Noncommutative Geometry
Chapter 11:
Birman-Solomyak's Weyl Laws. Applications

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Reminder: Weak Schatten Classes

Definition (Weak Schatten Classes $\mathcal{L}^{p,\infty}$)

Let $p \in (0, \infty)$.

1. The weak Schatten class $\mathcal{L}^{p,\infty}$ consists of all $T \in \mathcal{L}(\mathcal{H})$ such that

$$\mu_n(T) = \mathrm{O}\left(n^{-\frac{1}{p}}\right) \qquad \text{as } n \to \infty.$$

2. For $T \in \mathcal{L}(\mathcal{H})$, we set

$$\|T\|_{p,\infty} := \sup_{n \geq 0} (n+1)^{\frac{1}{p}} \mu_n(T).$$

Reminder: Weak Schatten Classes

Proposition

1. *$\mathcal{L}^{p,\infty}$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*
2. *$\|\cdot\|_{p,\infty}$ is a quasi-norm which respect to which $\mathcal{L}^{p,\infty}$ is a quasi-Banach ideal.*

Remark

1. If $p > 1$, then $\|\cdot\|_{p,\infty}$ is equivalent to the norm,

$$\|T\|'_{p,\infty} = \sup_{N\geq 1}\left\{N^{-1+\frac{1}{p}}\sum_{j<N}\mu_j(T)\right\}, \qquad T \in \mathcal{L}^{p,\infty}.$$

2. In this case, $\mathcal{L}^{p,\infty}$ is a Banach ideal (w.r.t. that norm).

Reminder: Weak Schatten Classes

Notation

$\mathcal{L}_0^{p,\infty}$ is the closure in $\mathcal{L}^{p,\infty}$ of the ideal of finite-rank operators.

Proposition

1. *We have*

$$\mathcal{L}_0^{p,\infty} = \left\{ T \in \mathcal{L}(\mathcal{H});\ \mu_n(T) = \mathrm{o}\left(n^{-\frac{1}{p}}\right)\right\}.$$

2. *We have a strict inclusion* $\mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty}$.
3. *In particular,* $\mathcal{L}^{p,\infty}$ *is not separable.*

Remark

For $0 < p < q$ we have strict inclusions,

$$\mathcal{L}^p \subsetneq \mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty} \subsetneq \mathcal{L}^q.$$

Reminder: Eigenvalue Sequences

Definition

An eigenvalue sequence $\lambda(T) = \{\lambda_j(T)\}_{j\geq 0}$ is any sequence s.t.:

1. $\lambda_j(T)$ is an eigenvalue of T and is repeated according to (algebraic) multiplicity.
2. $|\lambda_0(T)| \geq |\lambda_1(T)| \geq \cdots$.

Remarks

1. An eigenvalue sequence need not be unique.
2. If $T \geq 0$, then $\lambda_j(T) = \mu_j(T)$.
3. We shall denote by $\lambda(T)$ *any* eigenvalue sequence for T.

Reminder: Weyl Operators

Definition

If $A \in \mathcal{L}(\mathcal{H})$, its real and imaginary parts are

$$\Re A := \frac{1}{2}\left(A + A^*\right), \qquad \Im A := \frac{1}{2i}\left(A - A^*\right).$$

Definition

If $A^* = A$, its positive and negative parts are

$$A_+ := \frac{1}{2}\left(|A| + A\right), \qquad A_- := \frac{1}{2}\left(|A| - A\right).$$

Reminder: Weyl Operators

Definition

If $A = A^*$ is compact, then $\pm\lambda_j^{\pm}(A)$, $j \geq 0$, are the positive eigenvalues of A such that

$$\lambda_0^{\pm}(A) \geq \lambda_1^{\pm}(A) \geq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

Remark

In other words,

$$\lambda_j^{\pm}(A) = \lambda_j(A_{\pm}) = \mu_j(A_{\pm}), \qquad j \geq 0.$$

Reminder: Weyl Operators

Definition

We say that $A \in \mathcal{L}^{p,\infty}$ is a Weyl operator if one of the following conditions is satisfied:

(i) $A \geq 0$ and $\lim_{j\to\infty} j^{1/p}\lambda_j(A)$ exists.

(ii) $A = A^*$ and $\lim_{j\to\infty} j^{1/p}\lambda_j^{\pm}(A)$ both exist.

(iii) The real and imaginary parts both satisfy (ii).

Notation

The class of Weyl operators in $\mathcal{L}^{p,\infty}$ is denoted $\mathcal{W}^{p,\infty}$.

Reminder: Weyl Operators

Definition

Let A be a Weyl operator in $\mathcal{L}^{p,\infty}$.

1. If $A \geq 0$, then we set
$$\Lambda(A) := \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j(A).$$
2. If $A^* = A$, then we set
$$\Lambda^{\pm}(A) := \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^{\pm}(A).$$
3. In general, we set
$$\Lambda^{\pm}(A) := \Lambda^{\pm}(\Re A) + i\Lambda^{\pm}(\Im A).$$

Proposition (Birman-Solomyak '70s)

1. $\mathcal{W}^{p,\infty}$ *is a closed subset of* $\mathcal{L}^{p,\infty}$ *on which the functions* $\Lambda^{\pm}$ *are continuous.*
2. *If* $A \in \mathcal{W}^{p,\infty}$ *and* $B \in \mathcal{L}_0^{p,\infty}$, *then* $A + B \in \mathcal{W}^{p,\infty}$ *and* $\Lambda^{\pm}(A+B) = \Lambda^{\pm}(A)$.

Remark

In particular, if $A_\ell \to A$ in $\mathcal{L}^{p,\infty}$ with $A_\ell = A_\ell^* \in \mathcal{W}^{p,\infty}$, then $A \in \mathcal{W}^{p,\infty}$, and

$$\Lambda^{\pm}(A) = \lim_{\ell\to\infty} \Lambda^{\pm}(A_\ell).$$

That is,

$$\lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^{\pm}(A) = \lim_{\ell\to\infty} \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^{\pm}(A_\ell).$$

Reminder: Birman-Solomyak Perturbation Theory

Definition

$|\mathcal{W}|^{p,\infty}$ consists of operators $A \in \mathcal{L}^{p,\infty}$ s.t. $|A| \in \mathcal{W}^{p,\infty}$. That is,

$$\lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j(|A|) = \lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A) \text{ exists.}$$

Remark

If $|A| \in \mathcal{W}^{p,\infty}$, then

$$\Lambda\,(|A|) = \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j(|A|) = \lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A).$$

Reminder: Birman-Solomyak Perturbation Theory

Proposition (Birman-Solomyak '70s)

1. $|\mathcal{W}|^{p,\infty}$ *is a closed subset of* $\mathcal{L}^{p,\infty}$ *on which* $A \to \Lambda(|A|)$ *is continuous.*
2. *If* $A \in |\mathcal{W}|^{p,\infty}$ *and* $B \in \mathcal{L}_0^{p,\infty}$, *then* $A + B \in |\mathcal{W}|^{p,\infty}$ *and* $\Lambda(|A+B|) = \Lambda(|A|)$.

Remark

In particular, if $A_\ell \to A$ in $\mathcal{L}^{p,\infty}$ with $A_\ell \in |\mathcal{W}|^{p,\infty}$, then

$$\lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A) = \lim_{\ell\to\infty} \lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A_\ell).$$

Reminder: Weak-Schatten Class Properties of ΨDOs

Setup

- $(M^n, g) =$ closed Riemannian manifold.
- $(\mathcal{E}^r, \| \cdot \|_x) =$ Hermitian vector bundle over M.

Reminder

If $P \in \Psi^{-m}(M, \mathcal{E})$, $m > 0$, then $P \in \mathcal{L}^{p,\infty}$ with $p = nm^{-1}$.

Zeta Functions of Elliptic ΨDOs

We will only need the following result on zeta functions of elliptic ΨDOs from Chapter 9.

Proposition

Let $P \in \Psi^m(M, \mathcal{E})$, $m > 0$, *be elliptic, and let* $A \in \Psi^0(M, \mathcal{E})$. *Then the function* $z \to \operatorname{Tr}[A|P|^{-z}]$ *is holomorphic for* $\Re z > nm^{-1}$ *and has a meromorphic continuation to the half-plane* $\Re z > n(m+1)^{-1}$ *with at worst a simple pole at* $z = nm^{-1}$ *s.t.*

$$m \operatorname{Res}_{z=nm^{-1}} \operatorname{Tr}\left[A|P|^{-z}\right] = \operatorname{Res}\left[A|P|^{-\frac{n}{m}}\right]$$
$$= \iint_{S^*M} \operatorname{tr}_{\mathcal{E}}\left[\sigma_0(A)(x,\xi)|\sigma_m(P)(x,\xi)|^{-\frac{n}{m}}\right] dx đ\xi.$$

Theorem (Ikehara's Tauberian Theorem; see Shubin)

Assume $N(t)$, $t \geq 0$, is a non-decreasing function s.t.

(i) $N(t) = 0$ *near* $t = 0$.

(ii) *The integral* $\zeta(z) := \int t^{-z} dN(t)$ *converges for* $\Re z > q > 0$.

(iii) $\zeta(z)$ *admits a meromorphic extension to a half-plane* $\Re z > q - \epsilon$, $\epsilon > 0$ *with only a simple pole at* $z = q$ *s.t.*

$$\mathrm{Res}_{z=q}\, \zeta(z) = A > 0.$$

Then, we have

$$N(t) \sim \frac{1}{q} A t^q \quad \text{as } t \to \infty.$$

Reminder: Weyl's Law for Positive-Elliptic Operators

Setup

$P \in \Psi^m(M, \mathcal{E})$, $m > 0$, is positive-elliptic, i.e.,

(i) P is elliptic with $\sigma_m(P)(x,\xi) > 0$.

(ii) P is selfadjoint and ≥ 0.

(iii) P is invertible, i.e., $0 \notin \mathrm{Sp}(P)$.

Fact

The spectrum of P can be can be arranged as a non-decreasing sequence,

$$0 < \lambda_0(P) \leq \lambda_1(P) \leq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

Definition

The counting function of P then is

$$N_P(t) := \#\left\{j;\ 0 < \lambda_j(P) \leq t\right\}, \quad t > 0.$$

Reminder: Weyl's Law for Positive-Elliptic Operators

Fact

We have

$$\int t^{-z} dN_P(t) = \sum \lambda_j(P)^{-z} = \operatorname{Tr}\left[P^{-z}\right] = \operatorname{Tr}\left[|P|^{-z}\right]$$

Therefore, by applying the result on zeta functions of elliptic operators and Ikehara's Tauberian we get:

Theorem (Weyl's Law; 1st Version)

As $t \to \infty$, we have

$$N_P(t) \sim \frac{1}{n}\left(\int_{S^*M} \operatorname{Tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}}\right] dx đ\xi\right) t^{\frac{n}{m}}.$$

Reminder: Weyl's Law for Positive-Elliptic Operators

Lemma

We always have

$$\limsup_{j\to\infty} j\lambda_j(P)^{-\frac{m}{n}} = \limsup_{t\to\infty} t^{-\frac{n}{m}} N_P(t),$$
$$\liminf_{j\to\infty} j\lambda_j(P)^{-\frac{m}{n}} = \liminf_{t\to\infty} t^{-\frac{n}{m}} N_P(t).$$

Therefore, we obtain:

Theorem (Weyl's Law; 2nd Version)

As $j \to \infty$ we have

$$\lambda_j(P) \sim j^{\frac{m}{n}} \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}} \right] dx đ\xi \right)^{-\frac{m}{n}}.$$

Reminder: Weyl's Law for Positive-Elliptic Operators

Remark

- These Weyl's laws continue to hold if we only assume that P is selfadjoint and $\sigma_m(P)(x,\xi) > 0$.
- In this case P is bounded from below and has at most finitely many non-positive eigenvalues.
- Thus, if j is large enough, then

$$\lambda_j(P) = \lambda_j(|P| + \Pi_0(P)),$$

  where $\Pi_0(P)$ is the orthogonal projection onto $\ker P$.
- $|P| + \Pi_0(P)$ is positive-elliptic, and $\sigma_m(|P| + \Pi_0(P)) = |\sigma_m(P)| = \sigma_m(P)$
- Thus,

$$\lim_{j\to\infty} j^{-\frac{m}{n}}\lambda_j(P) = \lim_{j\to\infty} j^{-\frac{m}{n}}\lambda_j\big(|P| + \Pi_0(P)\big)$$
$$= \left(\frac{1}{n}\int_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}}\right] dx đ\xi\right)^{-\frac{m}{n}}$$

Weyl's Laws for Selfadjoint Elliptic Operators

Setup

$P \in \Psi^m(M, \mathcal{E})$, $m > 0$, is elliptic and selfadjoint.

Facts

- The spectrum of P consists of isolated real eigenvalues with finite multiplicity.
- The positive/negative eigenvalues $\pm\lambda_j(P)$ of P can be arranged as a non-decreasing sequence,

$$0 < \lambda_0^{\pm}(P) \leq \lambda_1^{\pm}(P) \leq \cdots ,$$

  where each eigenvalue is repeated according to multiplicity.

Remark

- The positive/negative parts of P are
$$P_{\pm} = \frac{1}{2}\left(|P| \pm P\right).$$
- As P is elliptic, $|P| \in \Psi^m(M, \mathcal{E})$, and
$$\sigma_m\left(|P|\right)(x,\xi) = \left|\sigma_m(P)(x,\xi)\right|.$$
- Thus, $P_{\pm} \in \Psi^m(M, \mathcal{E})$, and
$$\begin{aligned}\sigma_m\left(P_{\pm}\right)(x,\xi)s &= \frac{1}{2}\left(|\sigma_m(P)(x,\xi)| \pm \sigma_m(P)(x,\xi)\right)\\ &= \sigma_m(P)(x,\xi)_{\pm}.\end{aligned}$$

Definition

$\Pi_{\pm}(P)$ is the orthogonal projection onto the positive/negative eigenspace of P, i.e.,

$$E_{\pm}(P) := \bigoplus_{\pm\lambda>0} \ker(P-\lambda).$$

Lemma

$\Pi_{\pm}(P) \in \Psi^0(M,\mathcal{E})$, and

$$\sigma_0\left(\Pi_{\pm}(P)\right)(x,\xi) = \pi_{\pm}\left(\sigma_m(x,\xi)\right),$$

where $\pi_{\pm}\left(\sigma_m(x,\xi)\right)$, $(x,\xi) \in T^*M\setminus 0$, is the orthogonal projection onto the positive/negative eigenspace of $\sigma_m(P)(x,\xi)$.

Proof.

- Let $F(P) = \text{sign}(P) = P|P|^{-1}$ be the sign of P.
- Let $\Pi_0(P)$ be the orthogonal projection onto $\ker P$.
- We have
$$\Pi_\pm(P) = \frac{1}{2}(1 \pm F(P)) - \frac{1}{2}\Pi_0(P).$$
- Here $|P|^{-1} \in \Psi^{-m}(M)$, and
$$\sigma_{-m}(|P|^{-1})(x,\xi) = \sigma_m(|P|)(x,\xi)^{-1} = |\sigma_m(P)(x,\xi)|^{-1}.$$
- Thus, $F(P) = P|P|^{-1} \in \Psi^0(M,\mathcal{E})$, and
$$\sigma_0\left(F(P)\right) = \sigma_m(P)(x,\xi)\,|\sigma_m(P)(x,\xi)|^{-1} = \text{sign}\left(\sigma_m(P)(x,\xi)\right).$$
- In addition, $\Pi_0(P)$ is a smoothing operator.
- It follows that $\Pi_\pm(P) \in \Psi^0(M,\mathcal{E})$, and
$$\sigma_0\left(\Pi_\pm(P)\right)(x,\xi) = \frac{1}{2}\left(1 \pm \text{sign}\left(\sigma_m(P)(x,\xi)\right)\right) = \pi_\pm\left(\sigma_m(x,\xi)\right).$$

□

Weyl's Laws for Selfadjoint Elliptic Operators

Definition

The counting functions of P are

$$N_P^{\pm}(t) := \#\left\{j;\ 0 < \lambda_j^{\pm}(P) \leq t\right\}, \quad t > 0.$$

Theorem (Weyl's Laws for Selfadjoint Elliptic ΨDOs)

1. *We have*

$$\lim_{t\to\infty} t^{-\frac{n}{m}} N_P^{\pm}(t) = \frac{1}{n}\int_{S^*M} \operatorname{Tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)_{\pm}^{-\frac{n}{m}}\right] dx đ\xi.$$

2. *We have*

$$\lim_{j\to\infty} j^{-\frac{m}{n}} \lambda_j^{\pm}(P) = \left(\frac{1}{n}\int_{S^*M} \operatorname{Tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)_{\pm}^{-\frac{n}{m}}\right] dx đ\xi\right)^{-\frac{m}{n}}.$$

Proof.

The 2nd part follows from the 1st part, so we only have to prove the 1st part. □

Weyl's Laws for Selfadjoint Elliptic Operators

Proof.

- For $\Re z > nm^{-1}$, we have

$$\zeta^{\pm}(z) := \int t^{-z} dN_P^{\pm}(t) = \sum \lambda_j^{\pm}(P)^{-z} = \operatorname{Tr}\left[P_{\pm}^{-z}\right] = \operatorname{Tr}\left[\Pi_{\pm}(P)|P|^{-z}\right].$$

- Here $\Pi_{\pm}(P) \in \Psi^0(M,\mathcal{E})$ and P is elliptic.
- The result on zeta functions of elliptic ΨDOs then ensures that $\zeta^{\pm}(z)$ has a meromorphic continuation to the half-plane $\Re z > n(m+1)^{-1}$ with at worst a pole at $z = nm^{-1}$ such that

$$\operatorname{Res}_{z=nm^{-1}} \zeta^{\pm}(z) = \operatorname{Res}_{z=nm^{-1}} \operatorname{Tr}\left[\Pi_{\pm}(P)|P|^{-z}\right] = \frac{1}{m} \operatorname{Res}\left[\Pi_{\pm}(P)|P|^{-\frac{n}{m}}\right].$$

- Applying Ikehara's Tauberian theorem then gives

$$\lim_{t\to\infty} t^{-\frac{n}{m}} N_P^{\pm}(t) = \frac{1}{n} \operatorname{Res}\left[\Pi_{\pm}(P)|P|^{-\frac{n}{m}}\right].$$

□

Proof.

- It remains to evaluate $\operatorname{Res}[\Pi_{\pm}(P)|P|^{-n/m}]$.
- We have

$$\begin{aligned}\sigma_{-n}\left(\Pi_{\pm}(P)|P|^{-\frac{n}{m}}\right)(x,\xi) &= \sigma_0\left(\Pi_{\pm}(P)\right)(x,\xi)\sigma_{-n}\left(|P|^{-\frac{n}{m}}\right)(x,\xi)\\ &= \pi_{\pm}\left(\sigma_m(x,\xi)\right)|\sigma_m(P)(x,\xi)|^{-\frac{n}{m}}\\ &= \sigma_m(P)(x,\xi)_{\pm}^{-\frac{n}{m}}\end{aligned}$$

- Thus,

$$\begin{aligned}\operatorname{Res}\left[\Pi_{\pm}(P)|P|^{-\frac{n}{m}}\right] &= \iint_{S^*M} \operatorname{tr}_{\mathcal{E}}\left[\sigma_{-n}\left(\Pi_{\pm}(P)|P|^{-\frac{n}{m}}\right)(x,\xi)\right] dx đ\xi\\ &= \iint_{S^*M} \operatorname{tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)_{\pm}^{-\frac{n}{m}}\right] dx đ\xi\end{aligned}$$

- This gives the result.

□

Birman-Solomyak Weyl's Laws

Theorem (Birman-Solomyak '70s)

Let $P \in \Psi^{-m}(M, \mathcal{E})$, $m > 0$, *and set* $p = nm^{-1}$.

1. P *and* $|P|$ *are both Weyl operators in* $\mathcal{L}^{p,\infty}$.
2. *We have*

$$\lim_{j \to \infty} j^{\frac{1}{p}} \mu_j(P) = \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[|\sigma_{-m}(P)(x,\xi)|^p \right] dx đ\xi \right)^{\frac{1}{p}}.$$

3. *If* $P^* = P$, *then*

$$\lim_{j \to \infty} j^{\frac{1}{p}} \lambda_j^{\pm}(P) = \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[\sigma_{-m}(P)(x,\xi)_{\pm}^{p} \right] dx đ\xi \right)^{\frac{1}{p}}.$$

Birman-Solomyak Weyl's Laws

Remarks

- The details of the original proof of Birman-Solomyak are only available in Russian in a paper which is hard to find.
- Their proof is very condensed and can be hard to understand for people not familiar with their work.
- They actually assume little regularity on the symbols the ΨDOs.
- I'll present here the soft proof I got in a recent paper.

Proof.

Step 1: $P \in \Psi^{-(m+1)}(M, \mathcal{E})$.

- In this case $\sigma_{-m}(P)(x, \xi) = 0$.
- Moreover, $P \in \mathcal{L}^{q,\infty}$, with $q = n(m+1)^{-1} < p$.
- Thus $P \in \mathcal{L}_0^{p,\infty}$, and so we have

$$\lim_{j \to \infty} j^{\frac{1}{p}} \mu_j(P) = 0 = \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[|\sigma_{-m}(P)(x,\xi)|^p \right] dx đ\xi \right)^{\frac{1}{p}}.$$

□

Proof.

Step 2: $P = Q^{-1}$, where $Q \in \Psi^m(M, \mathcal{E})$ is positive-elliptic.

- Here $\sigma_{-m}(P)(x,\xi) = \sigma_m(Q)(x,\xi)^{-1}$.
- We also have $\mu_j(P) = \lambda_j(P^{-1}) = \lambda_j(P)^{-1}$.
- Thus, the Weyl's law for positive-elliptic ΨDOs gives

$$\begin{aligned}
\lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(P) &= \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j(Q)^{-1} \\
&= \big(\lim_{j\to\infty} j^{-\frac{1}{p}} \lambda_j(Q)\big)^{-1} \\
&= \left(\frac{1}{n}\int_{S^*M} \operatorname{Tr}_{\mathcal{E}}\left[\sigma_m(Q)(x,\xi)^{-p}\right] dx đ\xi\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{n}\int_{S^*M} \operatorname{Tr}_{\mathcal{E}}\left[\sigma_{-m}(P)(x,\xi)^{p}\right] dx đ\xi\right)^{\frac{1}{p}}.
\end{aligned}$$

□

Proof.

Step 3: $\sigma_{-m}(P)(x,\xi) > 0$.

- As $\sigma_{-m}(P)(x,\xi)^{-1} > 0$, there is a positive-elliptic operator $Q \in \Psi^m(M,\mathcal{E})$ such that $\sigma_m(Q) = \sigma_{-m}(P)^{-1}$.
- $Q^{-1} \in \Psi^{-m}(M,\mathcal{E})$ and $\sigma_{-m}(Q^{-1}) = \sigma_m(Q)^{-1} = \sigma_{-m}(P)$.
- Thus, $R := P - Q^{-1} \in \Psi^{-(m+1)}(M,\mathcal{E})$, and hence $R \in \mathcal{L}_0^{p,\infty}$ by Step 1.
- As Q is positive-elliptic, Step 2 shows that Q^{-1} is a Weyl operator in $\mathcal{L}^{p,\infty}$.
- It follows that $P = Q^{-1} + R$ and $|P|$ are Weyl operators in $\mathcal{L}^{p,\infty}$, and we have

$$\Lambda(P) = \Lambda(Q^{-1}) = \left(\frac{1}{n}\int_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(Q^{-1})(x,\xi)^p\right] dx đ\xi\right)^{\frac{1}{p}}$$
$$= \left(\frac{1}{n}\int_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(P)(x,\xi)^p\right] dx đ\xi\right)^{\frac{1}{p}}.$$

□

The BKS Inequality

Proposition (BKS Inequality, Birman-Koplienko-Solomyak '70)

Let T_1 *and* T_2 *be positive operators in* $\mathcal{L}(\mathcal{H})$ *such that* $T_1 - T_2 \in \mathcal{L}^{p,\infty}$. *Then* $\sqrt{T_1} - \sqrt{T_2} \in \mathcal{L}^{2p,\infty}$, *and*

$$\|\sqrt{T_1} - \sqrt{T_2}\|_{2p,\infty} \leq C_p \sqrt{\|T_1 - T_2\|_{p,\infty}},$$

where the constant C_p *does not depend on* T_1 *or* T_2.

Proof.

Step 4: Approximation of $|P|$.

- Let $Q_0 \in \Psi^m(M, \mathcal{E})$ be positive-elliptic with $\sigma_m(Q_0)(x,\xi) = |\xi|_g^m \operatorname{id}_{\mathcal{E}_x}$.
- For $\epsilon > 0$ set $A_\epsilon = \sqrt{P^*P + \epsilon^2 Q_0^{-2}}$.
- We have

$$A_\epsilon^2 - |P|^2 = P^*P + \epsilon^2 Q_0^{-2} - P^*P = \epsilon^2 Q_0^{-2} \in \Psi^{-2m}(M, \mathcal{E}).$$

- As $\Psi^{-2m}(M,\mathcal{E}) \subset \mathcal{L}^{p/2,\infty}$, we see that $A_\epsilon^2 - |P|^2 \in \mathcal{L}^{p/2,\infty}$.
- The BKS inequality then ensures that

$$\begin{aligned} \big\| A_\epsilon - |P| \big\|_{p,\infty} &\leq C_p \sqrt{\| A_\epsilon^2 - |P|^2 \|_{\frac{p}{2},\infty}} \\ &\leq C_p \epsilon \sqrt{\| Q_0^{-2} \|_{\frac{p}{2},\infty}} \longrightarrow 0. \end{aligned}$$

- That is, $A_\epsilon \to |P|$ in $\mathcal{L}_{p,\infty}$.

□

Proof.

Step 5: A_ϵ is a Weyl operator.

- Here $A_\epsilon^2 = P^*P + \epsilon^2 Q_0^{-2} \in \Psi^{-2m}(M, \mathcal{E})$, and
$$\begin{aligned}\sigma_{-2m}(A_\epsilon^2) &= \sigma_{-m}(P)^*\sigma_{-m} + \epsilon^2\sigma_{-2m}(Q_0)^{-2} \\ &= |\sigma_m(P)|^2 + \epsilon^2|\xi|_g^{-2m} > 0.\end{aligned}$$
- Let $Q_\epsilon \in \Psi^m(M, \mathcal{E})$ be positive-elliptic with
$$\sigma_m(Q_\epsilon)(x,\xi) = \left(|\sigma_m(P)(x,\xi)|^2 + \epsilon^2|\xi|_g^{-2m}\right)^{-\frac{1}{2}} = \sigma_{-2m}(A_\epsilon^2)(x,\xi)^{-\frac{1}{2}}.$$
- Here $Q_\epsilon^{-2} \in \Psi^{-2m}(M, \mathcal{E})$, and
$$\sigma_{-2m}(Q_\epsilon^{-2})(x,\xi) = \sigma_m(Q_\epsilon)(x,\xi)^{-2} = \sigma_{-2m}(A_\epsilon^2)(x,\xi).$$
- Thus, $A_\epsilon^2 - Q_\epsilon^{-2}$ is in $\Psi^{-(2m+1)}(M, \mathcal{E})$, and hence is in $\mathcal{L}^{q,\infty}$ with $q = n(2m+1)^{-1}$.

□

Proof.

- As $A_\epsilon^2 - Q_\epsilon^{-2} \in \mathcal{L}^{q,\infty}$ with $q = n(2m+1)^{-1} < \frac{1}{2}p$, the BKS inequality implies that
$$A_\epsilon - Q_\epsilon^{-1} \in \mathcal{L}^{2q,\infty} \subset \mathcal{L}_0^{p,\infty}.$$
- By Step 2 Q_ϵ^{-1} is a Weyl operator in $\mathcal{L}^{p,\infty}$.
- Birman-Solomyak's perturbation theory and Step 2 ensure that A_ϵ is a Weyl operator, and

$$\Lambda(A_\epsilon) = \Lambda\left(Q_\epsilon^{-1}\right) = \left(\frac{1}{n}\int_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(Q_\epsilon^{-1})(x,\xi)^p\right] dx đ\xi\right)^{\frac{1}{p}}$$

□

Proof.

Step 6: $|P|$ is a Weyl operator.

- By Step 4 $A_\epsilon \to |P|$ in $\mathcal{L}^{p,\infty}$.
- By Step 5 each operator A_ϵ is a Weyl operator in $\mathcal{L}^{p,\infty}$.
- Birman-Solomyak's perturbation theory then ensures that $|P|$ is a Weyl operator in $\mathcal{L}^{p,\infty}$, and we have

$$\begin{aligned}\Lambda(|P|) &= \lim_{\epsilon\to 0^+} \Lambda(A_\epsilon) \\ &= \lim_{\epsilon\to 0^+} \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[\sigma_{-m}(Q_\epsilon^{-1})(x,\xi)^p \right] dx đ\xi \right)^{\frac{1}{p}}\end{aligned}$$

□

Proof.

- We have

$$\begin{aligned}\sigma_{-m}(Q_\epsilon^{-1})(x,\xi) &= \sigma_{-m}(Q_\epsilon)(x,\xi)^{-1} \\ &= \left(|\sigma_{-m}(P)(x,\xi)|^2 + \epsilon^2|\xi|_g^{-2m}\right)^{\frac{1}{2}} \\ &\longrightarrow |\sigma_{-m}(P)(x,\xi)|.\end{aligned}$$

- Thus,

$$\begin{aligned}\Lambda(|P|) &= \lim_{\epsilon\to 0^+}\left(\frac{1}{n}\int_{S^*M}\mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(Q_\epsilon^{-1})(x,\xi)^p\right]dx\,đ\xi\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{n}\int_{S^*M}\mathrm{Tr}_{\mathcal{E}}\left[|\sigma_{-m}(P)(x,\xi)|^p\right]dx\,đ\xi\right)^{\frac{1}{p}}.\end{aligned}$$

- That is,

$$\lim_{j\to\infty} j^{\frac{1}{p}}\mu_j(P) = \left(\frac{1}{n}\int_{S^*M}\mathrm{Tr}_{\mathcal{E}}\left[|\sigma_{-m}(P)(x,\xi)|^p\right]dx\,đ\xi\right)^{\frac{1}{p}}.$$

□

Proof.

Step 6: $P^* = P$, approximation of P_+.

- From now on we assume that $P^* = P$.
- We have
$$P_+ = \frac{1}{2}\left(|P| + P\right).$$
- For $\epsilon > 0$ we set
$$B_\epsilon = \frac{1}{2}\left(A_\epsilon + P\right).$$
- By Step 4 as $\epsilon \to 0^+$ and in $\mathcal{L}^{p,\infty}$ we have
$$B_\epsilon = \frac{1}{2}\left(A_\epsilon + P\right) \longrightarrow \frac{1}{2}\left(|P| + P\right) = P_+.$$

□

Proof.

Step 7: B_ϵ is a Weyl operator.

- Set $\tilde{B}_\epsilon = \frac{1}{2}(Q_\epsilon^{-1} + P)$.
- Here $\tilde{B}_\epsilon \in \Psi^{-m}(M, \mathcal{E})$, and

$$\begin{aligned}\sigma_{-m}\left(\tilde{B}_\epsilon\right) &= \frac{1}{2}\left(\sigma_{-m}\left(Q_\epsilon^{-1}\right) + \sigma_{-m}(P)\right)\\ &= \frac{1}{2}\left(\sqrt{|\sigma_{-m}(P)|^2 + \epsilon^2|\xi|^{-2m}} + \sigma_{-m}(P)\right)\\ &\geq \frac{\epsilon}{2}|\xi|^{-m} > 0.\end{aligned}$$

- Step 3 then ensures that $\tilde{B}_\epsilon$ is a Weyl operator in $\mathcal{L}^{p,\infty}$, and

$$\Lambda\left(\tilde{B}_\epsilon\right) = \left(\frac{1}{n}\int_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(\tilde{B}_\epsilon)(x,\xi)^p\right] dx đ\xi\right)^{\frac{1}{p}}.$$

□

Proof.

- By Step 5 we have

$$\begin{aligned}\tilde{B}_\epsilon - B_\epsilon &= \frac{1}{2}\left(Q_\epsilon^{-1} + P\right) - \frac{1}{2}\left(A_\epsilon + P\right)\\ &= \frac{1}{2}\left(Q_\epsilon^{-1} - A_\epsilon\right) \in \mathcal{L}_0^{p,\infty}.\end{aligned}$$

- As $\tilde{B}_\epsilon$ is a Weyl operator, Birman-Solomyak's perturbation theory ensures that B_ϵ is a Weyl operator, and we have

$$\Lambda\left(B_\epsilon\right) = \Lambda\left(\tilde{B}_\epsilon\right) = \left(\frac{1}{n}\int_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(\tilde{B}_\epsilon)(x,\xi)^p\right] dx đ\xi\right)^{\frac{1}{p}}.$$

□

Proof.

Step 8: P_+ is a Weyl operator.

- By Step 6 $B_\epsilon \to P_+$ in $\mathcal{L}^{p,\infty}$.
- By Step 7 B_ϵ is a Weyl operator in $\mathcal{L}^{p,\infty}$.
- Thus, by Birman-Solomyak's perturbation theory P_+ is a Weyl operator in $\mathcal{L}^{p,\infty}$, and we have

$$\begin{aligned}\Lambda(P_+) &= \lim_{\epsilon \to 0^+} \Lambda(B_\epsilon) \\ &= \lim_{\epsilon \to 0^+} \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[\sigma_{-m}(\tilde{B}_\epsilon)(x,\xi)^p \right] dx đ\xi \right)^{\frac{1}{p}}.\end{aligned}$$

□

Proof.

- As shown in Step 7, we have

$$\sigma_{-m}(\tilde{B}_\epsilon) = \frac{1}{2}\left(\sqrt{|\sigma_{-m}(P)|^2 + \epsilon^2|\xi|^{-2m}} + \sigma_{-m}(P)\right)$$
$$\longrightarrow \frac{1}{2}\left(|\sigma_{-m}(P)| + \sigma_{-m}(P)\right) = \sigma_{-m}(P)_+.$$

- Thus,

$$\Lambda(P_+) = \lim_{\epsilon\to 0^+}\left(\frac{1}{n}\int_{S^*M}\mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(\tilde{B}_\epsilon)(x,\xi)^p\right]dx đ\xi\right)^{\frac{1}{p}}$$
$$= \left(\frac{1}{n}\int_{S^*M}\mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(P)(x,\xi)_+^p\right]dx đ\xi\right)^{\frac{1}{p}}.$$

- This means that

$$\lim_{j\to 0} j^{\frac{1}{p}}\lambda_j^+(P) = \left(\frac{1}{n}\int_{S^*M}\mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-m}(P)(x,\xi)_+^p\right]dx đ\xi\right)^{\frac{1}{p}}.$$

□

Birman-Solomyak Weyl's Laws

Proof.

Step 9: P is a Weyl operator.

- Replacing by P by $-P$ shows that $P_- = (-P)_+$ is a Weyl operator, and

$$\lim_{j\to 0} j^{\frac{1}{p}} \lambda_j^-(P) = \left(\frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[\sigma_{-m}(P)(x,\xi)_-^p \right] dx đ\xi \right)^{\frac{1}{p}}.$$

- Thus, P is a Weyl operator in $\mathcal{L}^{p,\infty}$.

□

Proof.

Step 10: $P \neq P^*$.

- If $P^* \neq P$, then $\Re P = \frac{1}{2}(P + P^*)$ and $\Im P = \frac{1}{2i}(P + P^*)$ are selfadjoint operators in $\Psi^{-m}(M, \mathcal{E})$.
- Therefore, they are Weyl operators in $\mathcal{L}^{p,\infty}$ by Step 9.
- This shows that P is a Weyl operator $\mathcal{L}^{p,\infty}$.

The proof is complete. □

Reminder: Weyl Operator and NC Integration

Proposition

Let A be a Weyl operator in $\mathcal{L}^{1,\infty}$.

1. *A is strongly measurable, and we have*
$$\fint A = \Lambda^{+}(A) - \Lambda^{-}(A).$$
2. *In particular, if $A^* = A$, then*
$$\fint A = \lim_{j\to\infty} j\lambda_j^{+}(A) - \lim_{j\to\infty} j\lambda_j^{-}(A).$$

Corollary

If $|A|$ is a Weyl operator in $\mathcal{L}^{1,\infty}$, then $|A|$ is strongly measurable, and
$$\fint |A| = \lim_{j\to\infty} j\mu_j(A).$$

Connes' Trace Theorem Revisited

For ΨDOs of order $-n$ Birman-Solomyak's Weyl's law yields:

Theorem (Birman-Solomyak '70s)

Let $P \in \Psi^{-n}(M, \mathcal{E})$.

1. P *and* $|P|$ *are both Weyl operators in* $\mathcal{L}^{1,\infty}$.
2. *We have*

$$\lim_{j \to \infty} j\mu_j(P) = \frac{1}{n} \iint_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[|\sigma_{-n}(P)(x,\xi)| \right] dx đ\xi.$$

3. *If* $P^* = P$, *then*

$$\lim_{j \to \infty} j\lambda_j^{\pm}(P) = \frac{1}{n} \iint_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[\sigma_{-n}(P)(x,\xi)_{\pm} \right] dx đ\xi.$$

Connes' Trace Theorem Revisited

Corollary (Connes' Trace Theorem)

If $P \in \Psi^{-n}(M, \mathcal{E})$, *then* P *is strongly measurable, and we have*

$$⨍ P = \frac{1}{n} \operatorname{Res}(P).$$

Proof.

- As P is a Weyl operator in $\mathcal{L}^{1,\infty}$, it is strongly measurable.
- Moreover, if $P^* = P$, then

$$\begin{aligned} ⨍ P &= \lim_{j\to\infty} j\lambda_j^+(P) - \lim_{j\to\infty} j\lambda_j^-(P) \\ &= \frac{1}{n}\iint_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-n}(P)(x,\xi)_+ - \sigma_{-n}(P)(x,\xi)_-\right] dx đ\xi \\ &= \frac{1}{n}\iint_{S^*M} \mathrm{Tr}_{\mathcal{E}}\left[\sigma_{-n}(P)(x,\xi)\right] dx đ\xi \\ &= \frac{1}{n}\mathrm{Res}(P). \end{aligned}$$

- By linearity we get the result for any $P \in \Psi^{-n}(M,\mathcal{E})$.

□

We also get a version of Connes' trace theorem for absolute values.

Corollary

If $P \in \Psi^{-n}(M, \mathcal{E})$, then $|P|$ is strongly measurable, and we have

$$\int\!\!\!\!\!- |P| = \frac{1}{n} \iint_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[|\sigma_{-n}(P)(x,\xi)| \right] dx đ\xi.$$

Proof.

- As $|P|$ is a Weyl operator in $\mathcal{L}^{1,\infty}$, it is strongly measurable, and we have
- Moreover, we have

$$\int\!\!\!\!\!- P = \lim_{j\to\infty} j\mu_j(P) \quad = \frac{1}{n} \iint_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[|\sigma_{-n}(P)(x,\xi)| \right] dx đ\xi.$$

□

Reminder: Birman-Schwinger Principle

Setup

- $H =$ selfadjoint (unbounded) operator with non-negative spectrum.
- $V =$ selfadjoint operator such that $(1 + H)^{-1/2} V (1 + H)^{-1/2}$ is compact.

Facts

- The operator $H_V := H + V$ is selfadjoint.
- The negative part of its spectrum is discrete, i.e., it consists of isolated eigenvalues with finite multiplicity.

Definition

$N^{-}(H_V)$ is the number of negative eigenvalues counted with multiplicity.

Remider: Birman-Schwinger Principle

Proposition (Birman-Schwinger Principle)

Assume that

- 0 *is an isolated eigenvalue of* H *with finite multiplicity.*
- $H^{-1/2}VH^{-1/2}$ *is a Weyl operator in* $\mathcal{L}^{p,\infty}$, $p>0$.

Then, we have

$$\lim_{h\to 0^+} h^{2p} N^-\left(h^2 H + V\right) = \left(\lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^-\left(H^{-\frac{1}{2}} V H^{-\frac{1}{2}}\right)\right)^p$$
$$= \int\!\!\!\!\!\!- \left(H^{-\frac{1}{2}} V H^{-\frac{1}{2}}\right)_-^p .$$

Semiclassical Weyl's Laws

Notation

$$c(n) = \frac{1}{n}(2\pi)^{-n}|\mathbb{S}^{n-1}| = (2\pi)^{-n}|\mathbb{B}^n|.$$

Theorem (Birman-Solomyak)

If $f \in C^\infty(M, \mathbb{R})$, then

$$\lim_{j\to\infty} j^{\frac{2}{n}} \lambda_j^{\pm}\left(\Delta_g^{-\frac{1}{2}} f \Delta_g^{-\frac{1}{2}}\right) = \left(c(n)\int_M f_{\pm}(x)^{\frac{n}{2}} d\nu_g(x)\right)^{\frac{2}{n}},$$

where $f_{\pm}(x) = \max(0, \pm f(x))$ are the positive/negative parts of f.

Proof.

- We apply Birman-Solomyak's result to $P = \Delta_g^{-1/2} f \Delta_g^{-1/2}$.
- This is a selfadjoint ΨDO of order -2.
- $\sigma_{-1}(\Delta_g^{-1/2}) = \sigma_2(\Delta_g)^{-1/2} = |\xi|_g^{-1}$.
- Thus,

  $$\sigma_{-2}(\Delta_g^{-1/2} f \Delta_g^{-1/2}) = \sigma_{-1}(\Delta_g^{-1/2}) f \sigma_{-1}(\Delta_g^{-1/2}) = f(x)|\xi|_g^{-2}.$$

- In particular $\sigma_{-2}(\Delta_g^{-1/2} f \Delta_g^{-1/2})(x,\xi)_\pm = f_\pm(x)|\xi|_g^{-2}$.
- Birman-Solomyak's Weyl's law then gives

$$\begin{aligned}\lim_{j\to\infty} j^{\frac{2}{n}} \lambda_j^\pm\left(\Delta_g^{-\frac{1}{2}} f \Delta_g^{-\frac{1}{2}}\right) &= \left(\frac{1}{n}\iint_{S^*M} f_\pm(x)|\xi|_g^{-2} dx đ\xi\right)^{\frac{2}{n}} \\ &= \left(c(n)\int_M f_\pm(x)^{\frac{n}{2}} d\nu_g(x)\right)^{\frac{2}{n}}.\end{aligned}$$

□

Remark

Cwikel's estimates and Birman-Solomyak's perturbation theory ensure that the previous spectral asymptotics continues to hold in the following situations:

- $n \geq 3$ and $f \in L^{\frac{n}{2}}(M)$.
- $n = 2$ and $f \in L\log L(M)$.
- $n = 1$ and $f \in L^1(M)$.

Semiclassical Weyl's Laws

Combining the previous spectral asymptotics with the Birman-Schwinger principle gives:

Theorem (Semiclassical Weyl's Law)

If $V \in C^\infty(M, \mathbb{R})$*, then*

$$\lim_{h \to 0^+} h^n N^- \left(h^2 \Delta_g + V\right) = c(n) \int_M V_-(x)^{\frac{n}{2}} d\nu_g(x).$$

More generally, we have:

Proposition (Birman-Solomyak)

Let $q > 0$. The following hold:

1. *For all $f \in C^\infty(M)$, have*

$$\lim_{j\to\infty} j^{\frac{q}{n}} \mu_j \left(\Delta_g^{-\frac{q}{4}} f \Delta_g^{-\frac{q}{4}} \right) = \left(c(n) \int_M |f(x)|^{\frac{n}{q}} d\nu_g(x) \right)^{\frac{q}{n}}.$$

2. *For all $f \in C^\infty(M, \mathbb{R})$, have*

$$\lim_{j\to\infty} j^{\frac{q}{n}} \lambda_j^{\pm} \left(\Delta_g^{-\frac{q}{4}} f \Delta_g^{-\frac{q}{4}} \right) = \left(c(n) \int_M f_\pm(x)^{\frac{n}{q}} d\nu_g(x) \right)^{\frac{q}{n}}.$$

3. *For all $V \in C^\infty(M, \mathbb{R})$, we have*

$$\lim_{h\to 0^+} h^n N^- \left(h^{2q} \Delta_g^q + V \right) = c(n) \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x).$$