Commutative Algebra
Chapter 2: Modules

Sichuan University, Fall 2023

Modules and Module Homomorphisms

Notation

Throughout this chapter A is a *ring* (which is commutative and has an identity element).

Definition (Modules)

An A-*module* is an Abelian group M on which A acts linearly. That is, there is a map $A \times M \ni (a, x) \to ax \in M$ such that

$$\begin{aligned} a(x+y) &= ax + ay, \\ (a+b)x &= ax + bx, \\ (ab)x &= a(bx), \\ 1x &= x, \qquad a, b \in A, \; x, y \in M. \end{aligned}$$

Remark

Equivalently, M is an Abelian group together with a ring homomorphism $A \to E(M)$, where $E(M)$ is the ring of homomorphisms of M.

Examples

1. Any ideal $\mathfrak{a}$ of A is an A-module. In particular, A itself is an A-module.
2. If A is a field k, then an A-module is exactly a vector space of over k.
3. If $A = \mathbb{Z}$, then a $\mathbb{Z}$-module is just an Abelian group.
4. If $A = k[x]$, k field, then an A-module is a k-vector space together with a linear transformation (which corresponds to the action of x).
5. If A is the group ring kG of a group G over a field k, then an A-module is exactly a k-representation of G, i.e., a k-vector space together with a group morphism $G \to \mathrm{End}(V)$.

Modules and Module Homomorphisms

Definition (Module Homorphisms)

Given modules M and M', an map $f : M \to M'$ is an *A-module homomorphism* (or is *A-linear*) if

$$f(x+y) = f(x) + f(y),$$
$$f(ax) = af(x), \qquad a \in A, \ x, y \in M.$$

Remark

In other words, f is a homomorphism of Abelian groups that commutes with the action of A.

Example

If A is a field k, then a k-module homomorphism is nothing but a linear transformation between k-vector spaces.

Modules and Module Homomorphisms

Definition

The set of all A-module homomorphisms $f : M \to M'$ is denoted $\mathrm{Hom}_A(M, N)$ (or simply $\mathrm{Hom}(M, N)$ when there is no ambiguity on the ground ring).

Fact

$\mathrm{Hom}_A(M, N)$ is an A-module. Given $f, g \in \mathrm{Hom}_A(M, N)$ and $a \in A$ we define $f + g$ and af by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \\ (af)(x) &= af(x), \qquad x \in M. \end{aligned}$$

Modules and Module Homomorphisms

Remarks

1. The composition of A-module homomorphisms is again an A-module homomorphism.
2. Given homomorphisms $u : M \to M'$ and $v : N \to N'$ we get maps,

$$\tilde{u} : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N), \quad \tilde{u}(f) = f \circ u,$$
$$\tilde{v} : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N'), \quad \tilde{v}(f) = v \circ f.$$

   These maps are A-module homomorphisms.
3. For any module M we have a natural isomorphism,

$$\mathrm{Hom}_A(A, M) \simeq M.$$

   Any $f \in \mathrm{Hom}_A(A, M)$ is uniquely determined by $f(1)$.

Submodules and Quotient Modules

Definition (Submodules)

A *submodule* M' of a module M is a subgroup that is closed under the action of A.

Fact

If M' is a submodule of M, then the Abelian group M/M' inherits an A-module structure given by

$$a(x + M') = ax + M', \qquad x \in M.$$

Definition (Quotient Modules)

M/M' is called the *quotient of M by M'*.

Facts

1. The canonical map $M \to M/M'$ is an A-module homomorphism.
2. There is a one-to-one correspondance between submodules of M that contains M' and submodules of M/M'.

Submodules and Quotient Modules

Definition

Let $f : M \to N$ be an A-module homomorphism.

1. The *kernel* of f is
$$\ker(f) = \{x \in M;\ f(x) = 0\}.$$
This is a submodule of M.
2. The *image* of f is
$$\operatorname{im}(f) = f(M).$$
This is a submodule of N.
3. The *cokernel* of f is
$$\operatorname{coker}(f) = N/\operatorname{im}(f).$$
This is a quotient module of N.

Facts

Let $f : M \to N$ be an A-module homomorphism and M' a submodule of M such that $M' \subseteq \ker(f)$.

- f gives rise to a homomorphism $\overline{f} : M/M' \to N$ defined by

  $$\overline{f}(\overline{x}) = f(x),$$

  where $\overline{x} \in M/M'$ is the image of $x \in M$.
- The kernel of $\overline{f}$ is $\ker(f)/M'$.

Definition (Induced Homomorphisms)

The homomorphism $\overline{f}$ is said to be *induced* by f.

Remark

For $M' = \ker(f)$ we get an isomorphism,

$$M/\ker(f) \simeq \text{im}(f).$$

Operations on Modules

Definition

Let M be an A-module, and $(M_i)_{i\in I}$ be a family of sub-modules of M. The *sum* $\sum M_i$ consists of all finite sums $\sum x_i$, where $x_i \in M_i$.

Remark

$\sum M_i$ is the smallest sub-module that contains all the M_i.

Facts

1. The intersection $\cap M_i$ is again a submodule of M.
2. The submodules form a lattice with respect to inclusion.

Proposition (Proposition 2.1)

1. *If $L \supseteq M \supseteq N$ are A-modules, then*

$$(L/M)/(M/N) \simeq L/N.$$

2. *If M_1 and M_2 are submodules of M, then*

$$(M_1 + M_2)/M_1 \simeq M_2/(M_1 \cap M_2).$$

Direct Sum and Product

Definition

If $\mathfrak{a}$ is an ideal of A and M is an A-module, then $\mathfrak{a}M$ consists of all finite sums $\sum a_i x_i$ with $a_i \in \mathfrak{a}$ and $x_i \in M$.

Remarks

1. $\mathfrak{a}M$ is a submodule of M.
2. In general we cannot define the product of two submodules.

Definition

1. If N and P are submodules of M, then $(N : P)$ is the set of all $a \in A$ such that $aP \subseteq N$.
2. $(0 : M)$ is called the *annihilator* of M and is denoted by $\mathrm{Ann}(M)$.

Remarks

1. $(N : P)$ is an ideal of A.
2. $\mathrm{Ann}(M)$ consists of all $a \in A$ such that $aM = 0$.

Operations on Modules

Fact

If $\mathfrak{a}$ is an ideal contained in $\mathrm{Ann}(M)$, then we may regard M as an $A/\mathfrak{a}$-module as follows: if $m \in M$ and $\overline{x} \in A/\mathfrak{a}$ is the class of $x \in A$, then

$$\overline{x}m = xm.$$

This definition makes sense since $\mathfrak{a}M = 0$.

Definition (Faithful Modules)

We say that an A-module M is *faithful* if $\mathrm{Ann}(M) = 0$.

Remark

If $\mathfrak{a} = \mathrm{Ann}(M)$, then M is always faithful as an $A/\mathfrak{a}$-module.

Exercise (Exercise 2.2)

(i) $\text{Ann}(M + N) = \text{Ann}(M) \cap \text{Ann}(N)$.

(ii) $(N : P) = \text{Ann}\left((N + P)/N\right)$.

Operations on Modules

Fact

If $x \in M$, the set of all multiples ax with $a \in A$ is a submodule of M denoted by Ax or (x).

Definition

- If $M = \sum Ax_i$, then we say that the x_i form a *set of generators* of M.
- We say that M is *finitely generated* if it admits a finite set of generators.

Remark

- That the x_i form a set of generators of M means that every $x \in M$ is a finite linear combination $\sum a_i x_i$ with $a_i \in A$.
- This linear combination need not be unique.

Direct Sum and Product

Definition (Direct Sum)

- If M and N are A-modules, their *direct sum* $M \oplus N$ consist of all pairs (x, y) with $x \in M$ and $y \in N$.
- This is an A-module with respect to the following addition and scalar multiplication,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$
$$a(x, y) = (ax, ay).$$

Direct Sum and Product

Definition (Direct Sum and Direct Product)

Let $(M_i)_{i \in I}$ be a family of A-modules.

1. The *direct sum* $\bigoplus M_i$ consists of all families $(x_i)_{i \in I}$ where all but finitely many of the x_i are zero.
2. The *direct product* $\prod M_i$ consists of all families $(x_i)_{i \in I}$.

Remarks

1. The direct sum $\bigoplus M_i$ and the direct product $\prod M_i$ are both A-modules.
2. They agree when the index set I is finite.

Direct Sum and Product

Facts

Suppose that the ring A is a direct product $\prod_{i=1}^{n} A_i$.

1. Let $\mathfrak{a}_i$ be the set of all elements of a of the form
$$(0, \dots, 0, a_i, 0, \dots, 0), \qquad a_i \in A.$$
This is an ideal of A.
2. The ring A, considered as an A-module, agrees with the direct sum $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$.

Direct Sum and Product

Facts

Conversely, suppose we have a module decomposition,

$$A = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n,$$

where $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are ideals.

1. We have

$$A \simeq \prod_{i=1}^{n} (A/\mathfrak{b}_i), \qquad \text{where } \mathfrak{b}_i = \bigoplus_{j \neq i} \mathfrak{a}_j.$$

2. Each ideal $\mathfrak{a}_i$ is a ring isomorphic to $A/\mathfrak{b}_i$.
3. The identity element e_i of $\mathfrak{a}_i$ is an idempotent in A and $\mathfrak{a}_i = (e_i)$.

Finitely Generated Modules

Definition (Free Modules)

A *free A-module* is an A-module of the form $\oplus_{i \in I} M_i$, where $M_i \simeq A$,

Example

- The direct sum $A^n = A \oplus \cdots \oplus A$ (n summands) is a free module.
- By convention A^0 is the zero module, denoted by 0.

Fact

Any finitely generated free module is isomorphic to A^n for some n.

Proposition (Proposition 2.3)

An A-module M is finitely generated if and only if it a quotient of A^n for some $n \geq 1$.

Finitely Generated Modules

Proposition (Cayley-Hamilton Theorem; Proposition 2.4)

Suppose that M is a finitely generated A-module and $\mathfrak{a}$ is ideal of A. Let $\phi : M \to M$ be an A-module endomorphism such that $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form,

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0, \qquad a_i \in \mathfrak{a}.$$

Remarks

- We identify A with its image in $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$.
- The above equality holds in $\mathrm{End}_A(M)$, which is an A-module.

Proof of Proposition 2.4.

- Let $x_1, \ldots, x_n$ be generators of M. Then:

$$(*) \qquad \phi x_j = a_{1j}x_1 + \cdots + a_{nj}x_n, \quad a_{ij} \in \mathfrak{a}.$$

- Let B be the sub-ring of $\mathrm{End}_A(M)$ generated by ϕ and A. This is a commutative ring.
- Set $a = [a_{ij}] \in M_n(\mathfrak{a})$ and $b = \phi I_n - a \in M_n(B)$. Note that $M_n(B)$ acts on M^n. Then $(*)$ means that

$$bx = 0 \qquad \text{with } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Proof of Proposition 2.4 (continued).

- Let c be the cofactor matrix of b. As B is a commutative ring, we have
$$cb = \det(b)I_n.$$
- As $bx = 0$, we get
$$0 = cbx = \det(b)x = \begin{bmatrix} \det(b)x_1 \\ \vdots \\ \det(b)x_n \end{bmatrix}.$$
- As $x_1, \ldots, x_n$ generate M, this gives $\det(b) = 0$ in $\mathrm{End}_A(M)$.
- Here $b = \phi I_n - a$ with $a \in M_n(\mathfrak{a})$. Expanding the equation $\det(\phi I_n - a) = 0$ shows there are $a_1, \ldots, a_n$ in $\mathfrak{a}$ such that
$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0.$$
- Equivalently, if $P(\lambda) = \det(\lambda I_n - a)$ is the characteristic polynomial of $a = [a_{ij}]$, then $P(\phi) = 0$.

□

Finitely Generated Modules

Corollary (Corollary 2.5)

Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal of A such that $\mathfrak{a}M = M$. Then there is $x \equiv 1 \bmod \mathfrak{a}$ such that $xM = 0$.

Proof.

- Apply Prop. 2.4 to $\phi = \mathrm{id}_M$. There are $a_1, \ldots, a_n$ in $\mathfrak{a}$ such that
$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0.$$
That is,
$$(1 + a_1 + \cdots + a_n)\,\mathrm{id}_M = 0.$$
- Thus, if we set $x = 1 + a_1 + \cdots + a_n$, then $x \equiv 1 \bmod \mathfrak{a}$, and
$$xy = 0 \qquad \forall y \in M.$$
That is, $xM = 0$.

The proof is complete. □

Reminder (Proposition 1.9)

If $\mathfrak{R}$ the Jacobson radical of A, then $x \in \mathfrak{R}$ if and only if $1 - xy$ is unit for all $y \in A$.

Proposition (Nakayama's Lemma; Proposition 2.6)

Let M be a finitely generated A module and $\mathfrak{a}$ an ideal of A contained in $\mathfrak{R}$. Then $\mathfrak{a}M = M$ implies that $M = 0$.

Proof.

- By Corollary 2.5 there is $x \equiv 1 \bmod \mathfrak{R}$ such that $xM = 0$.
- As $1 - x \in \mathfrak{R}$, by Proposition 1.9 $x = 1 - (1 - x)1$ is a unit.
- Thus,

$$M = x^{-1}(xM) = 0.$$

The proof is complete. □

Corollary (Corollary 2.7)

Let M be a finitely generated A module, N a submodule of A, and $\mathfrak{a}$ an ideal of A contained in $\mathfrak{R}$. Then $M = \mathfrak{a}M + N \Rightarrow M = N$.

Proof.

- If $M = \mathfrak{a}M + N$, then
$$\mathfrak{a}(M/N) = (\mathfrak{a}M + N)/N = M/N.$$
- Nakayama's lemma then implies that $M/N = 0$, i.e., $M = N$.

The result is proved. □

Finitely Generated Modules

Fact

Let A be a local ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field. Let M be a finitely generated A-module M. Then:

- The quotient module $V = M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, and hence this is an $A/\mathfrak{m}$-module, i.e., a vector space over k.
- This vector space has finite dimension.

Proposition (Proposition 2.8)

Let $x_1, \ldots, x_n$ be elements in M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then $x_1, \ldots, x_n$ generate M.

Proof.

- Let N be the module generated by $x_1, \ldots, x_n$, and let $\phi : M \to M/\mathfrak{m}M$ be the canonical homomorphism.
- As $\{\phi(x_1), \ldots, \phi(x_n)\}$ is a basis of $M/\mathfrak{m}M$, we see that $\phi(N) = M/\mathfrak{m}M$.
- Thus,

$$M = \phi^{-1}(M/\mathfrak{m}M) = \phi^{-1}(\phi(N)) = N + \mathfrak{m}M.$$

- Corollary 2.7 then implies that $M = N$, i.e., $x_1, \ldots, x_n$ generate M.

The result is proved. □

Exact Sequences

Definition (Exact Sequences)

A sequence of A-modules and A-homomorphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is said to be *exact at* M_i if $\operatorname{im}(f_i) = \ker(f_{i+1})$. It is called an *exact sequence* if it is exact at each M_i.

Examples

1. A sequence $0 \to M' \xrightarrow{f} M$ is exact if and only if f is injective.
2. A sequence $M \xrightarrow{g} M'' \to 0$ is exact if and only if g is surjective.
3. A sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is exact if and only if f is injective, g is surjective, and g induces an isomorphism of $\operatorname{coker}(f) = M/f(M')$ onto M''. Such a sequence is called a *short exact sequence*.

Remark

- Any long exact sequence,

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

  can be split up into short exact sequences.

- If we set $N_i = \mathrm{im}(f_i) = \ker(f_{i+1})$, then, for each i, we have a short exact sequence,

$$0 \longrightarrow N_i \longrightarrow M_i \xrightarrow{f_i} N_{i+1} \longrightarrow 0.$$

Proposition (Proposition 2.9; see Carlson)

1. *A sequence of the form*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact if and only if, for every A*-module* N*, the sequence*

$$0 \longrightarrow \mathrm{Hom}(M'', N) \xrightarrow{\tilde{v}} \mathrm{Hom}(M, N) \xrightarrow{\tilde{u}} \mathrm{Hom}(M', N)$$

is exact.

2. *A sequence of the form*

$$0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N''$$

is exact if and only if, for every A*-module* M*, the sequence*

$$0 \longrightarrow \mathrm{Hom}(M, N') \xrightarrow{\tilde{u}} \mathrm{Hom}(M, N) \xrightarrow{\tilde{v}} \mathrm{Hom}(M, N'')$$

is exact.

Proposition (Snake Lemma, Proposition 2.10; see Carlson)

Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of A-modules and A-homomorphisms with exact rows. Then there exists an exact sequence,

$$0 \longrightarrow \ker(f') \xrightarrow{\overline{u}} \ker(f) \xrightarrow{\overline{v}} \ker(f'') \xrightarrow{d}$$
$$\operatorname{coker}(f') \xrightarrow{\overline{u}'} \operatorname{coker}(f) \xrightarrow{\overline{v}'} \operatorname{coker}(f'') \longrightarrow 0,$$

where $\overline{u}, \overline{v}$ are restrictions of u, v, and $\overline{u}', \overline{v}'$ are induced by u', v'.

Remark

The map $d : \ker(f'') \to \operatorname{coker}(f')$ is called *boundary homomorphism*. It is constructed as follows:

- If $x'' \in \ker(f'')$, we have $x'' = v(x)$ for some $x \in M$ (since v is surjective).
- We have $v'(f(x)) = f''(v(x)) = f''(x'') = 0$, and so $f(x) \in \ker(v') = \operatorname{ran}(u')$.
- As u' is injective, there is a unique $y' \in N'$ such that $f(x) = u'(y')$.
- We define dx'' to be the class of y' in $\operatorname{coker}(f') = N'/\operatorname{im}(f')$.
- It can be shown that the class of y' does not depend on the choice of x, and dx'' is well defined.

Exact Sequences

Definition

Let $\mathscr{C}$ be a class of A-modules. A function $\lambda : \mathscr{C} \to \mathbb{Z}$ is *additive* if, for every short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{C}$, we have

$$\lambda(M') - \lambda(M) + \lambda(M'') = 0.$$

Example

Let A be a field k and $\mathscr{C}$ be the class of finite-dimension vector spaces over k. Then the function $V \to \dim V$ is additive.

Proposition (Proposition 2.11)

Let $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ *be an exact sequence in* $\mathscr{C}$*. Then, for any additive function* λ *on* $\mathscr{C}$*, we have*

$$\sum_{0 \leq i \leq n} (-1)^i \lambda(M_i) = 0.$$

Tensor Product of Modules

Definition (A-Bilinear Maps)

Given A-modules M, N, and P, an A-*bilinear map* is any map $f : M \times N \to P$ such that

(i) For every $x \in M$, the map $N \ni y \to f(x, y) \in P$ is A-linear.

(ii) For every $y \in N$, the map $M \ni x \to f(x, y) \in P$ is A-linear.

Fact

Given A-modules M and N, their *tensor product* $M \otimes_A N$ is an A-module such that A-bilinear maps $M \times N \to P$ are in one-to-one correspondance with A-linear maps $M \otimes_A N \to P$.

Proposition (Proposition 2.12)

Let M, N be A-modules.

1. *There exist an A-module $M \otimes_A N$ and an A-bilinear map $\otimes : M \times N \to M \otimes_A N$ satisfying the following universal property:*

   For any A-module P and A-bilinear map $f : M \times N \to P$ there is a unique A-linear map $f' : M \otimes_A N \to P$ such that

   $$f(x, y) = f'(x \otimes y) \qquad \text{for all } x \in M \text{ and } y \in N.$$

2. *If $(M \otimes' N, \otimes')$ is another pair satisfying the above universal property, then there is a unique isomorphism $j : M \otimes_A N \to M \otimes' N$ such that*

   $$j(x \otimes y) = x \otimes' y \qquad \text{for all } x \in M \text{ and } y \in N.$$

Remark

The A-module $M \otimes_A N$ is constructed as follows:

- Let C be the free A-module $\bigoplus_{(x,y)\in M\times N} A$ generated by all pairs $(x, y) \in M \times N$. It consists of finite formal linear combinations $\sum a_j(x_j, y_j)$ with $a_j \in A$ and $(x_j, y_j) \in M \times N$.
- Let D be the submodule generated by elements of the form,

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),$$
$$(ax, y) - a(x, y), \qquad (x, ay) - a(x, y).$$

- The A-module $M \otimes_A N$ is the quotient module C/D. If $(x, y) \in M \times N$, we denote by $x \otimes y$ the class of (x, y) in $M \otimes_A N$. From the definition we have

$$(x + x') \otimes y = x \otimes y - x' \otimes y, \quad x \otimes (y + y') = x \otimes y - x \otimes y',$$
$$(ax) \otimes y = a(x \otimes y), \qquad x \otimes (ay) = a(x \otimes y).$$

That is, $\otimes : M \times N \to M \otimes_A N$ is an A-bilinear map.

Remarks

1. We often denote $M \otimes_A N$ by $M \otimes N$ when the ring A is understood from context.
2. In practice, we will not need the construction of the tensor product. What is essential is to keep in mind its universal property.
3. If $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are generator sets of M and N, respectively, then the elements $x_i \otimes y_j$ generate $M \otimes N$. In particular, if M and N are finitely generated, then $M \otimes N$ is finitely generated as well.

Remark

- Let M' and N' are submodules of M and N, respectively, If $x \in M'$ and $y \in N'$, then it may happen that $x \otimes y$ is zero as an element of $M \otimes N$, but is not zero as an element of $M' \otimes N'$.
- Take $A = M = \mathbb{Z}$, $N = N' = \mathbb{Z}/2\mathbb{Z}$ and $M' = 2\mathbb{Z}$. Let x be the non-zero element of $\mathbb{Z}/2\mathbb{Z}$. Then $2 \otimes x$ is not zero in $M' \otimes N'$ since it generates $M' \otimes N'$. However, it is zero in $M \otimes N$, since we have

$$2 \otimes x = 2(1 \otimes x) = 1 \otimes (2x) = 0.$$

- Nevertheless, we have the following result:

Corollary (Corollary 2.13)

Let $x_i \in M$ and $y_i \in N$ be such that $\sum x_i \otimes y_i = 0$ in $M \otimes N$. Then there are finitely generated submodules M_0 of M and N_0 of N such that $\sum x_i \otimes y_i = 0$ in $M_0 \otimes N_0$.

Tensor Product of Modules

Remark

- We can also define multi-tensor products $M_1 \otimes \cdots \otimes M_r$ by using multilinear maps instead of
- A map $M_1 \times \cdots \times M_r \to P$ is multilinear if it is linear with respect to each argument.

Proposition (Proposition 2.12*; see Carlson)

1. *There exist an A-module $M_1 \otimes \cdots \otimes M_r$ and an multilinear map $\otimes \cdots \otimes : M_1 \times \cdots \times M_r \to M_1 \otimes \cdots \otimes M_r$ satisfying the following universal property:*

   For any A-module P and multilinear map $f : M_1 \times \cdots \times M_r \to P$ there is a unique A-linear map $f' : M_1 \otimes \cdots \otimes M_r \to P$ such that

   $$f(x_1, \cdots, x_r) = f'(x_1 \otimes \cdots \otimes x_r) \qquad \text{for all } x_i \in M_i.$$

2. *The pair $(M_1 \otimes \cdots \otimes M_r, \otimes \cdots \otimes)$ is unique up to isomorphism.*

Tensor Product of Modules

Proposition (Proposition 2.14; see Atiyah-MacDonald and Carlson)

Let M, N, P be A-modules. Then we have canonical isomorphisms:

(i) $M \otimes N \simeq N \otimes M$, *where* $x \otimes y \to y \otimes x$.

(ii) $(M \otimes N) \otimes P \simeq M \otimes (N \otimes P) \simeq M \otimes N \otimes P$, *where*

$$(x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z) \longrightarrow x \otimes y \otimes z.$$

(iii) $(M \oplus N) \otimes P \simeq (M \oplus P) \otimes (N \oplus P)$, *where*

$$(x + y) \otimes z \longrightarrow (x \otimes z) + (y \otimes z).$$

(iv) $A \otimes M \simeq M$, *where* $a \otimes x \to ax$.

Tensor Product of Modules

Definition (Bimodules)

Given rings A and B, an (A,B)-bimodule is an Abelian group N which is both an A-module and a B-module and the two structures are compatible in the sense that

$$a(xb) = (ax)b, \qquad a \in A, \ x \in M, \ b \in B.$$

Exercise (Exercise 2.15; see Carlson)

Suppose that M is A-module, P is an B-module, and N is an (A,B)-bimodule. Then:

- $M \otimes_A N$ is naturally a B-module.
- $N \otimes_B P$ is naturally an A-module.
- We have a natural isomorphism of (A,B)-bimodules,

$$(M \otimes_A N) \otimes_B P \simeq M \otimes_A (N \otimes_B P).$$

Tensor Product of Modules

Facts

- If $f : M \to M'$ and $g : N \to N'$ are A-linear maps, then $M \times N \ni (x, y) \to f(x) \otimes g(y) \in M' \otimes N'$ is an A-bilinear map.
- Therefore, there is a unique A-linear map $f \otimes g : M \otimes N \to M' \otimes N'$ such that

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y) \qquad \text{for all } x \in M \text{ and } y \in N.$$

- If $f' : M' \to M''$ and $g' : N' \to N''$ are A-linear maps, then

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$$

Restrictions and Extensions of Scalars

Fact

Let $f : A \to B$ be a ring homomorphism.

- Any B-module N can be turned into an A-module as follows: if $a \in A$ and $x \in N$, then ax is defined to be $f(a)x$.
- This A-module is said to be obtained from N by *restriction of scalars*.
- In particular, B is a module over A this way,

Proposition (Proposition 2.16)

If N is finitely generated as a B-module and B is finitely generated as an A-module, then N is finitely generated as an A-module.

Proof.

- Let $y_1, \ldots, y_n$ generate N over B, and let $x_1, \ldots, x_m$ generate B over A.
- If $y \in N$, then $y = \sum b_j y_j$ with $b_j \in B$.
- Write $b_j = \sum a_{ij} x_i$ with $a_{ij} \in A$. Then

$$y = \sum_{j=1}^{n} b_j y_j = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i y_j.$$

- Therefore N is generated by the $x_i y_j$ over A, and hence is finitely generated over A.

The proof is complete □

Restrictions and Extensions of Scalars

Fact

Let M be an A-module.

- As B is an A-module, we can form the tensor product $M_B = B \otimes_A M$.
- In fact, M_B is a B-module such that

$$b(b' \otimes x) = (bb') \otimes x, \qquad b, b' \in B, \ x \in M.$$

- We say that M_B is obtained from M by *extensions of the scalars*.

Proposition (Proposition 2.17)

If M is finitely generated as an A-module, then M_B is finitely generated as a B-module.

Proof.

If $x_1, \ldots, x_n$ generate M over A, then $1 \otimes x_1, \ldots, 1 \otimes x_n$ generate M_B over B. □

Fact

Let S be the set of all A-bilinear maps $f : M \times N \to P$.

- S is an A-module.
- If $f : M \times N \to P$ is bilinear, then, for every $x \in M$, we have an A-linear map $y \to f(x, y)$. It depends linearly on x and f, and so we get an A-linear map $S \to \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$.
- Conversely, any $\phi \in \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P))$ gives rise an A-bilinear map $(x, y) \to (\phi(x))(y)$.
- Therefore, we have a canonical isomorphism,

$$S \simeq \mathrm{Hom}_A\left(M, \mathrm{Hom}_A(N, P)\right).$$

Exactness Properties of the Tensor Product

Fact

Thanks to the defining property of the tensor product we also have a canonical isomorphism,

$$S \simeq \mathrm{Hom}_A(M \otimes N, P).$$

Consequence

We have a canonical isomorphism,

$$\mathrm{Hom}_A(M \otimes N, P) \simeq \mathrm{Hom}_A\left(M, \mathrm{Hom}_A(N, P)\right).$$

Remark

In the language of functors on the category of A-modules, the above result means that the functor $- \otimes_A N : M \rightarrow M \otimes N$ is the left adjoint of the functor $\mathrm{Hom}_A(N, -) : P \rightarrow \mathrm{Hom}_A(N, P)$ (and hence $\mathrm{Hom}_A(N, -)$ is the right adjoint of $- \otimes_A N$).

Proposition (Proposition 2.18)

Suppose we are given an exact sequence of A*-modules and homomorphisms of the form,*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

Then, for every A*-module* N*, we have an exact sequence,*

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \longrightarrow 0,$$

where we have denoted by 1 *the identity map of* N*.*

Proof of Proposition 2.18 (Sketch).

Let P be any A-module.

- As $M' \to M \to M'' \to 0$ is an exact sequence, by Proposition 2.9 we get an exact sequence,

$$0 \to \operatorname{Hom}(M'', \operatorname{Hom}(N, P)) \to \operatorname{Hom}(M, \operatorname{Hom}(N, P)) \to \operatorname{Hom}(M', \operatorname{Hom}(N, P)).$$

- As $\operatorname{Hom}(Q, \operatorname{Hom}(N, P)) \simeq \operatorname{Hom}(Q \otimes N, P)$, we get an exact sequence,

$$0 \longrightarrow \operatorname{Hom}(M'' \otimes N, P) \longrightarrow \operatorname{Hom}(M \otimes N, P) \longrightarrow \operatorname{Hom}(M' \otimes N, P).$$

- As this is true for any A-module P, by Proposition 2.9 again we get an exact sequence,

$$M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \longrightarrow 0.$$

This gives the result. □

Exactness Properties of the Tensor Product

Remarks

1. Proposition 2.18 means that, for every A-module N, the functor $-\otimes_A N$ is *left exact*.
2. More generally, it can be shown that any functor that has a left adjoint is left exact.
3. Likewise, any functor that has a right adjoint is right exact.

Exactness Properties of the Tensor Product

Remark

The functor $-\otimes_A N$ need not be *exact*, i.e., if $M' \to M \to M''$ is an exact sequence, then the sequence $M' \otimes N \to M \otimes N \to M'' \otimes N$ may fail to be exact.

Example

Take $A = \mathbb{Z}$ and consider the exact sequence,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}, \qquad f(x) = 2x.$$

- If we take tensor products with $N = \mathbb{Z}/2\mathbb{Z}$, then the sequence $0 \to \mathbb{Z} \otimes N \xrightarrow{f\otimes 1} \mathbb{Z} \otimes N$ is not exact.
- Indeed, given any $x \in \mathbb{Z}$ and $y \in \mathbb{Z}/2\mathbb{Z}$, we have
$$(f \otimes 1)(x \otimes y) = f(x) \otimes y = 2x \otimes y = x \otimes (2y) = 0.$$
- This implies that $(f \otimes 1) = 0$, i.e., $\ker(f \otimes 1) = \mathbb{Z} \otimes N \neq 0$. Therefore, the sequence is not exact.

Exactness Properties of the Tensor Product

Definition (Flat Module)

We say that an A-module is *flat* if the functor $-\otimes_A N$ is exact, i.e., for every exact sequence $M' \to M \to M''$, the sequence $M' \otimes N \to M \otimes N \to M'' \otimes N$ is again exact.

Proposition (Proposition 2.19; see also Gaillard)

Let N be an A-module. Then TFAE:

(i) *N is flat.*

(ii) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then so is the tensored sequence $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$.*

(iii) *If $f : M' \to M$ is injective, then so is $f \otimes 1 : M' \otimes N \to M \otimes N$.*

(iv) *If $f : M' \to M$ is injective and M and M' are finitely generated, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is again injective.*

Exercise (Exercise 2.20; see Carlson)

If $A \to B$ is a ring homomorphism, and M is a flat A-module, then $M_B = B \otimes_A M$ is a flat B-module.

Hint: Use the isomorphisms from Proposition 2.14 and Exercise 2.15.

Algebras

Definition

An A-*algebra* is an A-module B together with a multiplication $(b, b') \to bb'$ which is A-bilinear and with respect to which B is a ring.

Remark

The A-bilinearity of the multiplication means that

$$a(bb') = (ab)b' = b(ab') \qquad a \in A, \ b, b' \in B.$$

This accounts for the compatibility of the module and ring structures of B.

Algebras

Example

Let B be a ring and let $f : A \to B$ be a ring homomorphism.

- By restriction B is an A-module with $ab = f(a)b$.
- If B is commutative or the image of f is contained in the center of B, then B is an A-algebra, since

$$f(a)(bb') = (f(a)b)\, b' = b\left(f(a)b'\right), \quad a \in A,\ b, b' \in B.$$

Fact

Let B an A-algebra with an identity element 1_B, and define $f : A \to B$ by

$$f(a) = a1_B, \qquad a \in A.$$

Then f is a ring homomorphism whose image is contained in the center of A, since

$$(a1_B)(a'1_B) = a\left[1_B(a'1_B)\right] = a\left[a'(1_B 1_B)\right] = (aa')1_B,$$
$$(a1_B)b = a(1_B b) = a(b1_B) = b(a1_B), \quad a, a' \in A, \ b \in B.$$

Algebras

Consequences

- An A-algebra with identity is exactly a ring B with identity together with a ring homomorphism $f : A \to B$ whose image is contained in the center of B.
- A commutative A-algebra with identity is exactly a commutative ring B with identity together with a ring homomorphism $f : A \to B$.

Remark

- In Atiyah-MacDonald's book rings are assumed to be commutative and have an identity.
- Therefore, A-algebras are assumed to be commutative and to have an identity element.
- Atiyah-MacDonald then defines an A-algebra as a ring B together a ring homomorphism $f : A \to B$.

Algebras

Remark

Suppose that A is a field k and B is a k-algebra with identity element.

- As k is a field, the ring homomorphism $k \ni \lambda \to \lambda 1_B$ must be injective (see Proposition 1.2).
- Thus, k can be identified as a subring of B.

Therefore, a k algebra (with identity) is a ring that contains k as a subring.

Examples

1. $A = k$ and $B = k[x_1, \ldots, x_n]$ (polynomials with n variables).
2. $A = k$ and $B = kG$ (group ring of a group G). This is not a commutative algebra unless G is Abelian.

Remark

If B is a ring with identity, then $\mathbb{Z} \ni m \to m1 \in B$ is a ring homomorphism. Therefore, any such ring is automatically a $\mathbb{Z}$-algebra.

Definition (Algebra Homomorphisms)

Given A-algebras B and C, a map $h : B \to C$ is called an A-*algebra homomorphism* if it is both a ring homomorphism and an A-module homomorphism.

Remark

Suppose that B and C have identities with ring homomorphisms $f : A \to B$ and $g : A \to B$. Then, for every A-algebra homomorphism $h : B \to C$, we have $g = h \circ f$.

Algebras

Definition (Finite Algebras)

A *finite* A*-algebra* is an A-algebra which it is finitely generated as an A-module.

Algebras

Definition (Finitely Generated Algebras)

Let B be a commutative A-algebra with identity. We say that B is a *finitely generated A-algebra* if there is a finite set of elements $x_1, \ldots, x_n$ such that every element of B is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$.

Remark

The above condition means that we have surjective A-algebra homomorphism $A[X_1, \ldots, X_n] \to B$ that maps X_i to x_i. . Every $b \in B$ is a polynomial in the generators $x_1, \ldots, x_n$.

Definition (Finitely Generated Rings)

A ring A is said to be *finitely generated* if it is finitely generated as a $\mathbb{Z}$-algebra.

Tensor Product of Algebras

Proposition

Let B and C be A-algebras. Then their tensor product $B \otimes_A C$ is an A-algebra whose product is such that

$$(b \otimes c)(b' \otimes c') = (bb') \otimes (cc'), \qquad b, b' \in B, \quad c, c' \in C.$$

Remark

In general, we have

$$\Big(\sum_i b_i \otimes c_i\Big)\Big(\sum_j b'_j \otimes c'_j\Big) = \sum_{i,j}(b_i b'_j) \otimes (c_i c'_j).$$

Remark

The product of $B \otimes_A C$ is well defined:

- Consider the multilinear map,
$$B \times C \times B \times C \ni (b, c, b', c') \to (bb') \otimes (cc') \in B \otimes C.$$
- It gives rise to an A-module homomorphism,
$$B \otimes C \otimes B \otimes C \longrightarrow B \otimes C.$$
- As $B \otimes C \otimes B \otimes C \simeq (B \otimes C) \otimes (B \otimes C)$, we get an A-module homomorphism,
$$(B \otimes C) \otimes (B \otimes C) \longrightarrow B \otimes C.$$
- This then gives rise to an A-bilinear map,
$$(B \otimes C) \times (B \otimes C) \longrightarrow B \otimes C,$$
which is our product.

Remark

Suppose that B and C are commutative algebras with identity elements 1_B and 1_C.

- The algebra $B \otimes C$ is a commutative and has $1_B \otimes 1_C$ as identity element.
- We have natural algebra homomorphisms,

$$\begin{aligned} \mathrm{id}_B \otimes 1_C &: B \longrightarrow B \otimes C, & b &\rightarrow b \otimes 1_C, \\ 1_B \otimes \mathrm{id}_C &: C \longrightarrow B \otimes C, & c &\rightarrow 1_B \otimes c. \end{aligned}$$

Remark (Continued)

We actually have a commutative diagram,