# Noncommutative Geometry
# Chapter 13:
# Cyclic Cohomology and the Local Index Formula

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**Setup**

$\mathcal{A}$ is a unital algebra over $\mathbb{C}$.

**Definition (Hochschild Complex)**

1. The space of *$n$-cochains*, $n \geq 0$, is

$$C^n(\mathcal{A}) := \left\{(n+1)\text{-linear forms } \varphi : \mathcal{A}^{n+1} \to \mathbb{C}\right\}, \quad n \geq 0.$$

2. The *coboundary* $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ is given by

$$(b\varphi)(a^0, \ldots, a^{n+1}) = \sum_{0 \leq j \leq n} (-1)^j \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n).$$

# Hochschild Cohomology

**Lemma**

*We have* $b^2 = 0$.

**Definition**

The cohomology of the complex $(C^\bullet(\mathcal{A}), b)$ is called the Hoschschild cohomology of $\mathcal{A}$ and is denoted $HH^\bullet(\mathcal{A})$.

# Hochschild Cohomology

**Example**

Let $C$ be a $k$-dimensional current on a compact manifold $M$. Define a $k$-cochain on $\mathcal{A} = C^\infty(M)$ by

$$\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!}\left\langle C, f^0 df^1 \wedge \cdots \wedge df^k \right\rangle.$$

Then $b\varphi_C = 0$. In fact, we have:

**Theorem (Hochschild-Kostant-Rosenberg, Connes)**

*There is an isomorphism,*

$$HH^k(M) \simeq \mathcal{D}'_k(M).$$

# Cyclic Cohomology (Connes, Tsygan)

## Definition (Cyclic Cochains)

A cochain $\varphi \in C^n(\mathcal{A})$, $n \geq 0$, is *cyclic* when

$$\varphi(a^1, \ldots, a^n, a^0) = (-1)^n \varphi(a^0, \ldots, a^n) \quad \forall a^j \in \mathcal{A}.$$

We denote by $C^n_\lambda(\mathcal{A})$ the space of cyclic $n$-cochains.

## Example

Let $C$ be a $k$-dimensional current on a compact manifold $M$. We saw it defines a Hochschild cocycle. Then

$$C \text{ closed (i.e., } d^t C = 0) \implies \varphi_C \text{ cyclic.}$$

**Lemma**

$$\varphi \text{ cyclic} \implies b\varphi \text{ cyclic.}$$

**Definition**

The cohomology of the sub-complex $(C_\lambda^\bullet(\mathcal{A}), b)$ is called the *cyclic cohomology* of $\mathcal{A}$ and is denoted $HC^\bullet(\mathcal{A})$.

# Periodic Cyclic Cohomology (Connes, Tsygan)

## Definition

Define $B : C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$ by

$$B = AB_0,$$

where $B_0 : C^n(\mathcal{A}) \to C^{n-1}(\mathcal{A})$ and $A : C^{n-1}(\mathcal{A}) \to C^{n-1}(\mathcal{A})$ are given by

$$B_0\varphi(a^0, \ldots, a^{n-1}) = \varphi(1, a^0, \ldots, a^{n-1}) - (-1)^n\varphi(a^0, \ldots, a^{n-1}, 1),$$
$$A\psi(a^0, \ldots, a^{n-1}) = \sum(-1)^{j(n-1)}\psi(a^j, \ldots, a^{n-1}, a^0, \ldots, a^{j-1}).$$

## Remark

1. If $\varphi$ is a cyclic cochain, then $B_0\varphi = 0$, and hence $B\varphi = 0$.
2. An $n$-cochain $\varphi$ is cyclic if and only if $A\varphi = \frac{1}{n+1}\varphi$.

# Periodic Cyclic Cohomology

**Example**

Let $C$ be a $k$-dimensional current on a compact manifold $M$. It defines a Hochschild cocycle,

$$\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!} \left\langle C, f^0 df^1 \wedge \cdots \wedge df^k \right\rangle.$$

We then have

$$B\varphi_C = \varphi_{d^t C}.$$

**Lemma**

*We have*

$$B^2 = 0 \qquad \text{and} \qquad bB + Bb = 0.$$

**Definition (Even/Odd Cochains)**

Define

$$C^{\text{even}}(\mathcal{A}) := \bigoplus_{k\geq 0} C^{2k}(\mathcal{A})$$
$$= \left\{\varphi = (\varphi_0, \varphi_2, \ldots);\ \varphi_{2k} \in C^{2k}(\mathcal{A}),\ \varphi_{2k} = 0 \text{ for large } k\right\},$$

and

$$C^{\text{odd}}(\mathcal{A}) := \bigoplus_{k\geq 0} C^{2k+1}(\mathcal{A})$$
$$= \left\{\varphi = (\varphi_1, \varphi_3, \ldots);\ \varphi_{2k+1} \in C^{2k+1}(\mathcal{A}),\ \varphi_{2k+1} = 0 \text{ for large } k\right\}.$$

# Periodic Cyclic Cohomology

**Proposition**

*We have a 2-periodic complex,*

$$C^{\text{even}}(\mathcal{A}) \overset{b+B}{\leftrightarrows} C^{\text{odd}}(\mathcal{A}).$$

**Definition**

The cohomology of $\left(C^{\text{even/odd}}(\mathcal{A}), b+B\right)$ is called the *periodic cyclic cohomology* of $\mathcal{A}$ and is denoted $HC^{\text{even/odd}}(\mathcal{A})$.

**Example**

Let $C = C_0 + C_2 + \cdots$ be an even current on a compact manifold $M$. Then $C$ defines an even cochain,

$$\varphi_C = (\varphi_{C_0}, \varphi_{C_2}, \ldots),$$

$$\varphi_{C_{2k}}(f^0, f^1, \ldots, f^{2k}) = \frac{1}{(2k)!}\left\langle C_{2k}, f^0 df^1 \wedge \cdots \wedge df^{2k} \right\rangle.$$

Then

$$(b + B)\varphi_C = B\varphi_C = \varphi_{d^t C}.$$

$$C \text{ closed} \implies \varphi_C \text{ even cyclic cocycle.}$$

**Theorem (Connes)**

*The map $C \to \varphi_C$ gives rise to isomorphisms,*

$$H_{\text{even/odd}}(M) \simeq HC^{\text{even/odd}}(C^\infty(M)).$$

# Periodic Cyclic Cohomology

**Remark**

Assume $M$ is oriented, Riemannian and has even dimension.

- The $\hat{A}$-form $\hat{A}(R^M)$ defines an even cyclic cocycle by
$$\varphi_{\hat{A}(R^M)} = \varphi_{\hat{A}(R^M)^\vee},$$
i.e., $\varphi_{\hat{A}(R^M)} = (\varphi_0, \varphi_2, \ldots)$, with
$$\varphi_{2k}(f^0, f^1, \ldots, f^{2k}) = \frac{1}{(2k)!}\int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M).$$
- Likewise the Pfaffian $\mathrm{Pf}(R^M)$, the $L$-form $L(R^M)$ and the Todd form $\mathrm{Td}(R^M)$ define even cyclic cocycles.

# Morita Equivalence

## Definition

Let $\varphi \in C^n(\mathcal{A})$. The $n$-cochain $\varphi \# \operatorname{Tr}$ on $M_q(\mathcal{A}) = \mathcal{A} \otimes M_q(\mathbb{C})$ is defined by

$$\varphi \# \operatorname{Tr}(a^0 \otimes \mu^0, \ldots, a^n \otimes \mu^n) := \varphi(a^0, \ldots, a^n) \operatorname{Tr}\left[\mu^0 \mu^1 \cdots \mu^n\right]$$

for all $a^j \in \mathcal{A}$ and $\mu^j \in M_q(\mathbb{C})$.

## Lemma

*We have*

$$b\,(\varphi \# \operatorname{Tr}) = (b\varphi) \# \operatorname{Tr}\,.$$

## Theorem (Connes)

*The map* $\varphi \to \varphi \# \operatorname{Tr}$ *gives rise to isomorphisms,*

$$HC^\bullet(\mathcal{A}) \simeq HC^\bullet\left(M_q(\mathcal{A})\right).$$

**Example**

Let $C$ be a $k$-dimensional current on a compact manifold $M$. For the cochain,

$$\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!}\left\langle C, f^0 df^1 \wedge \cdots \wedge df^k \right\rangle,$$

we have

$$\varphi_C \# \operatorname{Tr}(a^0, a^1, \ldots, a^k) = \frac{1}{k!}\left\langle C, \operatorname{Tr}\left[a^0 da^1 \wedge \cdots \wedge da^k\right]\right\rangle$$

for all $a^j$ in $M_q\left(C^\infty(M)\right) = C^\infty\left(M, M_q(\mathbb{C})\right)$.

# Pairing with $K$-Theory ($\mathcal{A} = C^\infty(M)$)

**Setup**

- $M$ is a compact manifold.
- $E = \operatorname{ran} e$, $e = e^2 \in M_q(C^\infty(M))$.
- $F^E$ is the curvature of the Grassmanian connection $\nabla^E$ of $E$.

**Lemma**

1. $F^E = e(de)^2 = e(de)^2 e$.
2. $\operatorname{Ch}(F^E) = \sum \frac{(-1)^k}{k!} \operatorname{Tr}\left[e(de)^k\right]$.

# Pairing with $K$-Theory ($\mathcal{A} = C^\infty(M)$)

**Proposition**

*Let* $C = C_0 + C_2 + \cdots$ *be a closed even current on* $M$ *with associated even cocycle* $\varphi_C = (\varphi_{C_0}, \varphi_{C_2}, \ldots)$. *Then*

$$\begin{aligned}\langle C, E\rangle &= \sum (-1)^k \frac{(2k)!}{k!} \left(\varphi_{C_{2k}} \# \operatorname{Tr}\right)(e, e, \ldots, e),\\ &= \varphi_{C_0}(e) + \sum_{k\geq 1} (-1)^k \frac{(2k)!}{k!} \left(\varphi_{C_{2k}} \# \operatorname{Tr}\right)\left(e - \frac{1}{2}, e, \ldots, e\right).\end{aligned}$$

# Pairing with $K$-Theory (General Case)

**Setup**

$\mathcal{A}$ is a unital algebra over $\mathbb{C}$.

**Definition**

A cochain $\varphi \in C^n(\mathcal{A})$, $n \geq 1$, is *normalized* when

$$\varphi(a^0, a^1, \ldots, a^n) = 0 \quad \text{whenever } a^j = 1 \text{ for some } j \geq 1.$$

**Lemma**

*Any class in $HC^{\text{even}}(\mathcal{A})$ contains a normalized representative.*

**Example**

Let $C$ be a $k$-dimensional current on a compact manifold $M$ with associated cochain,

$$\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!} \left\langle C, f^0 df^1 \wedge \ldots \wedge df^k \right\rangle .$$

Then $\varphi_C$ is a normalized cochain.

**Definition**

Let $\varphi = (\varphi_0, \varphi_2, \ldots)$ be an even cyclic cocycle and let $\mathcal{E} = e\mathcal{A}^q$, $e = e^2 \in M_q(\mathcal{A})$, a finitely generated projective module. The pairing of $\varphi$ and $\mathcal{E}$ is

$$\langle \varphi, \mathcal{E} \rangle := \varphi_0(e) + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} (\varphi_{2k} \# \operatorname{Tr}) \left( e - \frac{1}{2}, e, \ldots, e \right).$$

**Theorem (Connes)**

*The above pairing descends to a bilinear pairing,*

$$\langle \cdot, \cdot \rangle : HC^{\text{even}}(\mathcal{A}) \times K_0(\mathcal{A}) \longrightarrow \mathbb{C}.$$

# Pairing with $K$-Theory

**Example**

Let $C = C_0 + C_2 + \cdots$ be a closed even current on a compact manifold $M$ and let $E = \operatorname{ran} e$, $e = e^2 \in M_q(C^\infty(M))$, so that $\mathcal{E} = C^\infty(M, E) \simeq C^\infty(M)^q$. Then

$$\langle \varphi_C, \mathcal{E} \rangle = \langle C, E \rangle .$$

# The Atiyah-Singer Index Theorem

## Example

Assume $M$ is spin, oriented, Riemannian and has even dimension.

1. For $C = \hat{A}(R^M)^\vee$ and the Dirac operator,
$$\left\langle \varphi_{\hat{A}(R^M)}, \mathcal{E} \right\rangle = \left\langle \hat{A}(R^M)^\vee, E \right\rangle \quad \text{and} \quad \mathrm{ind}_{\not{D}}[\mathcal{E}] = \mathrm{ind}_{\not{D}}[E].$$
2. By the $K$-theoretic version of the Atiyah-Singer Index Theorem explained in Chapter 12,
$$\mathrm{ind}_{\not{D}}[E] = (2i\pi)^{-\frac{n}{2}} \left\langle \hat{A}(R^M)^\vee, E \right\rangle .$$
3. Therefore, the Atiyah-Singer Index Theorem can be further restated as

## Theorem

$$\mathrm{ind}_{\not{D}}[\mathcal{E}] = (2i\pi)^{-\frac{n}{2}} \left\langle \varphi_{\hat{A}(R^M)}, \mathcal{E} \right\rangle \quad \forall \mathcal{E} \in K_0(C^\infty(M)).$$

# The Connes-Chern Character

## Setup

- $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$
- $\gamma := 1_{\mathcal{H}^+} - 1_{\mathcal{H}^-}$ is the grading operator, $\gamma^2 = 1$, $\gamma^* = \gamma$.
- $F := D|D|^{-1}$ is the sign of $D$.

## Assumption

$(\mathcal{A}, \mathcal{H}, D)$ is $p$-*summable* with $p \geq 1$, i.e.,

$$\mu_k(D^{-1}) = \mathrm{O}(k^{-\frac{1}{p}}).$$

That is, $D^{-1}$ is an infinitesimal operator of order $1/p$.

# The Connes-Chern Character

**Lemma**

*Let $q > p$. Then*

$$\mathrm{Tr}\left[\left|[F, a^1] \cdots [F, a^q]\right|\right] < \infty \qquad \forall a^j \in \mathcal{A}.$$

**Definition (Connes)**

For $n > \frac{1}{2}(p+1)$ let $\tau_{2n}$ be the $2n$-cochain defined by

$$\tau_{2n}(a^0, \ldots, a^{2n}) = \frac{1}{2}\frac{n!}{(2n)!}\,\mathrm{Tr}\left[\gamma F[F, a^0] \cdots [F, a^{2n}]\right], \quad a^j \in \mathcal{A}.$$

# The Connes-Chern Character

**Lemma (Connes)**

1. *$\tau_{2n}$ is a normalized cyclic cocycle.*
2. *The class of $\tau_{2n}$ in $HC^{\text{even}}(\mathcal{A})$ does not depend on $n$.*

**Definition**

The class of $\tau_{2n}$ in $HC^{\text{even}}(\mathcal{A})$ is called the *Connes-Chern character* and is denoted $\mathrm{Ch}(\mathcal{A}, D)$.

**Theorem (Connes)**

*For all $\mathcal{E} \in K_0(\mathcal{A})$,*

$$\mathrm{ind}_D[\mathcal{E}] = \langle \mathrm{Ch}(\mathcal{A}, D), \mathcal{E} \rangle .$$

**Remark**

Connes' cocycle $\tau_{2n}$ is difficult to compute in practice, because it definition involves

1. The operator $F$ which is like a $\psi$DO.
2. The operator trace which is not a local functional, i.e., it does not vanish on infinitesimals of a given order.

Therefore, it was sought for a more convenient representative of the Connes-Chern character.

# The JLO Cocycle

## Assumption

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is $\theta$-*summable*, i.e.,

$$\mathrm{Tr}\left[e^{-tD^2}\right] < \infty \qquad \forall t > 0.$$

## Remark

$$p\text{-summability} \implies \theta\text{-summability}.$$

# The JLO Cocycle

**Definition (Jaffe-Lesniewski-Osterwlader)**

For $t > 0$ define $\varphi^t_{\mathrm{JLO}} = (\varphi_0, \varphi_2, \ldots)$ by

$$\varphi^t_{2k}(a^0, \ldots, a^{2k}) =$$
$$t^k \int_{\Delta_{2k}} \mathrm{Tr}\left[a^0 e^{-ts_0 D^2}[D, a^1]e^{-ts_1 D^2} \cdots [D, a^{2k}]e^{-ts_{2k} D^2}\right] ds, \;\; a^j \in \mathcal{A},$$

where $\Delta_{2k}$ is the $2k$-simplex

$$\Delta_{2k} := \{(s_0, \ldots, s_{2k}) \in \mathbb{R}^{2k+1};\; s_0 + \cdots + s_{2k} = 1,\; s_j \geq 0\}.$$

**Remark**

As observed by Quillen, $\varphi^t_{\mathrm{JLO}}$ can be interpreted as the Chern character of a superconnection on the algebra of cochains.

# The JLO Cocycle

**Proposition (Jaffe-Lesniewski-Osterwlader, Connes, Getzler-Szenes)**

1. $(b+B)\varphi^t_{\text{JLO}} = 0.$
2. *For all* $t > 0$ *and* $\mathcal{E} \in K_0(\mathcal{A})$,
$$\text{ind}_D[\mathcal{E}] = \langle \varphi^t_{\text{JLO}}, \mathcal{E} \rangle .$$

**Remark**

1. $\varphi^t_{2k} \neq 0$ for large $k$, so $\varphi^t_{\text{JLO}}$ is NOT a cochain in $C^{\text{even}}(\mathcal{A})$.
2. This is a cocycle in *entire cyclic cohomology*, i.e., in the cohomology of infinite cochains $\varphi = (\varphi_0, \varphi_2, \ldots)$ such that, for any finite subset $S \subset \mathcal{A}$, the power series,
$$\sum_{k\geq 0} \frac{z^k}{k!} \varphi_{2k}(a^0, \ldots, a^{2k}), \qquad a^j \in S,$$
are entire functions.

# Retraction of the JLO Cocycle

**Assumption**

$(\mathcal{A}, \mathcal{H}, D)$ is $p$-summable.

**Remark**

This assumption ensures us the existence of the Connes-Chern character.

**Theorem (Connes)**

*Connes's cocycle* $\tau_{2n}^{D}$ *and the JLO cocycle* $\varphi_{\mathrm{JLO}}^{t}$ *are cohomologous in entire cyclic cohomology.*

# Retraction of the JLO Cocycle

**Assumption**

As $t \to 0^+$,

$$\varphi_{2k}^t = \sum_{\substack{\alpha, l \geq 0 \\ \alpha + l > 0}} t^{-\alpha} (\log^l t) \varphi_{2k}^{(\alpha,l)} + \varphi_{2k}^{(0,0)} + \mathrm{o}(t),$$

where the $\varphi_k^{(\alpha,l)}$ are $2k$-cochains.

**Definition**

The finite part of the JLO cocycle is

$$\mathrm{FP}_{t \to 0^+} \varphi_{\mathrm{JLO}}^t := \left( \varphi_0^{(0,0)}, \varphi_2^{(0,0)}, \ldots \right).$$

# Retraction of the JLO Cocycle

**Theorem (Connes-Moscovici)**

1. $\mathrm{FP}_{t\to 0^+}\,\varphi^t_{\mathrm{JLO}}$ *is an even periodic cyclic cocycle representing the Connes-Chern character.*
2. *For all* $\mathcal{E} \in K_0(\mathcal{A})$,

$$\mathrm{ind}_D[\mathcal{E}] = \left\langle \mathrm{FP}_{t\to 0^+}\,\varphi^t_{\mathrm{JLO}}, \mathcal{E} \right\rangle .$$

**Example**

For a Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$,

$$\mathrm{FP}_{t\to 0^+}\,\varphi^t_{\mathrm{JLO}} = (\varphi_0, \varphi_{2k}, \ldots),$$

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^k \wedge \hat{A}(R^M).$$

For $T \in \mathcal{L}(\mathcal{H})$ set

$$\delta^0(T) = T, \quad \delta^1(T) := [|D|, T], \qquad \delta^2(T) := [|D|, [|D|, T]],$$

$$\delta^j(T) = \underbrace{[|D|, [|D|, \dots, [|D|, T] \cdots ]}_{j \text{ times}}.$$

**Definition**

$\mathcal{B}$ is the algebra generated by $\gamma$ and the $\delta^j(a)$ and $\delta^j([D, a])$, $a \in \mathcal{A}$.

**Fact**

*For any $b \in \mathcal{B}$, the function $\zeta_b(z) := \mathrm{Tr}\,[b|D|^{-z}]$ is analytic for $\Re z \gg 1$.*

# Dimension Spectrum

**Definition**

The *dimension spectrum* is the maximal subset $\Sigma \subset \mathbb{C}$ such that all the functions $\zeta_b(z)$, $b \in \mathcal{B}$, have an analytic continuation to $\mathbb{C} \setminus \Sigma$.

**Example**

A Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$ has dimension spectrum,

$$\Sigma = \{k \in \mathbb{Z};\ k \leq \dim M\}.$$

# Key Assumptions

**Assumption**

The dimension spectrum $\Sigma$ is *discrete* and is *simple*, i.e., the zeta functions $\zeta_b(z)$, $b \in \mathcal{B}$, have at worst simple pole singularities on $\Sigma$.

**Assumption**

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *regular*, i.e., all the operators in $\mathcal{B}$ are bounded.

# Residual Trace

**Definition**

$\Psi_D^q(\mathcal{A})$, $q \in \mathbb{C}$, is the space of operators such that

$$P \simeq b_0|D|^m + b_1|D|^{m-1} + \ldots, \qquad b_j \in \mathcal{B},$$

where $\simeq$ means that, for all $N \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$|D|^{s-m}\Big(P - \sum_{j<N} b_j|D|^{q-j}\Big)|D|^{N-s} \text{ is bounded.}$$

**Proposition (Connes-Moscovici)**

*Under the previous assumptions:*

1. $\Psi_D^\bullet(\mathcal{A}) := \bigcup_{q\in\mathbb{C}} \Psi_D^q(\mathcal{A})$ *is an algebra.*
2. *The following formula defines a trace on* $\Psi_D^\bullet(\mathcal{A})$,

$$⨍ P := \text{Res}_{z=0} \text{Tr}\left[P|D|^{-z}\right], \qquad P \in \Psi^\bullet(\mathcal{A}).$$

# The CM Cocycle

**Theorem (Connes-Moscovici)**

1. *The following formulas define a normalized even cocycle* $\varphi_{\mathrm{CM}} = (\varphi_0, \varphi_2, \ldots)$ *in the periodic cyclic cohomology of* $\mathcal{A}$,

$$\varphi_0(a^0) = \mathrm{Res}_{z=0} \left\{ \Gamma(z)\, \mathrm{Tr} \left[ \gamma a^0 |D|^{-z} \right] \right\},$$

$$\varphi_{2k}(a^0, \ldots, a^{2k}) = \sum c_{k,\alpha} ⨍ a^0 [D, a^1]^{[\alpha_1]} \cdots [D, a^{2k}]^{[\alpha_{2k}]} |D|^{-2(|\alpha|+k)},$$

   *where the sum is finite, the* $c_{k,\alpha}$ *are universal constants, and we have set*

$$T^{[j]} := \overbrace{[D^2, [D^2, \ldots [D^2, T] \cdots ]]}^{j \text{ times}}.$$

2. *The CM cocycle represents the Connes-Chern character, and so we have*

$$\mathrm{ind}_D[\mathcal{E}] = \langle \varphi_{\mathrm{CM}}, \mathcal{E} \rangle \qquad \forall \mathcal{E} \in K_0(\mathcal{A}).$$

# The CM Cocycle

**Example**

For a Dirac spectral triple $\left(C^\infty(M), L^2(M,\not{S}), \not{D}\right)$,

$$\varphi_{\mathrm{CM}} = (\varphi_0, \varphi_{2k}, \ldots),$$

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M).$$

# Summary

| Classical | NCG |
|---|---|
| Manifold $M$ | Spectral triple $(\mathcal{A}, \mathcal{H}, D)$ |
| Vector bundles over $M$ | F.g. projective modules over $\mathcal{A}$ |
| $\mathrm{ind}\,\not{D}_E$ | $\mathrm{ind}_D[\mathcal{E}]$ |
| Differential forms | Cyclic cocycles |
| Atiyah-Singer Index Formula | Connes-Chern character & CM cocycle |
| Characteristic classes | Cyclic cohomology for Hopf algebras |