimes am = (a/s) \otimes m.$$

Thus, $f : S^{-1}A \otimes_A M \to S^{-1}M$ is an A-module isomorphism.

Modules of Fractions

Corollary (Corollary 3.6)

$S^{-1}A$ is a flat A-module, i.e., the functor $S^{-1}A \otimes -$ preserves exactness of A-module sequences.

Proposition (Proposition 3.7)

If M and N are A-modules, then we have a canonical isomorphism,

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \simeq S^{-1}(M \otimes_A N), \quad (m/s) \otimes (n/t) \longrightarrow (m \otimes n)/st.$$

In particular, for any prime ideal $\mathfrak{p}$ of A we get an $A_{\mathfrak{p}}$-module isomorphism,

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \simeq (M \otimes_A N)_{\mathfrak{p}}.$$

Remarks

The proof is similar to that of Proposition 3.5.

1. Due to the $S^{-1}A$-bilinearity of $(m/s) \times (n/t) \to (m \otimes n)/st$ there is a unique $S^{-1}A$-module homomorphism $f : S^{-1}M \otimes_{S^{-1}A} S^{-1}N \to S^{-1}(M \otimes_A N)$ such that
$$f\left((m/s) \otimes (n/t)\right) = (m \otimes n)/st.$$

2. We also observe that
$$(m/s)\otimes(n/t) = [(1/s)(m/1)]\otimes[(1/t)(n/1)] = \frac{1}{st}\left[(m/1)\otimes(n/1)\right].$$
In particular, we have
$$(m/st) \otimes (n/1) = \frac{1}{st}\left[(m/1) \otimes (n/1)\right] = (m/s) \otimes (n/t).$$

3. Using this it can be checked that $(m \otimes n)/s \to (m/s) \otimes (n/1)$ is an inverse of f, and hence f is an isomorphism.

Local Properties

Definition

1. We say that a property P of a ring A is a *local property* if

$$A \text{ has } P \iff A_{\mathfrak{p}} \text{ has } P \text{ for each prime ideal } \mathfrak{p} \text{ of } A.$$

2. Similarly, a property P of an A-module M is a *local property* if

$$M \text{ has } P \iff M_{\mathfrak{p}} \text{ has } P \text{ for each prime ideal } \mathfrak{p} \text{ of } A.$$

The next propositions provide examples of local properties.

Proposition (Proposition 3.8)

Let M be an A-module. Then TFAE:

1. $M = 0$.
2. $M_{\mathfrak{p}} = 0$ *for each prime ideal* $\mathfrak{p}$ *of* A.
3. $M_{\mathfrak{m}} = 0$ *for each maximal ideal* $\mathfrak{m}$ *of* A.

Proof.

- It is immediate that (i) implies (ii), and (ii) implies (iii).
- Suppose that $M \neq 0$. Let $x \in M \setminus 0$.
- As $\mathrm{Ann}(x) \neq (1)$, there is a maximal ideal $\mathfrak{m} \supseteq \mathrm{Ann}(x)$.
- If $x/1 = 0$ in $M_{\mathfrak{m}}$, then $\exists u \in A \setminus \mathfrak{m}$ such that $ux = 0$
- This means that $u \in \mathrm{Ann}(x) \subseteq \mathfrak{m}$ (impossible).
- Thus $x/1 \neq 0$ in $M_{\mathfrak{m}}$, and hence $M_{\mathfrak{m}} \neq 0$ for some maximal $\mathfrak{m}$.
- By contraposition this shows that (iii) implies (i).

The proof is complete. □

Proposition (Proposition 3.9; 1st Part)

Let $\phi : M \to N$ be an A-module homomorphism. Then TFAE:

1. *ϕ is injective.*
2. *$\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective for every prime ideal $\mathfrak{p}$ of A.*
3. *$\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective for every maximal ideal $\mathfrak{m}$ of A.*

Proof.

- It is immediate that (ii) implies (iii).
- If (i) holds, then $0 \to M \xrightarrow{\phi} N$ is exact, and so $0 \to M_{\mathfrak{p}} \xrightarrow{\phi_{\mathfrak{p}}} N_{\mathfrak{p}}$ is exact, i.e., $\phi_{\mathfrak{p}}$ is injective. Thus (i) $\Rightarrow$ (ii).
- Suppose that $\phi_{\mathfrak{m}}$ is injective for all $\mathfrak{m}$. Set $M' = \ker(\phi)$.
- As $0 \to M' \to M \xrightarrow{\phi} N$ is exact, $0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \xrightarrow{\phi_{\mathfrak{m}}} N_{\mathfrak{m}}$ is exact.
- As $\phi_{\mathfrak{m}}$ is injective, $M'_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$, and hence $M' = 0$ by Prop. 3.8. This shows that (iii) implies (i).

$\square$

Proposition (Proposition 3.9; 2nd Part)

Let $\phi : M \rightarrow N$ be an A-module homomorphism. Then TFAE:

1. *ϕ is surjective.*
2. *$\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is surjective for every prime ideal $\mathfrak{p}$ of A.*
3. *$\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is surjective for every maximal ideal $\mathfrak{m}$ of A.*

Proof.

- It is immediate that (ii) implies (iii).
- If (i) holds, then $M \xrightarrow{\phi} N \rightarrow 0$ is exact, and so $M_{\mathfrak{p}} \xrightarrow{\phi_{\mathfrak{p}}} N_{\mathfrak{p}} \rightarrow 0$ is exact, i.e., $\phi_{\mathfrak{p}}$ is surjective. Thus (i) $\Rightarrow$ (ii).
- Suppose that $\phi_{\mathfrak{m}}$ is surjective for all $\mathfrak{m}$. Set $N' = N/\operatorname{ran}(\phi)$.
- As $M \xrightarrow{\phi} N \rightarrow N' \rightarrow 0$ is exact, $M_{\mathfrak{m}} \xrightarrow{\phi_{\mathfrak{m}}} N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}} \rightarrow 0$ is exact.
- As $\phi_{\mathfrak{m}}$ is surjective, $N'_{\mathfrak{m}} = 0$ for all $\mathfrak{m}$, and hence $\operatorname{ran}(\phi) = N$ by Prop. 3.8, i.e., ϕ is surjective. Thus, (iii) implies (i). ∎

Local Properties

As the following result shows, flatness is a local property.

Proposition (Proposition 3.10)

Let M be an A-module. TFAE:

1. M *is a flat* A*-module.*
2. $M_{\mathfrak{p}}$ *is a flat* $A_{\mathfrak{p}}$*-module for every prime ideal* $\mathfrak{p}$ *of* A*.*
3. $M_{\mathfrak{m}}$ *is a flat* $A_{\mathfrak{m}}$*-module for every maximal ideal* $\mathfrak{m}$ *of* A*.*

Extended and Contracted Ideals in Rings of Fractions

Reminder

Let $f : A \to B$ be a ring homomorphism.

- If $\mathfrak{a}$ is an ideal in A, then its *extension* $\mathfrak{a}^e$ is the ideal in B generated by $f(\mathfrak{a})$. Thus, it consists of all finite sums,

$$\sum f(a_i)b_i, \qquad a_i \in \mathfrak{a}, \quad b_i \in B.$$

- If $\mathfrak{b}$ is an ideal in B, then its *contraction* $\mathfrak{b}^c$ is the ideal $f^{-1}(\mathfrak{b})$ in A.
- If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A, their *ideal quotient* is the ideal

$$(\mathfrak{a} : \mathfrak{b}) = \left\{x \in A; x\mathfrak{b} \subseteq \mathfrak{a}\right\}.$$

When $\mathfrak{b} = (b)$ we write $(\mathfrak{a} : b)$ for $(a : (b))$.

Extended and Contracted Ideals in Rings of Fractions

Facts

Let $f : A \to S^{-1}A$ be the natural homomorphism $a \to a/1$.

- If $\mathfrak{a}$ is an ideal in A, then any $y \in \mathfrak{a}^e$ is of the form

$$y = \sum f(a_i)(b_i/s_i) = \sum (a_i/1)(b_i/s_i) = \sum a_i b_i/s_i,$$

  where $a_i \in \mathfrak{a}$, $b_i \in B$ and $s_i \in S$.
- Set $s = \prod s_i$ and $t_i = \prod_{j \neq i} s_j$, so that $1/s_i = t_i/s$. Then

$$y = \sum (a_i b_i t_i/s) = \left(\sum a_i b_i t_i\right)/s.$$

- Set $a' = \sum a_i b_i t_i$. Then $a' \in \mathfrak{a}$, and so $y = a'/s \in S^{-1}\mathfrak{a}$.
- We then deduce that

$$\mathfrak{a}^e = S^{-1}\mathfrak{a}.$$

Extended and Contracted Ideals in Rings of Fractions

Proposition (Proposition 3.11)

1. *If* $\mathfrak{b}$ *is an ideal in* $S^{-1}A$*, then* $\mathfrak{b} = S^{-1}(\mathfrak{b}^c)$*. Thus, any ideal in* $S^{-1}A$ *is an extended ideal.*
2. *If* $\mathfrak{a}$ *is an ideal in* A*, then* $\mathfrak{a}^{ec} = \cup_{s \in S}(\mathfrak{a} : s)$*. In particular,* $\mathfrak{a}^e = (1)$ *if and only if* $S \cap \mathfrak{a} \neq \emptyset$*.*
3. *An ideal* $\mathfrak{a}$ *in* A *is a contracted ideal if and only if no element of* S *is a zero-divisor in* $A/\mathfrak{a}$*.*
4. *We have a one-to-one correspondance* $\mathfrak{p} \leftrightarrow S^{-1}\mathfrak{p}$ *between prime ideal in* $S^{-1}A$ *and prime ideals in* A *that don't meet* S*.*
5. *The operation* S^{-1} *on ideals commutes with taking finite sums, products, intersections, and radicals.*

Proof of Proposition 3.11 (Part 1).

- Let $\mathfrak{b}$ be an ideal in $S^{-1}A$, and let $x/s \in \mathfrak{b}$.
- $x/1 = (x/s)(s/1) \in \mathfrak{b}$, and so $x \in \mathfrak{b}^c$.
- Then $x/s = (x/1)(1/s) \in (\mathfrak{b}^c)^e$, and hence $\mathfrak{b} \subseteq \mathfrak{b}^{ce}$.
- As $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$, we deduce that $\mathfrak{b} = \mathfrak{b}^{ce} = S^{-1}(\mathfrak{b}^c)$.

This proves Part 1. □

Proof of Proposition 3.11 (Part 2).

- Let $\mathfrak{a}$ be an ideal in A. We have

$$x \in \mathfrak{a}^{ec} \Leftrightarrow x/1 \in \mathfrak{a}^{e} = S^{-1}(\mathfrak{a}) \Leftrightarrow x/1 = a/s \text{ with } (a,s) \in \mathfrak{a} \times S.$$

- If $x/1 = a/s$ with $(a,s) \in \mathfrak{a} \times S$, then $\exists t \in S$ such that $t(xs - a \cdot 1) = 0$, i.e., $x(st) = a \in \mathfrak{a}$, and hence $x \in (\mathfrak{a} : st)$.
- If $x \in (\mathfrak{a} : s)$, then $xs \in \mathfrak{a}$, and so $x/1 = xs/s \in S^{-1}(\mathfrak{a})$.
- This shows that $\mathfrak{a}^{ec} = \cup_{s \in S}(\mathfrak{a} : s)$.
- Next, if $\mathfrak{a}^{e} = (1)$, then $\mathfrak{a}^{ec} = (1) = A$, and so $1 \in (\mathfrak{a} : s)$ for some $s \in S$, i.e., $s = 1 \cdot s \in \mathfrak{a}$.
- Conversely, if $s \in S \cap \mathfrak{a} \neq \emptyset$, then $1 = s/s \in S^{-1}(\mathfrak{a}) = \mathfrak{a}^{e}$, and hence $\mathfrak{a}^{e} = (1)$.

This proves Part 2. $\square$

Proof of Proposition 3.11 (Part 3).

- An ideal $\mathfrak{a}$ in A is a contracted ideal if and only if $\mathfrak{a}^{ec} \subseteq \mathfrak{a}$ (since we always have $\mathfrak{a}^{ec} \supseteq \mathfrak{a}$).
- As $\mathfrak{a}^{ec} = \cup_{s \in S}(\mathfrak{a} : s)$, we get

$$\begin{aligned} \mathfrak{a}^{ec} \subseteq \mathfrak{a} &\Leftrightarrow (\mathfrak{a} : s) \subseteq \mathfrak{a} \ \forall s \in S \\ &\Leftrightarrow \big(xs \in \mathfrak{a} \Rightarrow x \in \mathfrak{a}\big) \ \forall s \in S \\ &\Leftrightarrow \big[(xs = 0 \text{ in } A/\mathfrak{a}) \Rightarrow (x = 0 \text{ in } A/\mathfrak{a})\big] \ \forall s \in S \end{aligned}$$

- Thus, $\mathfrak{a}$ in A is a contracted ideal if and only if no $s \in S$ is a zero-divisor in $A/\mathfrak{a}$.

This proves Part 3. □

Proof of Proposition 3.11 (Part 4).

- Let $\mathfrak{q}$ be a prime ideal in $S^{-1}A$. Then $\mathfrak{q} = S^{-1}(\mathfrak{q}^c)$.
- Here $(\mathfrak{q}^c)^e = \mathfrak{q} \neq (1)$, so by Part 2 $\mathfrak{q}^c \cap S = \emptyset$.
- $\mathfrak{q}^c$ is a prime ideal, since

$$\begin{aligned} xy \in \mathfrak{q}^c &\iff (xy)/1 \in \mathfrak{q} \\ &\iff (x/1)(y/1) \in \mathfrak{q} \\ &\iff \big(x/1 \in \mathfrak{q} \text{ or } y/1 \in \mathfrak{q}\big) \\ &\iff \big(x \in \mathfrak{q}^c \text{ or } y \in \mathfrak{q}^c\big). \end{aligned}$$

- Thus $\mathfrak{q} = S^{-1}(\mathfrak{p})$, where $\mathfrak{p} = \mathfrak{q}^c$ is a prime ideal that does not meet S.

Remark

The fact that the contraction of a prime ideal is prime is true for any homomorphism.

Proof of Proposition 3.11 (Part 4, continued).

- Let $\mathfrak{p}$ be a prime ideal s.t. $\mathfrak{p} \cap S = \emptyset$.
- As $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is an integral domain.
- Let $\overline{S}$ be the image of S in $A/\mathfrak{p}$, then $\overline{S}^{-1}(A/\mathfrak{p})$ is contained in the fraction field $\operatorname{Frac}(A/\mathfrak{p})$, and hence is either zero or an integral domain.
- As $S^{-1}A/S^{-1}\mathfrak{p} \simeq \overline{S}^{-1}(A/\mathfrak{p})$, we see that $S^{-1}A/S^{-1}\mathfrak{p}$ is either zero or an integral domain.
- That is, either $S^{-1}\mathfrak{p} = (1)$ or $S^{-1}\mathfrak{p}$ is prime.
- By Part 2 $S^{-1}\mathfrak{p} = \mathfrak{p}^e \neq (1)$ since $\mathfrak{p} \cap S = \emptyset$, so $S^{-1}\mathfrak{p}$ is prime.

This completes the proof of Part 4. $\square$

Proof of Proposition 3.1 (Part 5, intersections).

- If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in A, then $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) \subseteq S^{-1}(\mathfrak{a}) \cap S^{-1}(\mathfrak{b})$.
- Let $a/s \in S^{-1}(\mathfrak{a})$ with $a \in \mathfrak{a}$ and $s \in S$. If $a/s \in S^{-1}\mathfrak{b}$, then $\exists b \in \mathfrak{b}$ and $t \in S$ such that $a/s = b/t$.
- This means there is $u \in S$ such that $(at - bs)u = 0$.
- Thus, $atu = bsu \in \mathfrak{a} \cap \mathfrak{b}$, and we have

$$a/s = (a/s)(tu/tu) = (atu)/(stu) \in S^{-1}(\mathfrak{a} \cap \mathfrak{b}).$$

- It follows that $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) = S^{-1}(\mathfrak{a}) \cap S^{-1}(\mathfrak{b})$.

This shows that S^{-1} commutes with intersection. $\square$

Reminder

- The *nilradical* of A is the ideal

$$\mathfrak{N}(A) = \{x \in A;\ x^n = 0 \text{ for some } n \geq 1\}.$$

- Equivalently, $\mathfrak{N}(A)$ is the intersection of all the prime ideals of A (see Proposition 1.8).

Corollary (Corollary 3.12)

The nilradical of $S^{-1}A$ is precisely $S^{-1}(\mathfrak{N}(A))$.

Proof.

- By Part 4 of Proposition 3.11,

$$\begin{aligned}\mathfrak{N}\left(S^{-1}A\right) &= \bigcap\left\{\mathfrak{q};\ \mathfrak{q} \text{ prime in } S^{-1}A\right\} \\ &= \bigcap\left\{S^{-1}\mathfrak{p};\ \mathfrak{p} \text{ prime in } A \ \&\ \mathfrak{p}\cap S=\emptyset\right\}.\end{aligned}$$

- By Part 2 if $\mathfrak{p}\cap S\neq\emptyset$, then $S^{-1}\mathfrak{p}=(1)=S^{-1}A$. Thus,

$$\mathfrak{N}\left(S^{-1}A\right)=\bigcap\left\{S^{-1}\mathfrak{p};\ \mathfrak{p} \text{ prime in } A\right\}.$$

- As S^{-1} commutes with intersection, we get

$$\mathfrak{N}\left(S^{-1}A\right)=S^{-1}\left(\bigcap\left\{\mathfrak{p};\ \mathfrak{p} \text{ prime in } A\right\}\right)=S^{-1}\left(\mathfrak{N}(A)\right).$$

The result is proved. $\square$

Extended and Contracted Ideals in Rings of Fractions

Corollary (Corollary 3.13)

Let $\mathfrak{p}$ be a prime ideal of A. Then the prime ideals of the local ring $A_{\mathfrak{p}}$ are in one-to-one correspondence with the prime ideals of A that are contained in $\mathfrak{p}$.

Remarks

- By this corollary, passing from A to $A_{\mathfrak{p}}$ cuts out all prime ideals except those contained in $\mathfrak{p}$.
- By Proposition 1.1, passing from A to $A/\mathfrak{p}$ cuts out all prime ideals except those containing $\mathfrak{p}$.
- Thus, if $\mathfrak{q}$ is a prime ideal contained in $\mathfrak{p}$, then passing to $(A_{\mathfrak{p}})/\mathfrak{q} \simeq (A/\mathfrak{q})_{\mathfrak{p}}$ restricts ourselves to those prime ideals between $\mathfrak{q}$ and $\mathfrak{p}$.
- For $\mathfrak{q} = \mathfrak{p}$ we obtain the *residual field* of $\mathfrak{p}$. It can be realized either as the fraction field of the integral domain $A/\mathfrak{p}$, or as the residue field of the local ring $A_{\mathfrak{p}}$.

Reminder

- If N and P are sub-modules of an A-module M, then

$$(N : P) = \{x \in A; xP \subseteq N\}.$$

  This is an ideal of A.

- The *annihilator* of M, denoted $\mathrm{Ann}(M)$, is the ideal $(0 : M)$. That is,

$$\mathrm{Ann}(M) = \{x \in A; xM = 0\}.$$

- By Exercise 2.2 we have

$$\mathrm{Ann}(N + P) = \mathrm{Ann}(N) \cap \mathrm{Ann}(P),$$
$$(N : P) = \mathrm{Ann}\left((N + P)/N\right).$$

Extended and Contracted Ideals in Rings of Fractions

Proposition (Proposition 3.14)

Let M be a finitely generated A-module. Then

$$S^{-1}(\mathrm{Ann}(M)) = \mathrm{Ann}\left(S^{-1}M\right).$$

Remark

- If M is single generated, i.e., $M = Ax$. Then we have an exact sequence of A-modules,

$$0 \longrightarrow \mathrm{Ann}(M) \longrightarrow A \xrightarrow{a \to ax} M \longrightarrow 0.$$

- By exactness of the functor S this gives an exact sequence of $S^{-1}A$-modules,

$$0 \longrightarrow S^{-1}(\mathrm{Ann}(M)) \longrightarrow S^{-1}A \xrightarrow{a/s \to ax/s} S^{-1}M \longrightarrow 0,$$

which shows that $\mathrm{Ann}(S^{-1}M) = S^{-1}(\mathrm{Ann}(M))$.

Extended and Contracted Ideals in Rings of Fractions

Corollary (Corollary 3.15)

If N and P are sub-modules of M with P finitely generated, then

$$S^{-1}(N:P) = \left(S^{-1}N : S^{-1}P\right).$$

Remarks

- The fact that P is finitely generated implies that $(N+P)/N$ is finitely generated as well.
- As $(N:P) = \mathrm{Ann}((N+P)/N)$ by applying the previous proposition we get

$$S^{-1}(N:P) = \mathrm{Ann}\left[S^{-1}\left((N+P)/N\right)\right].$$

- We have

$$S^{-1}((N+P)/P) = S^{-1}(N+P)/S^{-1}N = (S^{-1}N+S^{-1}P)/S^{-1}N.$$

- Thus,

$$S^{-1}(N:P) = \mathrm{Ann}\left[\left(S^{-1}N+S^{-1}P\right)/S^{-1}N\right] = \left(S^{-1}N : S^{-1}P\right).$$

Proposition (Proposition 3.16)

Let $g : A \to B$ be a ring homomorphism, and $\mathfrak{p}$ a prime ideal in A. Then TFAE:

(i) $\mathfrak{p}$ *is the contraction of a prime ideal in* B.

(ii) $\mathfrak{p}^{ec} = \mathfrak{p}$.

Proof of Proposition 3.16.

- If $\mathfrak{p} = \mathfrak{q}^c$, then $\mathfrak{p}^{ec} = \mathfrak{q}^{cec} = \mathfrak{q}^c = \mathfrak{p}$ by Proposition 1.17.
- Suppose that $\mathfrak{p}^{ec} = \mathfrak{p}$. Let S be the image of $A \setminus \mathfrak{p}$ in B.
- As $\mathfrak{p}^e \cap S = \emptyset$, we see that $S^{-1}(\mathfrak{p}^e) \subsetneq S^{-1}B$.
- Thus, there is a maximal ideal $\mathfrak{m}$ in $S^{-1}B$ containing $S^{-1}(\mathfrak{p}^e)$.
- Let $\mathfrak{q}$ be the contraction of $\mathfrak{m}$ in B. This is a prime ideal such that $\mathfrak{q} \cap S = \emptyset$. Thus,
$$\mathfrak{q}^c \cap (A \setminus \mathfrak{p}) \subseteq g^{-1}(\mathfrak{q}) \cap g^{-1}(S) = g^{-1}(\mathfrak{q} \cap S) = \emptyset.$$
That is, $\mathfrak{q}^c \subseteq \mathfrak{p}$.
- As $\mathfrak{q}$ contains the contraction of $S^{-1}(p^e)$ in B, it contains $\mathfrak{p}^e$, and hence $\mathfrak{q}^c \supseteq \mathfrak{p}^{ec} = \mathfrak{p}$.
- Thus, $\mathfrak{p} = \mathfrak{q}^c$.

The proof is complete. $\square$