Noncommutative Geometry
Chapter 12:
Spectral Triples, Dirac Operators, K-Theory, and the Atiyah-Singer Index Theorem

Sichuan University, Fall 2022

What is a Noncommutative Manifold?

Definition

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$, where

- $\mathcal{H}$ is a *super* Hilbert space with a $\mathbb{Z}_2$-grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
- $\mathcal{A}$ is an (even) algebra of *bounded* operators on $\mathcal{H}$.
- D is a selfadjoint (unbounded) operator such that:
  - D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
  - $[D, a]$ is *bounded* for all $a \in \mathcal{A}$.
  - $(D + i)^{-1}$ is a *compact* operator.

The de Rham Spectral Triple

Setup

- M^n is a compact oriented Riemannian manifold (n even).
- $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ is the de Rham differential with adjoint d^*.

Remark

$$\Lambda^* T^* M = \Lambda^{\text{even}} T^* M \oplus \Lambda^{\text{odd}} T^* M.$$

Proposition

The following is a spectral triple

$$\left(C^\infty(M), L^2(M, \Lambda^* T^* M), d + d^*\right),$$

with $L^2(M, \Lambda^* T^* M) = L^2(M, \Lambda^{\text{even}} T^* M) \oplus L^2(M, \Lambda^{\text{odd}} T^* M)$.

The Chern-Gauss-Bonnet Theorem

Definition

The *Fredholm index* of the operator $d + d^*$ is

$$\operatorname{ind}(d + d^*) := \dim\ker\left[(d + d^*)_{|\Lambda^{\text{even}}}\right] - \dim\ker\left[(d + d^*)_{|\Lambda^{\text{odd}}}\right].$$

Definition (Euler Characteristic $\chi(M)$)

$$\chi(M) := \sum_{k=0}^{n} (-1)^k \dim H^k(M),$$

where $H^k(M)$ is the de Rham cohomology of M.

The Chern-Gauss-Bonnet Theorem

Theorem (Chern-Gauss-Bonnet)

$$\chi(M) = \operatorname{ind}(d + d^*) = \int_M \operatorname{Pf}\left(R^M\right),$$

where $\operatorname{Pf}\left(R^M\right)$ *is the Pfaffian form of the curvature* R^M *of* M*.*

The Signature Spectral Triple

Setup

- (M^n, g) compact oriented Riemannian manifold (n even).

Definition (Hodge Operator)

The operator $\star : \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ is defined by

$$\star\alpha \wedge \beta = \langle \alpha, \beta \rangle \, \mathrm{Vol}_g(x) \quad \forall \alpha, \beta \in \Lambda^k T_x^*M,$$

where $\mathrm{Vol}_g(x)$ is the volume form of M.

Remark

As $\star^2 = 1$, there is a splitting

$$\Lambda^* T^*M = \Lambda^+ \oplus \Lambda^-, \quad \text{with } \Lambda^\pm := \{\alpha;\ \star\alpha = \pm\alpha\}.$$

The Signature Spectral Triple

Proposition

The following is a spectral triple,

$$\left(C^\infty(M), L^2(M, \Lambda^* T^* M), d - \star d \star\right),$$

with $L^2(M, \Lambda^* T^* M) = L^2(M, \Lambda^+) \oplus L^2(M, \Lambda^-)$.

The Signature Theorem

Definition

The Fedholm index of $d - \star d \star$ is

$$\mathrm{ind}(d - \star d \star) := \dim\ker\left[(d - \star d \star)_{|\Lambda^+}\right] - \dim\ker\left[(d - \star d \star)_{|\Lambda^-}\right].$$

Definition (Signature $\sigma(M)$)

If $n = 4p$, then $\sigma(M)$ of M is the signature of the bilinear form,

$$H^{\frac{n}{2}}(M) \times H^{\frac{n}{2}}(M) \ni (\alpha, \beta) \to \int_M \alpha \wedge \beta.$$

The Signature Theorem

Theorem (Hirzebruch)

$$\begin{aligned}\sigma(M) &= \mathrm{ind}(d - \star d \star) \quad \textit{if } n = 4p, \\ &= 2^{\frac{n}{2}} \int_M L\left(R^M\right),\end{aligned}$$

where $L\left(R^M\right) := \det^{\frac{1}{2}}\left[\frac{R^M/2}{\tanh\left(R^M/2\right)}\right]$ *is called the* L*-form of the curvature* R^M.

Setup

- M^n compact Kälher manifold (n complex dim.).
- $\Lambda^{0,q} T^* M := \mathrm{Span}\left\{ d\overline{z_{k_1}} \wedge \cdots \wedge d\overline{z_{k_q}} \right\}$ is the bundle of anti-holomorphic q-forms.
- $\overline{\partial} : C^\infty(M, \Lambda^{0,q} T^* M) \to C^\infty(M, \Lambda^{0,q+1} T^* M)$ is the Dolbeault differential with adjoint $\overline{\partial}^*$.

Remark

$$\Lambda^{0,*} T^* M = \Lambda^{0,\mathrm{even}} T^* M \oplus \Lambda^{0,\mathrm{odd}} T^* M.$$

The Dolbeault Spectral Triple

Proposition

The following is a spectral triple,

$$\left(C^\infty(M), L^2\left(M, \Lambda^{0,*}T^*M\right), \overline{\partial} + \overline{\partial}^*\right),$$

with $L^2\left(M, \Lambda^{0,*}T^*M\right) = L^2\left(M, \Lambda^{0,\text{even}}T^*M\right) \oplus L^2\left(M, \Lambda^{0,\text{odd}}T^*M\right)$.

The Hirzebruch-Riemann-Roch Theorem

Definition

The *Fredholm index* of the operator $\overline{\partial} + \overline{\partial}^*$ is

$$\mathrm{ind}(\overline{\partial} + \overline{\partial}^*) := \dim\ker\left[(\overline{\partial} + \overline{\partial}^*)_{|\Lambda^{0,\mathrm{even}}}\right] - \dim\ker\left[(\overline{\partial} + \overline{\partial}^*)_{|\Lambda^{0,\mathrm{odd}}}\right].$$

Definition (Holomorphic Euler Characteristic)

$$\chi(M) := \sum_{q=0}^{n} (-1)^q \dim H^{0,q}(M),$$

where $H^{0,q}(M)$ is the Dolbeault cohomology of M.

The Hirzebruch-Riemann-Roch Theorem

Theorem (Hirzebruch-Riemann-Roch)

$$\chi(M) = \operatorname{ind}\left(\overline{\partial} + \overline{\partial}^*\right) = \int_M \operatorname{Td}\left(R^{1,0}\right),$$

where $\operatorname{Td}\left(R^{1,0}\right) := \det\left[\frac{R^{1,0}}{e^{R^{1,0}}-1}\right]$ *is called the Todd form of the holomorphic curvature* $R^{1,0}$ *of* M*.*

The Dirac Operator

Fact

On $\mathbb{R}^n$ the square root $\sqrt{\Delta}$ is a ΨDO, but not a differential operator.

Dirac's Idea

Seek for a square root of Δ as a differential operator with *matrix* coefficients,

$$\not{D} = \sum c^j \partial_j.$$

The Dirac Operator

Definition

The Clifford algebra of $\mathbb{R}^n$ is the $\mathbb{C}$-algebra $\mathrm{Cl}(\mathbb{R}^n)$ generated by the canonical basis vectors $e^1, \cdots, e^n$ of $\mathbb{R}^n$ with relations,

$$e^i e^j + e^j e^i = -2\delta^{ij}.$$

Remark

Any Euclidean space $(V, \langle\cdot,\cdot\rangle)$ defines a Clifford algebra.

The Quantization Map

Denote by $\Lambda^\bullet_{\mathbb{C}}\mathbb{R}^n$ the complexified exterior algebra of $\mathbb{R}^n$.

Proposition

There is a linear isomorphism $c : \Lambda^\bullet_{\mathbb{C}}\mathbb{R}^n \to \mathrm{Cl}(\mathbb{R}^n)$ *given by*

$$c(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{i_1} \cdots e^{i_k}, \qquad 1 \leq i_1 < \cdots < i_k \leq n.$$

Remark

This is not an isomorphism of algebras, e.g., for all $\xi, \eta \in \mathbb{R}^n$, we have $c^{-1}\left(c(\xi)c(\eta)\right) = \xi \wedge \eta - \langle \xi, \eta \rangle$.

The Quantization Map

Corollary

There is a $\mathbb{Z}_2$-grading,

$$\mathrm{Cl}(\mathbb{R}^n) = \mathrm{Cl}^+(\mathbb{R}^n) \oplus \mathrm{Cl}^-(\mathbb{R}^n), \quad \mathrm{Cl}^\pm(\mathbb{R}^n) := c(\Lambda_{\mathbb{C}}^{\mathrm{even}/odd}\mathbb{R}^n).$$

Remark

$\mathrm{Cl}^+(\mathbb{R}^n)$ is a sub-algebra of $\mathrm{Cl}(\mathbb{R}^n)$.

The Spinor Representation

Theorem

1. $\mathrm{Cl}(\mathbb{R}^n)$ *has a unique irreducible representation,*
$$\rho : \mathrm{Cl}(\mathbb{R}^n) \to \mathrm{End}(\$_n),$$
where $\$_n$ *is the space of spinors of* $\mathbb{R}^n$*.*
2. *If* n *is even, then* $\$$ *has a splitting* $\$_n = \$_n^+ \oplus \$_n^-$*, which is preserved by the action of* $\mathrm{Cl}^+(\mathbb{R}^n)$*.*
3. *If* n *is even, the spinor representation gives rise to an isomorphism,*
$$\mathrm{Cl}(\mathbb{R}^n) \simeq \mathrm{End}\,\$_n.$$

The Spin Group Spin(n)

Definition

The spin group $\mathrm{Spin}(n)$ is the double cover of $\mathrm{SO}(n)$,

$$\{\pm 1\} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to \{1\}.$$

Remark

The spin group $\mathrm{Spin}(n)$ can be realized as the Lie group of some Lie algebra contained in $\mathrm{Cl}^+(\mathbb{R}^n)$.

Proposition

The spinor representation splits into the half-spin representations,

$$\rho_\pm : \mathrm{Spin}(n) \longrightarrow \mathrm{End}(\$_n^\pm).$$

Setup

(M^n, g) is a compact oriented Riemannian manifold (n even).

Definition

The *Clifford bundle* of M is the bundle of algebras,

$$\mathrm{Cl}(M) = \bigsqcup_{x \in M} \mathrm{Cl}(T_x^* M),$$

where $\mathrm{Cl}(T_x^* M)$ is the Clifford algebra of $(T_x^* M, g^{-1})$.

The Dirac Operator

Remarks

- There is a quantization map,

$$c : \Lambda^{\bullet}_{\mathbb{C}} T^* M \longrightarrow \mathrm{Cl}(M).$$

- This an isomorphism of vector bundles, but not an isomorphism of algebra bundles.
- There is also a splitting,

$$\mathrm{Cl}(M) = \mathrm{Cl}^{+}(M) \oplus \mathrm{Cl}^{-}(M), \quad \mathrm{Cl}^{\pm}(M) = c\left(\Lambda^{\mathrm{even/odd}} T^*_{\mathbb{C}} M\right).$$

- Here $\mathrm{Cl}^{+}(M)$ is a sub-bundle of algebras of $\mathrm{Cl}(M)$.

Spin Structure

Definition

A *spin structure* on M is a reduction of its structure group from $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$.

Theorem

If M has a spin structure, then there is an associated spinor bundle $\$ = \$^+ \oplus \$^-$ such that

1. $\mathrm{Cl}(M) \simeq \operatorname{End} \$$ *and the action of* $\mathrm{Cl}^+(M)$ *preserves the* $\mathbb{Z}_2$*-grading* $\$ = \$^+ \oplus \$^-$.
2. *The Riemannian metric lifts to a Hermitian metric on* $\$$.
3. *The Levi-Civita connection lifts to a connection* $\nabla^{\$}$ *on* $\$$ *preserving its* $\mathbb{Z}_2$*-grading and Hermitian metric.*

The Dirac Operator

Setup

(M^n, g) is a compact spin oriented Riemannian manifold (n even).

Definition (Dirac operator)

The Dirac operator $D\!\!\!/ : C^\infty(M, \$) \to C^\infty(M, \$)$ is the composition,

$$\begin{aligned} D\!\!\!/ : C^\infty(M,\$) \xrightarrow{\nabla^{\$}} C^\infty(M, \$ \otimes T^*M) &\longrightarrow C^\infty(M,\$) \\ \sigma \otimes \xi &\longrightarrow c(\xi)\sigma, \end{aligned}$$

where $c(\xi) \in \mathrm{Cl}_x(M)$ is identified with an element of $\mathrm{End}\,\$_x$.

Proposition

The following is a spectral triple,

$$\left(C^\infty(M), L^2\left(M, \$\right), D\!\!\!/\right),$$

with $L^2(M, \$) = L^2(M, \$^+) \oplus L^2(M, \$^-)$.

The Atiyah-Singer Index Theorem

Setup

- (M^n, g) is a compact spin oriented Riemannian manifold (n even).
- E is a Hermitian vector bundle over M with connection ∇^E.

Definition (Twisted Dirac Operator)

The operator $\not{D}_E = \not{D}_{E,\nabla^E} : C^\infty(M, \not{S} \otimes E) \to C^\infty(M, \not{S} \otimes E)$ is

$$\not{D}_E = \not{D} \otimes 1_E + c \circ \nabla^E,$$

where $c \circ \nabla^E$ is given by the composition,

$$\begin{array}{ccccc} C^\infty(M, \not{S} \otimes E) & \overset{1 \otimes \nabla^E}{\to} & C^\infty(M, \not{S} \otimes T^*M \otimes E) & \overset{c \otimes 1}{\to} & C^\infty(M, \not{S} \otimes E) \\ & & \sigma \otimes \xi \otimes s & \longrightarrow & (c(\xi)\sigma) \otimes s. \end{array}$$

Definition

The *Fredholm index* of $\not{D}_E$ is

$$\operatorname{ind} \not{D}_E := \dim\ker\left[(\not{D}_E)_{|\$^+\otimes E}\right] - \dim\ker\left[(\not{D}_E)_{|\$^-\otimes E}\right].$$

Theorem (Atiyah-Singer)

$$\operatorname{ind} \not{D}_E = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \operatorname{Ch}(F^E),$$

where:

- $\hat{A}(R^M) := \det^{\frac{1}{2}}\left[\frac{R^M/2}{\sinh\left(R^M/2\right)}\right]$ *is called the* $\hat{A}$*-class of the curvature* R^M *of* M.
- $\operatorname{Ch}(F^E) := \operatorname{Tr}\left[e^{-F^E}\right]$ *is called the Chern form of the curvature* F^E *of* ∇^E.

Remark

The index formula can be proved by heat kernel arguments.

- By the McKean-Singer formula,

$$\begin{aligned}\operatorname{ind} D\!\!\!/_E &= \operatorname{Tr}\left[\gamma e^{-tD\!\!\!/_E^2}\right] \quad \forall t > 0\\ &= \int_M \operatorname{Tr}\left[\gamma e^{-tD\!\!\!/_E^2}(x,x)\right] \operatorname{vol}_g(x) \quad \forall t > 0.\end{aligned}$$

  where $\gamma := 1_{\$^+ \otimes E} - 1_{\$^- \otimes E}$ is the grading operator.
- The proof is then completed by using:

Theorem (Local Index Theorem; Atiyah-Bott-Patodi, Gilkey)

$$\operatorname{Tr}\left[\gamma e^{-tD\!\!\!/_E^2}(x,x)\right] \operatorname{vol}_g(x) \xrightarrow[t \to 0^+]{} \left[\hat{A}(R^M) \wedge \operatorname{Ch}(F^E)\right]^{(n)}.$$

K-Theory

Setup

- M is a compact manifold.

Definition

Two vector bundles E_1 and E_2 over M are *stably equivalent* when there exists a vector bundle F such that

$$E_1 \oplus F \simeq E_2 \oplus F.$$

Remark

There is an addition on stable equivalence classes of vector bundles given by

$$[E_1] + [E_2] := [E_1 \oplus E_2].$$

This turns the set of stable equivalence classes into a monoid.

K-Theory

Definition

$K^0(M)$ is the Abelian group of formal differences

$$[E_1] - [E_2]$$

of stable equivalence classes of vector bundles over M.

Remark

Let G be an Abelian group and $\varphi : \mathrm{Vect}(M) \to G$ a map such that

$$\varphi(E_1 \oplus E_2) = \varphi(E_1) + \varphi(E_2) \qquad \forall E_j \in \mathrm{Vect}(M).$$

Then φ gives rise to a unique additive map,

$$\varphi : K^0(M) \longrightarrow G,$$
$$\varphi([E]) := \varphi(E) \qquad \forall E \in \mathrm{Vect}(M).$$

Index Map of a Dirac Operator

Setup

- M^n is a compact spin oriented Riemannian manifold (n even).
- $\not{D} : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator of M.

Lemma

If E_1 and E_2 are vector bundles over M, then

$$\mathrm{ind}\,\not{D}_{E_1 \oplus E_2} = \mathrm{ind}\,\not{D}_{E_1} + \mathrm{ind}\,\not{D}_{E_2}.$$

Proposition

The Dirac operator gives rise to a unique additive index map,

$$\mathrm{ind}_{\not{D}} : K^0(M) \longrightarrow \mathbb{Z},$$

$$\mathrm{ind}_{\not{D}}[E] := \mathrm{ind}\,\not{D}_E.$$

de Rham Currents

Setup

M is a compact manifold.

Definition

$\mathcal{D}'_k(M)$ is the space of de Rham currents of dimension k, i.e., continuous linear forms on $C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M)$.

Example

Let N be an oriented submanifold of dimension k. Then N defines a k-dimensional current C_N on M by

$$\langle C_N, \eta \rangle := \int_N \iota^* \eta \qquad \forall \eta \in C^\infty(M, \Lambda^k_{\mathbb{C}} T^*M),$$

where $\iota : N \to M$ is the inclusion of N into M.

Poincaré Duality

Definition

Assume M oriented and set $n = \dim M$. The *Poincaré dual* of an $n-k$- form ω on M is the k-dimensional current $\omega^{\wedge}$ defined by

$$\langle \omega^{\wedge}, \eta \rangle := \int_M \omega \wedge \eta \qquad \forall \eta \in C^{\infty}(M, \Lambda^k_{\mathbb{C}} T^* M).$$

Example

The Poincaré dual of $\hat{A}(R^M)$ is

$$\left\langle \hat{A}(R^M)^{\wedge}, \eta \right\rangle = \int_M \hat{A}(R^M) \wedge \eta.$$

This is an even (resp., odd) current if $\dim M$ is even (resp., odd).

de Rham Homology

Definition (de Rham Boundary)

The *de Rham boundary* $d^t : \mathcal{D}'_k(M) \to \mathcal{D}'_{k-1}(M)$ is defined by

$$\langle d^t C, \eta \rangle := \langle C, d\eta \rangle \qquad \forall \eta \in C^\infty(M, \Lambda^{k-1}_{\mathbb{C}} T^* M).$$

Definition

The *de Rham homology of M* is the homology of the complex $(\mathcal{D}'_\bullet(M), d^t)$. It is denoted $H_\bullet(M)$.

Remark

When M is oriented, the Poincaré duality yields an isomorphism,

$$H^{n-k}(M) \simeq H_k(M).$$

Pairing with K-Theory

Definition

Let $C = C_0 + C_2 + \cdots$ be an even current and let E be a vector bundle over M. The pairing of C and E is

$$\langle C, E \rangle := \left\langle C, \mathrm{Ch}(F^E) \right\rangle,$$

where F^E is the curvature of any connection on E.

Lemma

The value of $\langle C, E \rangle$ depends only the homology class of C and the K-theory class of E.

Proposition

The above pairing descends to a bilinear pairing,

$$\langle \cdot, \cdot \rangle : H_{\mathrm{even}}(M) \times K^0(M) \longrightarrow \mathbb{C}.$$

The Atiyah-Singer Index Theorem (K-Theoretic Version)

Setup

- M^n is a compact spin oriented Riemannian manifold (n even).
- $\not{D} : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ is the Dirac operator of M.
- E is a vector bundle over M.

- For the Poincaré dual $C = \hat{A}(R^M)^\wedge$ we get
$$\left\langle \hat{A}(R^M)^\wedge, E \right\rangle = \left\langle \hat{A}(R^M)^\wedge, \mathrm{Ch}(F^E) \right\rangle = \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E).$$
- By the Atiyah-Singer index theorem,
$$\mathrm{ind}_{\not{D}}[E] = \mathrm{ind}\, \not{D}_E = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(F^E).$$
- Therefore, the Atiyah-Singer index theorem can be restated as

Theorem (Atiyah-Singer)

$$\mathrm{ind}_{\not{D}}[E] = (2i\pi)^{-\frac{n}{2}} \left\langle \hat{A}(R^M)^\wedge, E \right\rangle \qquad \forall E \in K^0(M).$$

Remark

$\mathrm{Ch}(\not{D}) := (2i\pi)^{-\frac{n}{2}} \left[\hat{A}(R^M)^\wedge\right] \in H_{\mathrm{even}}(M)$ is called the *Chern character* of $\not{D}$.

Noncommutative Vector Bundles

Definition

A *finitely generated projective module* over an algebra $\mathcal{A}$ is a (right-)module of the form,

$$\mathcal{E} = e\mathcal{A}^N, \qquad e \in M_N(\mathcal{A}),\ e^2 = e.$$

Theorem (Serre-Swan)

For $\mathcal{A} = C^\infty(M)$ (with M compact manifold), there is a one-to-one correspondence:

$$\begin{array}{rcl} \{\textit{Vector Bundles over } M\} & \longleftrightarrow & \{\textit{f.g. proj. modules over } C^\infty(M)\} \\ E & \longrightarrow & C^\infty(M, E). \end{array}$$

Suppose that $E = \mathrm{ran}(e)$ with $e = e^* = e^2 \in C^\infty(M, M_q(\mathbb{C}))$. Then

$$C^\infty(M, E) = \{\xi = (\xi_j) \in C^\infty(M, \mathbb{C}^q);\ e\xi = \xi\} = eC^\infty(M)^q.$$

Thus,

$$C^\infty(M, \$ \otimes E) = C^\infty(M, \$) \otimes_{C^\infty(M)} C^\infty(M, E) = eC^\infty(M, \$)^q.$$

Definition

The *Grassmanian connection* ∇_0^E of E is defined by

$$\nabla_0^E \xi := e(d\xi_j) \qquad \forall \xi = (\xi_j) \in C^\infty(M, E).$$

Twisted Dirac Operators

Lemma

Under the identification $C^\infty(M, \not{S} \otimes E) = eC^\infty(M, \not{S})^q$, *the twisted Dirac operator* $\not{D}_E = \not{D}_{E,\nabla_0^E}$ *agrees with*

$$e(\not{D} \otimes 1) : eC^\infty(M, \not{S})^q \longrightarrow eC^\infty(M, \not{S})^q,$$
$$[e(\not{D} \otimes 1)]\, s := e(\not{D} s_j) \qquad \forall s = (s_j) \in eC^\infty(M, \not{S})^q.$$

Index Map of a Spectral Triple

Setup

- $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with $\mathcal{A}$ unital.
- $\mathcal{E} = e\mathcal{A}^q$, $e^2 = e \in M_q(\mathcal{A})$, is a f.g. projective module.

Remark

$e\mathcal{H}^q$ is a Hilbert space with grading $e\mathcal{H}^q = e\left(\mathcal{H}^+\right)^q \oplus e\left(\mathcal{H}^-\right)^q$.

Definition

$D_{\mathcal{E}}$ is the (unbounded) operator of $e\mathcal{H}^q$ with domain $e\left(\operatorname{dom} D\right)^q$ and defined by

$$D_{\mathcal{E}}\sigma := e(D_{\sigma_j}) \qquad \forall \sigma = (\sigma_j) \in e\left(\operatorname{dom} D\right)^q .$$

Index Map of a Spectral Triple

Lemma

The operator $D_{\mathcal{E}}$ is Fredholm.

Definition

The index of $D_{\mathcal{E}}$ is

$$\operatorname{ind} D_{\mathcal{E}} := \dim\ker(D_{\mathcal{E}})_{|e(\mathcal{H}^{+})^{q}} - \dim\ker(D_{\mathcal{E}})_{|e(\mathcal{H}^{-})^{q}}.$$

Example

For a Dirac spectral triple $\left(C^{\infty}(M), L^{2}(M, \$), \not{D}\right)$, as we saw before

$$\not{D}_{\mathcal{E}} = \not{D}_{E} \quad \text{with } E := \operatorname{ran}(e).$$

Thus,

$$\operatorname{ind}\not{D}_{\mathcal{E}} = \operatorname{ind}\not{D}_{E}.$$

K-Theory of $\mathcal{A}$

Definition

Two f.g. projective modules $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathcal{A}$ are *stably equivalent* when there exists a f.g. projective module such that

$$\mathcal{E}_1 \oplus \mathcal{F} \simeq \mathcal{E}_2 \oplus \mathcal{F}.$$

Definition

$K_0(\mathcal{A})$ is the Abelian group of formal differences

$$[\mathcal{E}_1] - [\mathcal{E}_2]$$

of stable equivalence classes of f.g. projective modules over $\mathcal{A}$.

Remark

When $\mathcal{A} = C^\infty(M)$, the Serre-Swan theorem implies that

$$K_0\left(C^\infty(M)\right) \simeq K^0(M).$$

The Index Map of a Spectral Triple

Lemma

If $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are f.g. projective modules over* $\mathcal{A}$*, then*

$$\mathrm{ind}\, D_{\mathcal{E}_1 \oplus \mathcal{E}_2} = \mathrm{ind}\, D_{\mathcal{E}_1} + \mathrm{ind}\, D_{\mathcal{E}_2}$$

Proposition

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *defines a unique additive index map,*

$$\mathrm{ind}_D : K_0(\mathcal{A}) \longrightarrow \mathbb{Z},$$

such that, for any f.g. projective module $\mathcal{E}$ *over* $\mathcal{A}$*,*

$$\mathrm{ind}_D[\mathcal{E}] = \mathrm{ind}\, D_{\mathcal{E}}.$$

The Index Map of a Spectral Triple

Example

For a Dirac spectral triple $\left(C^\infty(M), L^2(M, \not{S}), \not{D}\right)$, under the Serre-Swan isomorphism

$$K_0\left(C^\infty(M)\right) \simeq K^0(M),$$

the index map $\mathrm{ind}_{\not{D}} : K_0\left(C^\infty(M)\right) \to \mathbb{Z}$ agrees with the Atiyah-Singer index map,

$$\mathrm{ind}_{\not{D}} : K^0(M) \longrightarrow \mathbb{Z}.$$