Differentiable Manifolds
§23. Integration on Manifolds

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The Integral of an n-Form on $\mathbb{R}^n$

Remark

Throughout this section we assume familiarity with measure theory and Lebesgue's integral on $\mathbb{R}^n$.

Definition

Let $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$ be a smooth n-form on an open $U \subset \mathbb{R}^n$ with coordinates $x^1, \ldots, x^n$. The *integral* of ω over a Borel set $A \subset U$ is defined by

$$\int_A \omega = \int_A f(x)dx^1 \wedge \cdots \wedge dx^n := \int_A f(x)dx.$$

Reminder (see Section 21)

Let $\phi : V \to U$ be a diffeomorphism between open subsets of $\mathbb{R}^n$.

- ϕ is orientation-preserving if and only if $\det(J(\phi)) > 0$ on V.
- It is orientation-reversing if and only if $\det(J(\phi)) < 0$ on V.

Here $J(\phi)$ is the Jacobian of ϕ.

Facts

Let $\phi : V \to U$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. We use coordinates $(x^1, \ldots, x^n)$ on U and coordinates $(y^1, \ldots, y^n)$ on V. Set $\phi^i = x^i \circ \phi = \phi^* x^i$.

- Let $\omega = f(x) dx^1 \wedge \cdots \wedge dx^n$ be a C^∞ n-form on U. As pullback commutes with wedge product and differential,

$$\begin{aligned}\phi^*\omega = \phi^*\left(f dx^1 \wedge \cdots \wedge dx^n\right) &= (\phi^* f)\left(\phi^* dx^1\right) \wedge \cdots \wedge \left(\phi^* dx^n\right),\\ &= (f \circ \phi) d\left(\phi^* x^1\right) \wedge \cdots \wedge d\left(\phi^* x^n\right),\\ &= (f \circ \phi) d\phi^1 \wedge \cdots \wedge d\phi^n.\end{aligned}$$

- By using the local expression for wedge of differentials (Proposition 18.3), we get

$$\begin{aligned}\phi^*\omega &= (f \circ \phi) \frac{\partial(\phi^1, \ldots, \phi^n)}{\partial(y^1, \ldots, y^n)} dy^1 \wedge \cdots \wedge dy^n\\ &= (f \circ \phi) \det\left(J(\phi)\right) dy^1 \wedge \cdots \wedge dy^n.\end{aligned}$$

The Integral of an n-Form on $\mathbb{R}^n$

Facts (Continued)

- Assume that the diffeomorphism ϕ is orientation-preserving or orientation-reversing. Then

$$\begin{aligned}\phi^*\omega &= (f\circ\phi)\det\big(J(\phi)\big)\,dy^1\wedge\cdots\wedge dy^n,\\ &= \pm(f\circ\phi)\big|\det\big(J(\phi)\big)\big|dy^1\wedge\cdots\wedge dy^n,\end{aligned}$$

  where the sign $\pm$ depends on whether ϕ is orientation-preserving or orientation-reversing.

- By using the usual change of variable formula, we get

$$\int_V \phi^*\omega = \pm\int_{\phi^{-1}(U)}(f\circ\phi)\big|\det\big(J(\phi)\big)\big|dy = \pm\int_U f dx = \pm\int_U \omega.$$

Therefore, we obtain:

Lemma

Let $\phi : V \to U$ be a diffeomorphism between open subsets of $\mathbb{R}^n$, and ω a smooth n-form on U.

- *If ϕ is orientation-preserving, then*
$$\int_V \phi^*\omega = \int_U \omega.$$
- *If ϕ is orientation-reversing, then*
$$\int_V \phi^*\omega = -\int_U \omega.$$

Integral of a Differential Form over a Manifold

Definition

If M is a smooth manifold, we denote by $\Omega_c^k(M)$ the space of smooth k-forms with compact support.

Definition

Assume M is oriented and is equipped with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$. Set $n = \dim M$. Let (U, ϕ) be chart in this atlas. The integral of any top-form $\omega \in \Omega_c^n(U)$ is defined by

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^* \omega.$$

Integral of a Differential Form over a Manifold

Remark

Let (U, ψ) be another chart with same domain in the oriented atlas.

- The transition map $\phi \circ \psi^{-1} : \psi(U) \to \phi(U)$ is an orientation-preserving diffeomorphism, since the charts (U, ϕ) and (U, ψ) belong to the same oriented atlas.
- Thus, by the previous lemma we have

$$\int_{\psi(U)} (\psi^{-1})^* \omega = \int_{\psi(U)} \left(\phi \circ \psi^{-1}\right)^* \left[(\phi^{-1})^* \omega\right] = \int_{\phi(U)} (\phi^{-1})^* \omega$$

- This shows that the integral $\int_U \omega$ is well-defined and independent of the choice of the coordinate system ϕ on U.

Facts

Let $\omega \in \Omega_c^n(M)$ and $\{\rho_\alpha\}$ a C^∞ partition of unity subordinated to the open cover $\{U_\alpha\}$.

- As ω has compact support, we have
$$\omega = \sum_\alpha \rho_\alpha \omega,$$
where the sum is actually finite (see Problem 18.6).
- By Problem 18.4 $\operatorname{supp}(\rho_\alpha \omega) = \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \omega$, and so $\rho_\alpha \omega$ has compact support.
- Thus, the integral $\int_{U_\alpha} \rho_\alpha \omega$ is well defined.

Integral of a Differential Form over a Manifold

Definition

Let $\omega \in \Omega_c^n(M)$. The integral of ω over M is defined by

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega.$$

Remark

The integral $\int_M \omega$ is well defined and independent of the partition of unity $\{\rho_\alpha\}$ (see Tu's book).

Integral of a Differential Form over a Manifold

Proposition (Proposition 23.10)

Let $-M$ be the manifold M with the opposite orientation. Then, for every $\omega \in \Omega_c^n(M)$, we have

$$\int_{-M} \omega = -\int_M \omega.$$

Remark

The treatment of integration of differential forms on oriented manifolds extends *verbatim* to differential forms on oriented manifolds with boundary.

Integral of a Differential Form over a Manifold

Definition (Domain of Integration; see Definition 23.6)

A subset $D \subset \mathbb{R}^n$ is called a *domain of integration* if it is bounded and its topological boundary has measure zero.

Definition (Parametrized Set)

A *parametrized set* in an oriented n-manifold M is a subset A together with a C^∞-map $F : D \to M$, where D is a compact domain of integration in $\mathbb{R}^n$ such that:

(i) $F(D) = A$.

(ii) F restricts to an orientation-preserving diffeomorphism from $\mathrm{Int}(D)$ to $F(\mathrm{Int}(D))$.

The map $F : D \to A$ is called a *parametrization* of A.

Remark

By smooth invariance of domain for manifolds, $F(\mathrm{Int}(D))$ must be an open subset of M (see Remark 22.5).

Integral of a Differential Form over a Manifold

Definition

Let A be a parametrized set in M and $F : D \to M$ a parametrization. For any $\omega \in \Omega^n(M)$, the integral of ω over A is defined by

$$\int_A \omega := \int_D F^*\omega.$$

Remarks

1. The integral $\int_A \omega$ is well defined and independent of the parametrization F.
2. We don't need to assume ω to have compact support in the above definition.

Integration over a Zero-Dimensional Manifold

Remarks

- A zero-dimensional manifold is a discrete countable set of points.
- A connected zero-dimensional manifold is just a point. In this case there are two classes $[1]$ and $[-1]$ of non-zero 0-forms.
- More generally, an orientation on a 0-dimensional manifold is given by a function on M that assigns the values ± 1.

Integration over a Zero-Dimensional Manifold

Facts

- A compact oriented 0-dimensional manifold M is a finite unions of points oriented by $+1$ and -1.
- We write $M = \sum_i p_i - \sum_j q_j$.
- The integral of a function $f : M \to \mathbb{R}$ is then defined by

$$\int_M f = \sum_i f(p_i) - \sum_j f(q_j).$$

Stokes's Theorem

Theorem (Stokes's Theorem; Theorem 23.12)

Let M be an oriented manifold with boundary. We endow ∂M with its boundary orientation. Then, for every $\omega \in \Omega_c^{n-1}(M)$, we have

$$\int_M d\omega = \int_{\partial M} \omega.$$

Notation

If $\boldsymbol{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$ and $\boldsymbol{r} = \langle x, y, z \rangle$ is the radial vector field, then $\boldsymbol{F} \cdot d\boldsymbol{r}$ is the 1-form $Pdx + Qdy + Rdz$.

Theorem (Fundamental theorem for line integrals; Theorem 23.13)

Let C be a smooth curve in $\mathbb{R}^3$ with parametrization $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$. For any smooth function f on $\mathbb{R}^3$ we have

$$\int_C \operatorname{grad} f \cdot d\boldsymbol{r} = f\big(\boldsymbol{r}(b)\big) - f\big(\boldsymbol{r}(a)\big).$$

Proof.

- Apply Stokes's theorem to $M = C$ and $\omega = f$ to get:

$$\int_C df = \int_{\partial C} f.$$

- We have

$$\int_C df = \int_C \left\{ \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right\} = \int_C \operatorname{grad} f \cdot d\boldsymbol{r},$$

$$\int_{\partial C} f = f \Big|_{\boldsymbol{r}(a)}^{\boldsymbol{r}(b)} = f\big(\boldsymbol{r}(b)\big) - f\big(\boldsymbol{r}(a)\big).$$

This gives the result. □

Line Integrals and Green's Theorem

Theorem (Green's Theorem; Theorem 23.14)

Let D be a planar region with boundary ∂D. For any smooth functions P and Q near D we have

$$\int_{\partial D} (Pdx + Qdy) = \int_D \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dxdy.$$

Proof.

- Stokes's theorem for $M = D$ and $\omega = Pdx + Qdy$ gives
$$\int_{\partial D} \left(Pdx + Qdy\right) = \int_D d\left(Pdx + Qdy\right).$$
- We have
$$\begin{aligned} d\left(Pdx + Qdy\right) &= dP \wedge dx + dQ \wedge dy \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\ &= \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dx \wedge dy. \end{aligned}$$
- Thus, by the very definition of the integral of a top form,
$$\int_D d\left(Pdx + Qdy\right) = \int_D \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dxdy.$$

This gives the result. □