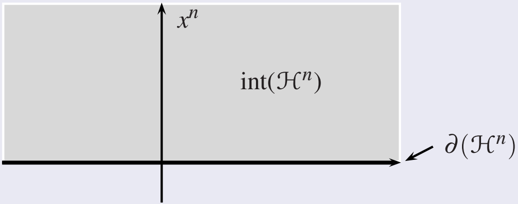
Differentiable Manifolds
§22. Manifolds with Boundary

Sichuan University, Fall 2022

The Upper Half-Space $\mathbb{H}^n$

Definition

- The (closed) *upper half-space* is
$$\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n;\ x^n \geq 0\right\}.$$
- The points $(x^1, \ldots, x^n) \in \mathbb{H}^n$ with $x^n > 0$ are called *interior points*. The set of interior points is denoted $\mathrm{Int}(\mathbb{H}^n)$.
- The points $(x^1, \ldots, x^n) \in \mathbb{H}^n$ with $x^n = 0$ are called *boundary points*. The set of boundary points is denoted $\partial(\mathbb{H}^n)$.



The Upper Half-Space $\mathbb{H}^n$

Remark

There are two types of open subsets of $\mathbb{H}^n$, depending on whether they intersect with the boundary $\partial\mathbb{H}^n$:

These open sets are the local models for manifolds with boundary.

Remark

- $\mathbb{H}^1$ is the right half-line $[0, \infty)$.
- It also convenient to consider the left-half line $\mathbb{L}^1 = (-\infty, 0]$.

Smooth Invariance of Domain in $\mathbb{R}^n$

Definition (Definition 22.1)

Let S be a subset of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$ be a map.

- We say that f is *smooth at a point* $p \in S$ if there is an open set $U \subset \mathbb{R}^n$ containing p and a smooth map $\tilde{f} : U \to \mathbb{R}^m$ such that $\tilde{f} = f$ on $U \cap S$.
- We say that f is *smooth on* S if it is smooth at every point $p \in S$.

Definition

We say that subsets $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ are *diffeomorphic* if there are smooth maps $f : S \to T \subset \mathbb{R}^m$ and $g : T \to S \subset \mathbb{R}^n$ which are inverses of each other.

Smooth Invariance of Domain in $\mathbb{R}^n$

Exercise (Exercise 22.2)

Let S be a subset of $\mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$ be a map. Using a partition of unity show that TFAE:

(i) f is smooth on S.

(ii) There is an open $U \subset \mathbb{R}^n$ containing S and a smooth map $\tilde{f} : U \to \mathbb{R}^m$ such that $\tilde{f}_{|S} = f$.

Consequence

Assume that S is an immersed submanifold in $\mathbb{R}^n$, and let $f : S \to \mathbb{R}^m$ be a map. Then TFAE:

(i) f is smooth in the sense of the previous slide.

(ii) f is smooth as a map from the manifold S to $\mathbb{R}^m$.

Smooth Invariance of Domain in $\mathbb{R}^n$

Theorem (Smooth invariance of domain; Theorem 22.3)

Let S be a subset of $\mathbb{R}^n$ which is diffeomorphic to an open of $\mathbb{R}^n$. Then S is an open of $\mathbb{R}^n$.

Proposition (Proposition 22.4)

Let $f : U \to V$ be a diffeomorphism between open subsets of $\mathbb{H}^n$. Then

$$f(U \cap \mathrm{Int}(\mathbb{H}^n)) = V \cap \mathrm{Int}(\mathbb{H}^n), \quad f(U \cap \partial(\mathbb{H}^n)) = V \cap \partial(\mathbb{H}^n).$$

Smooth Invariance of Domain in $\mathbb{R}^n$

Definition (Definition 22.1)

Let S be any subset of a manifold M, and let $f : S \to N$ be a map, where N is a manifold.

- We say that f is *smooth at a point* $p \in S$ if there is an open set $U \subset M$ containing p and a smooth map $\tilde{f} : U \to N$ such that $\tilde{f} = f$ on $U \cap S$.
- We say that f is *smooth on* S if it is smooth at every point $p \in S$.

Smooth Invariance of Domain in $\mathbb{R}^n$

Definition

We say that subsets $S \subset M$ and $T \subset N$ are *diffeomorphic* if there are smooth maps $f : S \to T \subset N$ and $g : T \to S \subset M$ that are inverse of each other.

Theorem (Smooth invariance of domain)

Let S be a subset of M which is diffeomorphic to an open of M. Then S is an open of M.

Manifolds with Boundary

Definition (Definition 22.6)

We say that a topological space M is *locally* $\mathbb{H}^n$ if every $p \in M$ has a neighborhood which is homeomorphic to an open subset of $\mathbb{H}^n$.

Definition (Topological manifolds with boundary; Definition 22.6)

A *topological* n*-manifold with boundary* is a Hausdorff second-countable topological space which is locally $\mathbb{H}^n$.

Manifolds with Boundary

Definition

Let M be a topological n-manifold with boundary.

- If $n \geq 2$, a *chart* is a pair (U, ϕ), where $U \subset M$ is an open set and $\phi : U \to \phi(U) \subset \mathbb{H}^n$ is a homeorphism onto an open set of $\mathbb{H}^n$.
- If $n = 1$ we allow a chart to be a pair (U, ϕ), where U is an open set of M and $\phi : U \to \phi(U)$ is a homeomorphism onto an open set of $\mathbb{H}^1 = [0, \infty)$ or $\mathbb{L}^1 = (-\infty, 0]$.

Remark

- With this convention, if $(U, x^1, x^2, \ldots, x^n)$ is a chart, then $(U, -x^1, x^2, \ldots, x^n)$ is a chart as well.
- In particular, for $n = 1$, if (U, x^1) is chart, then so is $(U, -x^1)$.

Manifolds with Boundary

Definition

If M is a topological manifold with boundary, a C^∞ atlas is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ covering M such that the transition maps,

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

are diffeomorphisms between open subsets of $\mathbb{H}^n$ (or $\mathbb{L}^1$).

Definition (Differentiable manifold with boundary)

A *differentiable manifold with boundary* (or *smooth manifold with boundary*) is a topological manifold with boundary with a maximal C^∞ atlas.

Manifolds with Boundary

Definition

Let M be a smooth manifold with boundary.

- We say that a point $p \in M$ is an *interior point* if there is a chart (U, ϕ) near p such that $\phi(p) \in \mathrm{Int}(\mathbb{H}^n)$.
- We say that p is a *boundary point* if there is a chart (U, ϕ) near p such that $\phi(p) \in \partial\mathbb{H}^n$.

Definition

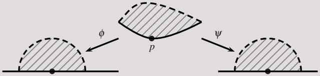
- The set of interior points is called the interior of M and is denoted $\mathrm{Int}(M)$.
- The set of boundary points is called the boundary of M and is denoted $\partial(M)$.

Manifolds with Boundary

Remark

Let p be an interior (resp., boundary) point, and (U, ϕ) a chart near p such that $\phi(p)$ is an interior (resp., boundary) point of $\mathbb{H}^n$.

- Let (V, ψ) be another chart near p. Then the transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism between open subsets of $\mathbb{H}^n$ (or $\mathbb{L}^1$).
- Thus it maps interior (resp., boundary) points of $\phi(U \cap V)$ to interior (resp., boundary) points of $\psi(U \cap V)$.
- In particular, $\psi(p) = \psi \circ \phi^{-1}(\phi(p))$ is an interior (resp., a boundary) point of $\mathbb{H}^n$.



This shows that the notions of interior and boundary points are independent of the chart.

Manifolds with Boundary

Remark

Most of the concepts introduced for manifolds extend *verbatim* for manifolds with boundary.

For instance:

Definition

Let M be a C^∞ manifold with boundary. A function $f : M \to \mathbb{R}$ is smooth if, for every chart (U, ϕ), the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth.

Remark

If p is a boundary point in U, this means that $f \circ \phi^{-1}$ has a C^∞ extension to an open neighborhood of $\phi(p)$.

Manifolds with Boundary

Remark

The interior and boundary in the sense of manifolds need not agree with the topological interior or the topological boundary.

Example (Example 22.7)

Let $D = \{x \in \mathbb{R}^2;\ \|x\| < 1\}$ be the open unit disk in $\mathbb{R}^2$

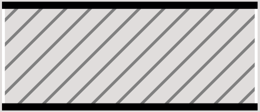
- D is a manifold without boundary, i.e., its manifold boundary is empty.
- However, its topological boundary is the unit circle $\mathbb{S}^1$.
- For the closed disk $\overline{D} = \{x \in \mathbb{R}^2;\ \|x\| \leq 1\}$ the manifold and topological boundaries agree; they are both equal to $\mathbb{S}^1$.

Example (Example 22.8)

Consider the band,

$$B = \{(x, y) \in \mathbb{H}^2;\ y \leq 1\} = \mathbb{R} \times [0, 1].$$

- The topological interior of B in $\mathbb{H}^2$ is $\mathbb{R} \times [0, 1)$.
- Its manifold interior is $\mathbb{R} \times (0, 1)$.



The Boundary of a Manifold with Boundary

Remark

The boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ is $\{x_n = 0\} = \mathbb{R}^{n-1} \times \{0\}$. This is just $\mathbb{R}^{n-1}$ under its standard embedding into $\mathbb{R}^n$.

Facts

Let M be a manifold with boundary and (U, ϕ) a chart for M. Denote by ϕ' the restriction of ϕ to $U \cap \partial M$.

- ϕ' maps $U \cap \partial M$ to $\partial\mathbb{H}^n \simeq \mathbb{R}^{n-1}$, since the points of $U \cap \partial M$ are precisely the points that are mapped to $\partial\mathbb{H}^n$ under ϕ.
- Thus, we get a homeomorphism,

$$\phi' : U \cap \partial M \longrightarrow \phi(U) \cap \partial\mathbb{H}^n \subset \mathbb{R}^{n-1}.$$

The Boundary of a Manifold with Boundary

Lemma

Let (V, ψ) be a another chart and let $\psi' : V \cap \partial M \to \psi(V) \cap \partial \mathbb{H}^n \subset \mathbb{R}^{n-1}$ the induced homeomorphism on $V \cap \partial M$. Then the transition map,

$$\psi' \circ (\phi')^{-1} : \phi(U \cap V) \cap \partial \mathbb{H}^n \longrightarrow \psi(U \cap V) \cap \partial \mathbb{H}^n$$

is a diffeomorphism between open subsets of $\mathbb{R}^{n-1}$.

As a consequence we obtain:

Proposition

Let $\{(U_\alpha, \phi_\alpha)\}$ be a C^∞ atlas of M. Then the collection $\{(U_\alpha \cap \partial M, \phi_{\alpha|U_\alpha \cap \partial M})\}$ is a C^∞ atlas of ∂M. In particular, ∂M is a smooth manifold (without of boundary) of dimension $n-1$.

The Boundary of a Manifold with Boundary

Remark

- It can also be shown that if $\{(U_\alpha, \phi_\alpha)\}$ is a C^∞ atlas of M, then $\{(U_\alpha \cap \mathrm{Int}(M), \phi_{\alpha|U_\alpha \cap \mathrm{Int}(M)})\}$ is a C^∞ atlas of $\mathrm{Int}(M)$.
- It follows that the interior $\mathrm{Int}(M)$ is a smooth manifold without boundary of dimension n.

Definition

Let M be a manifold with boundary and $p \in M$.

- Two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ on open neighborhoods of p in M are said to be equivalent if they agree on a (possibly smaller) open neighborhood of p.
- Equivalence classes of such functions are called *germs at* p.
- The set of germs at p is denoted $C_p^\infty(M)$.

Remark

The addition and multiplication of functions induce an addition and a multiplication on $C_p^\infty(M)$ with respect to which $C_p^\infty(M)$ is an $\mathbb{R}$-algebra.

Tangent Vectors, Differential Forms, and Orientations

Definition

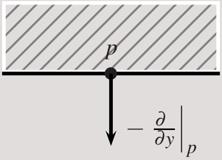
Let M be a manifold with boundary and $p \in M$. The tangent space T_pM is the space of point-derivations on $C_p^\infty(M)$, i.e., linear maps $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg \qquad \forall f, g \in C_p^\infty(M).$$

Example

Let p be a boundary point of $\mathbb{H}^2$.

- The tangent space $T_p(\mathbb{H}^2)$ is a 2-dimensional vector space with basis $\{\partial/\partial x\big|_p, \partial/\partial y\big|_p\}$.
- In particular, $-\partial/\partial y\big|_p$ is a tangent vector at p.
- However, there is no curve in $\mathbb{H}^2$ which starts at p and has initial velocity $-\partial/\partial y\big|_p$.



Remarks

- We can define smooth vector bundles over a manifold with boundary in the same way as with manifolds without boundary.
- In this context, a C^∞ vector bundle E over a manifold with boundary M is itself a manifold with boundary whose boundary is $E_{|\partial M}$.

Remarks

Let M be a manifold with boundary.

- In the same way as with manifolds without boundary, the tangent spaces T_pM, $p \in M$, can be organized as a C^∞-vector bundle,
$$TM = \bigsqcup_{p \in M} T_pM.$$
- We call TM the *tangent bundle of* M.
- A *vector field* on M is a section of the tangent bundle TM.
- A vector field is *smooth* if it is smooth section of TM.

Definition

Let M be a manifold with boundary and $p \in M$.

- The *cotangent space* T_p^*M is the dual of the tangent space T_pM.
- We denote by $\Lambda^k(T_p^*M)$ the space of k-covectors on T_pM.

Remarks

- As with manifolds without boundary, we get C^∞ vector bundles,

$$T^*M = \bigsqcup_{p \in M} T_p^*M, \qquad \Lambda^k(T^*M) = \bigsqcup_{p \in M} \Lambda^k(T_p^*M).$$

- The bundle T^*M is called the *cotangent bundle* of M.
- A k*-form* on M is a section of $\Lambda^k(T^*M)$.
- A *smooth* k*-form* is a smooth section of $\Lambda^k(T^*M)$.

Remarks

- As with manifolds without boundary, we define the orientation of a manifold with boundary M by the datum of a *continuous* pointwise orientation.
- A *pointwise orientation* is the assignment for each $p \in M$ to an orientation of the tangent space T_pM.
- In the same way as with manifolds without boundary we have one-to-one correspondences:

$$\{\text{orientations of } M\} \longleftrightarrow \left\{ \begin{array}{c} \text{equivalence classes of} \\ C^\infty \text{ nowhere-vanishing } n\text{-forms} \end{array} \right\},$$

$$\{\text{orientations of } M\} \longleftrightarrow \left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of oriented atlases} \end{array} \right\}$$

Outward-Pointing Vector Fields

Remarks

Let M be manifold with boundary.

- The boundary ∂M is an embedded submanifold of M, in the sense that the inclusion $\imath : \partial M \hookrightarrow M$ is both a topological embedding and an immersion.
- In particular, for every $p \in \partial M$, the differential $\imath_{*,p} : T_p(\partial M) \rightarrow T_pM$ is injective.
- This allows us to identify $T_p(\partial M)$ with a subspace of T_pM.

Outward-Pointing Vector Fields

Definition

Let p be a boundary point of M and let $X \in T_p(M)$ be a tangent vector.

- We say that X is *inward-pointing* if $X \notin T_p(\partial M)$ and there is a smooth curve $c : [0, \epsilon) \to M$ such that

$$c(0) = p, \qquad c\big((0, \epsilon)\big) \subset \mathrm{Int}(M), \qquad c'(0) = X.$$

- We say that X is *outward-pointing* if $-X$ is inward-pointing.

Example

On $\mathbb{H}^2$ if $p \in \partial\mathbb{H}^2$, then $\partial/\partial y\big|_p$ is inward-pointing and $-\partial/\partial y\big|_p$ is outward-pointing.

Outward-Pointing Vector Fields

Definition

A vector field *along* ∂M is a map $X : \partial M \to TM$ such that $X_p \in T_pM$ for all $p \in \partial M$. We say that such a vector field is smooth if it is smooth as a map from ∂M to TM.

Remark

A vector field *along* ∂M should not be confused with a vector field on ∂M, since it takes values in TM, not in $T(\partial M)$.

Outward-Pointing Vector Fields

Remarks

Let X be a vector field along ∂M.

- If $p \in \partial M$ and $(U, x^1, \ldots, x^n)$ is a chart for M near p, then

$$X_q = \sum a^j(q) \frac{\partial}{\partial x^j}\bigg|_q, \qquad q \in U \cap \partial M.$$

- Then X is smooth on $U \cap \partial M$ if and only if the coefficients $a^j(q)$ are smooth functions on $\partial M \cap U$.
- The vector field is outward-pointing along U if and only if $a^n(q) < 0$.

Outward-Pointing Vector Fields

Proposition (Proposition 22.10; see also Problem 22.4)

If M is a smooth manifold with boundary, then there always exists a smooth outward-pointing vector field along ∂M.

Remark

An outward-pointing vector field is always non-vanishing since $X_p \in TM \setminus T(\partial M)$ for all $p \in \partial M$.

Boundary Orientation

Proposition (Proposition 22.11)

Assume that M is oriented manifold with boundary of dimension n. Let ω be an orientation form on M and X a smooth outward-pointing vector field along ∂M. Then $\imath_X(\omega)$ is a non-vanishing smooth $(n-1)$-form on ∂M, and hence ∂M is orientable.

Remark

It can be shown that the orientation class of $\imath_X(\omega)$ is independent of ω and X.

Definition

The orientation class of $\imath_X(\omega)$ is called the *boundary orientation* on ∂M.

Boundary Orientation

Proposition (Proposition 22.11)

Suppose that M is an oriented manifold with boundary of dimension n. Let $p \in \partial M$ and let X_p be an outward-pointing vector in T_pM. If $(v_1, \dots, v_{n-1})$ is a basis of $T_p(\partial M)$ representing the boundary orientation on ∂M at p, then $(X_p, v_1, \dots, v_{n-1})$ is a basis of $T_p(M)$ and represents the orientation of M at p.

Boundary Orientation

Example (Example 22.13; Boundary orientation of $\mathbb{H}^n$)

- An orientation form for $\mathbb{H}^n$ is $\omega = dx^1 \wedge \cdots \wedge dx^n$ and a smooth outward-pointing vector field on $\partial\mathbb{H}^n$ is $X = -\partial/\partial x^n$.
- Thus, an orientation form on $\partial\mathbb{H}^n$ is

$$\begin{aligned} \imath_X \omega &= -\imath_{\partial/\partial x^n}\left(dx^1 \wedge \cdots \wedge dx^n\right) \\ &= -(-1)^{n-1}\imath_{\partial/\partial x^n}\left(dx^n \wedge dx^1 \wedge \cdots \wedge dx^{n-1}\right) \\ &= (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}. \end{aligned}$$

- For $n = 2$ we the boundary orientation is given by dx^1, which is the standard orientation on $\mathbb{R}^1 = \partial\mathbb{H}^2$.
- For $n = 3$ the boundary orientation is given by $-dx^1 \wedge dx^2$, which is the clockwise orientation on $\mathbb{R}^2 = \partial\mathbb{H}^3$.