Noncommutative Geometry
Chapter 10:
Connes' Trace Theorem, Integration Formula, and Lower Dimensional Volumes

Sichuan University, Fall 2022

Reminder: Weak Schatten Classes

Definition (Weak Schatten Classes $\mathcal{L}^{p,\infty}$)

Let $p \in (0, \infty)$.

1. The weak Schatten class $\mathcal{L}^{p,\infty}$ consists of all $T \in \mathcal{L}(\mathcal{H})$ such that
$$\mu_n(T) = \mathrm{O}\left(n^{-\frac{1}{p}}\right) \qquad \text{as } n \to \infty.$$
2. For $T \in \mathcal{L}(\mathcal{H})$, we set
$$\|T\|_{p,\infty} := \sup_{n \geq 0} (n+1)^{\frac{1}{p}} \mu_n(T).$$

Reminder: Weak Schatten Classes

Proposition

1. *$\mathcal{L}^{p,\infty}$ is a two-sided ideal of $\mathcal{L}(\mathcal{H})$.*
2. *$\|\cdot\|_{p,\infty}$ is a quasi-norm which respect to which $\mathcal{L}^{p,\infty}$ is a quasi-Banach ideal.*

Remark

1. If $p > 1$, then $\|\cdot\|_{p,\infty}$ is equivalent to the norm,
$$\|T\|'_{p,\infty} = \sup_{N\geq 1}\left\{N^{-1+\frac{1}{p}}\sum_{j<N}\mu_j(T)\right\}, \qquad T \in \mathcal{L}^{p,\infty}.$$
2. In this case, $\mathcal{L}^{p,\infty}$ is a Banach ideal (w.r.t. that norm).

Reminder: Weak Schatten Classes

Notation

$\mathcal{L}_0^{p,\infty}$ is the closure in $\mathcal{L}^{p,\infty}$ of the ideal of finite-rank operators.

Proposition

1. *We have*

$$\mathcal{L}_0^{p,\infty} = \left\{ T \in \mathcal{L}(\mathcal{H});\ \mu_n(T) = \mathrm{o}\left(n^{-\frac{1}{p}}\right) \right\}.$$

2. *We have a strict inclusion* $\mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty}$.
3. *In particular,* $\mathcal{L}^{p,\infty}$ *is not separable.*

Remark

For $0 < p < q$ we have strict inclusions,

$$\mathcal{L}^p \subsetneq \mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty} \subsetneq \mathcal{L}^q.$$

Reminder: Quantized Calculus

Classical	**Quantum** (Connes)
Complex variable	Operator on Hilbert space $\mathcal{H}$
Real variable	Selfadjoint operator on $\mathcal{H}$
Infinitesimal variable	Compact operator on $\mathcal{H}$
Infinitesimal of order α	Compact operator s.t. $\mu_j(T) = \mathrm{O}(j^{-\alpha})$
Integral $\int f(x)dx$	NC integral $⨍ T$

Here the $\mu_j(A)$ are the singular values of A.

Reminder: Quantized Calculus

Definition (Infinitesimal Operator)

An operator $T \in \mathcal{L}(\mathcal{H})$ is infinitesimal if, for all $\epsilon > 0$, there is a subspace $E \subset \mathcal{H}$, $\dim E < \infty$, such that

$$\left\| T_{|E^\perp} \right\| < \epsilon.$$

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. Then TFAE

1. *T is an infinitesimal operator.*
2. *$\mu_j(T) \to 0$ as $j \to \infty$.*
3. *T is a compact operator.*

Reminder: Quantized Calculus

Definition

A (compact) operator T is an infinitesimal of order α, $\alpha > 0$, if

$$\mu_j(T) = O(j^{-\alpha}) \quad \text{as } j \to \infty.$$

That is,

$$T \in \mathcal{L}^{p,\infty} \qquad \text{with } p = \alpha^{-1}.$$

Reminder: NC Integral

Ansatz for a NC Integral (Connes)

The NC integral should have the following properties:

1. It is defined on infinitesimals of order 1, i.e., on the weak trace class $\mathcal{L}^{1,\infty}$.
2. It should take non-negative values on positive operators.
3. It vanishes on infinitesimals of order > 1.
4. It vanishes on the commutator space,

$$\mathrm{Com}(\mathcal{L}^{1,\infty}) = \mathrm{Span}\left\{[A, T];\ A \in \mathcal{L}(\mathcal{H}),\ T \in \mathcal{L}^{1,\infty}\right\}.$$

That is, it should be a positive trace on $\mathcal{L}^{1,\infty}$.

Remark

- It can be shown that $\text{Com}(\mathcal{L}^{1,\infty})$ contains trace-class operators, including infinitesimals of order > 1.
- Thus, the 3rd condition is encapsulated by the 4th condition.

Remarks

1. Any positive trace on $\mathcal{L}^{1,\infty}$ is continuous.
2. Any continuous trace is linear combination of positive traces.

Reminder: NC Integral

Definition

An eigenvalue sequence $\lambda(T) = \{\lambda_j(T)\}_{j\geq 0}$ is any sequence s.t.:

1. $\lambda_j(T)$ is an eigenvalue of T and is repeated according to (algebraic) multiplicity.
2. $|\lambda_0(T) \geq |\lambda_1(T)| \geq \cdots$.

Remarks

1. An eigenvalue sequence need not be unique.
2. If $T \geq 0$, then $\lambda_j(T) = \mu_j(T)$.
3. We shall denote by $\lambda(T)$ *any* eigenvalue sequence for T.

Reminder: NC Integral

Notation

- $\ell^\infty = C^*$-algebra of bounded complex-valued sequences.
- $\mathfrak{c}_0 =$ closed ideal of sequences converging to 0.

Definition

An extended limit is any positive linear map $\lim_\omega : \ell^\infty \to \mathbb{C}$ s.t.:

(i) $\lim_\omega 1 = 1$.

(ii) $\lim_\omega a_j = 0$ if $(a_j) \in \mathfrak{c}_0$.

Remarks

1. If $a_j \to L$, then $\lim_w a_j = L$.
2. We have a one-to-one correspondence,

$$\{\text{extended limits}\} \longleftrightarrow \{\text{states on } \ell^\infty/\mathfrak{c}_0\}.$$

Reminder: NC Integral

Definition

If $\lim_\omega$ is an extended limit, then $\mathrm{Tr}_\omega : \mathcal{L}^{1,\infty} \to \mathbb{C}$ is given by

$$\mathrm{Tr}_\omega(T) := \lim_\omega \left\{ \frac{1}{\log N} \sum_{j<N} \lambda_j(T) \right\}, \qquad T \in \mathcal{L}^{1,\infty}.$$

Proposition (Dixmier)

1. *Tr_ω is a positive linear trace on $\mathcal{L}^{1,\infty}$.*
2. *It is annihilated by $\mathcal{L}_0^{1,\infty}$, and hence it vanishes on infinitesimals of order > 1.*

Definition

Tr_ω is called the Dixmier trace associated with the extended limit $\lim_\omega$.

Reminder: NC Integral

Definition (Connes)

1. An operator $T \in \mathcal{L}^{1,\infty}$ is called measurable if the value of $\mathrm{Tr}_\omega(T)$ does not depend on the extended limit.
2. We denote by $\mathcal{M}$ the class of measurable operators.
3. If $T \in \mathcal{M}$, we define its NC integral by

$$\fint T := \mathrm{Tr}_\omega(A),$$

where Tr_ω is any Dixmier trace.

Proposition (Connes, Lord-Sukochev-Zanin)

Given $T \in \mathcal{L}^{1,\infty}$, TFAE:

1. *T is measurable and $\fint T = L$.*
2. *We have*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(T) = L.$$

Reminder: NC Integral

Proposition

1. *$\mathcal{M}$ is a closed subspace of $\mathcal{L}^{1,\infty}$ that contains $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.*
2. *$⨍ : \mathcal{M} \to \mathbb{C}$ is a positive linear functional that vanishes on $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.*
3. *In particular, this is a positive trace that annihilates infinitesimals of order > 1.*

Reminder: Strong Measurability

Definition

A trace $\varphi : \mathcal{L}^{1,\infty} \to \mathbb{C}$ is called normalized if

$$\left(T \geq 0 \text{ and } \lambda_j(T) = (j+1)^{-1}\right) \Longrightarrow \varphi(T) = 1.$$

Remarks

1. Every Dixmier trace Tr_ω is a normalized trace.
2. There are (uncountably) many normalized positive traces on $\mathcal{L}^{1,\infty}$ that are not Dixmier traces.

Definition

An operator $T \in \mathcal{L}^{1,\infty}$ is called strongly measurable (or positively measurable) if $\varphi(T)$ takes the same value as φ ranges over all normalized positive traces.

Reminder: Strong Measurability

Notation

$\mathcal{M}_s$ = class of strongly measurable operators.

Proposition

1. *$\mathcal{M}_s$ is a closed subspace of $\mathcal{L}^{1,\infty}$.*
2. *It contains $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$. In particular, it contains all infinitesimals of order > 1.*
3. *It does not depend on the inner product of $\mathcal{L}(\mathcal{H})$.*

Proposition

Let $T \in \mathcal{L}^{1,\infty}$ be such that

$$\sum_{j<N} \lambda_j(T) = L \cdot \log N + \mathrm{O}(1).$$

Then T is strongly measurable and $⨍ T = L$.

Definition

If $A \in \mathcal{L}(\mathcal{H})$, its real and imaginary parts are

$$\Re A := \frac{1}{2}\left(A + A^*\right), \qquad \Im A := \frac{1}{2i}\left(A - A^*\right).$$

Definition

If $A^* = A$, its positive and negative parts are

$$A^+ := \frac{1}{2}\left(|A| + A\right), \qquad A^- := \frac{1}{2}\left(|A| - A\right).$$

Reminder: Weyl Operators

Definition

If $A = A^*$ is compact, then $\pm\lambda_j^{\pm}(A)$, $j \geq 0$, are the positive eigenvalues of A such that

$$\lambda_0^{\pm}(A) \geq \lambda_1^{\pm}(A) \geq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

Remark

In other words,

$$\lambda_j^{\pm}(A) = \lambda_j(A^{\pm}) = \mu_j(A^{\pm}), \qquad j \geq 0.$$

Reminder: Weyl Operators

Definition

We say that $A \in \mathcal{L}^{p,\infty}$ is a Weyl operator if one of the following conditions is satisfied:

(i) $A \geq 0$ and $\lim_{j\to\infty} j^{1/p}\lambda_j(A)$ exists.

(ii) $A = A^*$ and $\lim_{j\to\infty} j^{1/p}\lambda_j^{\pm}(A)$ both exist.

(iii) The real and imaginary parts both satisfy (ii).

Reminder: Weyl Operators

Definition

Let A be a Weyl operator in $\mathcal{L}^{p,\infty}$.

1. If $A \geq 0$, then we set
$$\Lambda(A) := \lim_{j \to \infty} j^{\frac{1}{p}} \lambda_j(A).$$
2. If $A^* = A$, then we set
$$\Lambda^{\pm}(A) := \lim_{j \to \infty} j^{\frac{1}{p}} \lambda_j^{\pm}(A).$$
3. In general, we set
$$\Lambda^{\pm}(A) := \Lambda^{\pm}(\Re A) + i\Lambda^{\pm}(\Im A).$$

Reminder: Weyl Operators

Proposition

Let A be a Weyl operator in $\mathcal{L}^{1,\infty}$.

1. *A is strongly measurable, and we have*
$$\fint A = \Lambda^{+}(A) - \Lambda^{-}(A).$$
2. *In particular, if $A^* = A$, then*
$$\fint A = \lim_{j\to\infty} j\lambda_j^{+}(A) - \lim_{j\to\infty} j\lambda_j^{-}(A).$$

Corollary

If $|A|$ is a Weyl operator in $\mathcal{L}^{1,\infty}$, then $|A|$ is strongly measurable, and
$$\fint |A| = \lim_{j\to\infty} j\mu_j(A).$$

Positive-Elliptic Operators

Setup

- (M^n, g) = closed Riemannian manifold.
- $(\mathcal{E}, \|\cdot\|_x)$ = smooth Hermitian vector bundle over M.

Definition

We say that $P \in \Psi^m(M, \mathcal{E})$, $m > 0$, is positive-elliptic if the following conditions are satisfied:

(i) P is elliptic with $\sigma_m(P)(x, \xi) > 0$.

(ii) P is selfadjoint and ≥ 0.

(iii) P is invertible, i.e., $0 \notin \mathrm{Sp}(P)$.

Lemma

For every $\sigma(x, \xi) \in S_m(T^*M, \mathcal{E})$ such that $\sigma(x, \xi) > 0$ we can find $P \in \Psi^m(M, \mathcal{E})$ such that $\sigma_m(P)(x, \xi) = \sigma(x, \xi)$ and P is positive-elliptic.

Positive-Elliptic Operators

Proof.

- Let $Q \in \Psi^m(M, \mathcal{E})$ be such that $\sigma_m(Q)(x,\xi) = \sigma(x,\xi)$.
- By assumption $\sigma(x,\xi) > 0$. Thus, $\sigma(x,\xi)$ is invertible for all $(x,\xi) \in T^*M \setminus 0$, and hence Q is elliptic.
- Therefore, $|Q| = \sqrt{Q^*Q}$ is an operator in $\Psi^m(M,\mathcal{E})$ such that

$$\sigma_m(|Q|)(x,\xi) = |\sigma_m(Q)(x,\xi)| = \sigma(x,\xi) > 0.$$

- By construction $|Q|$ is selfadjoint and ≥ 0.
- Thus, the conditions (i)–(ii) are satisfied.
- We get condition (iii) by taking $P = |Q| + \Pi_0$, where Π_0 is the orthogonal projection onto $\ker |Q| = \ker Q$.

□

Positive-Elliptic Operators

Remark

Let $P \in \Psi^m(M, \mathcal{E})$ be positive-elliptic.

1. The spectrum of P can be can be arranged as a non-decreasing sequence,
$$0 < \lambda_0(P) \leq \lambda_1(P) \leq \cdots .$$
where each eigenvalue is repeated according to multiplicity.
2. As $j \to \infty$, we have the Weyl's law,
$$\lambda_j(P) \sim j^{\frac{m}{n}} \left(\frac{1}{n} \int_{S^*M} \mathrm{tr}_{\mathcal{E}} \left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}} \right] dx đ\xi \right)^{-\frac{m}{n}}.$$

Lemma

Let $P \in \Psi^m(M, \mathcal{E})$ *be positive-elliptic.*

(i) P^{-1} *is a Weyl operator in* $\mathcal{L}^{nm^{-1},\infty}$*, and we have*

$$\lim_{j\to\infty} j^{\frac{m}{n}} \lambda_j\left(P^{-1}\right) = \left(\frac{1}{n}\int_{S^*M} \mathrm{tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}}\right] dx đ\xi\right)^{\frac{m}{n}}.$$

(ii) *If* $m = n$*, then* P^{-1} *is strongly measurable, and we have*

$$⨍ P^{-1} = \frac{1}{n}\int_{S^*M} \mathrm{tr}_{\mathcal{E}}\left[\sigma_n(P)(x,\xi)^{-1}\right] dx đ\xi.$$

Positive-Elliptic Operators

Proof.

- The Weyl's law for P gives

$$j^{\frac{m}{n}}\lambda_j\left(P^{-1}\right)=\left(j^{-\frac{m}{n}}\lambda_j(P)\right)^{-1}$$
$$\longrightarrow\left(\frac{1}{n}\int_{S^*M}\mathrm{tr}_{\mathcal{E}}\left[\sigma_m(P)(x,\xi)^{-\frac{n}{m}}\right]dx đ\xi\right)^{\frac{m}{n}}.$$

- This shows that P^{-1} is a Weyl operator in $\mathcal{L}^{nm^{-1},\infty}$.
- In particular, as $P^{-1}\geq 0$, we have $\mu_j(P^{-1})=\lambda_j(P^{-1})=\mathrm{O}(j^{m/n})$, and hence $P^{-1}\in\mathcal{L}^{nm^{-1},\infty}$.
- For $m=n$, we get that P^{-1} is a Weyl operator in $\mathcal{L}^{1,\infty}$.
- Thus P^{-1} is strongly measurable, and we have

$$⨍ P^{-1}=\lim_{j\to\infty}j\lambda_j\left(P^{-1}\right)$$
$$=\frac{1}{n}\int_{S^*M}\mathrm{tr}_{\mathcal{E}}\left[\sigma_n(P)(x,\xi)^{-1}\right]dx đ\xi.$$

□

Proposition

If $P \in \Psi^{-m}(M, \mathcal{E})$, $m > 0$, then P is in the weak Schatten class $\mathcal{L}^{nm^{-1},\infty}$. That is, P is an infinitesimal of order $\geq mn^{-1}$.

Proof.

- Let $Q \in \Psi^{m}(M, \mathcal{E})$ be positive-elliptic.
- Q^{-1} is in $\mathcal{L}^{nm^{-1},\infty}$.
- PQ is a ΨDO of order ≤ 0, and hence is bounded.
- It then follows that $P = (PQ)Q^{-1}$ is in the ideal $\mathcal{L}^{nm^{-1},\infty}$.

□

Noncommutative Residue

Reminder

- If $P \in \Psi^{\mathbb{Z}}(M)$, then there is a unique density $c_P(x) \in C^\infty(M, |\Lambda|(M))$ such that, for every chart $\kappa : U \to V$, we have

$$\kappa_*\left(c_P(x)\right) = c_{\kappa_*(P_{|U})}(x) := \int_{\mathbb{S}^{n-1}} p_{-n}(x,\xi) đ\xi,$$

  where $p_{-n}(x,\xi)$ is the symbol of degree $-n$ of $\kappa_*(P_{|U})$.

- If $P \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$, then there is a unique density $c_P(x) \in C^\infty(M, |\Lambda|(M) \otimes \mathrm{End}(\mathcal{E}))$ such that, for every trivialization $\tau : U \to V$, we have

$$\tau_*\left(c_P(x)\right) = c_{\tau_*(P_{|U})}(x).$$

- If $P \in \Psi^{-n}(M, \mathcal{E})$, then

$$c_P(x) = \Big(\int_{|\xi|_g = 1} \sigma_{-n}(P)(x,\xi) đ\xi \Big) \nu(g)(x).$$

Noncommutative Residue

Reminder

- The noncommutative residue $\mathrm{Res}: \Psi^{\mathbb{Z}}(M,\mathcal{E}) \rightarrow \mathbb{C}$ is given by

$$\mathrm{Res}(P) = \int_M \mathrm{tr}_{\mathcal{E}}[c_P(x)], \qquad P \in \Psi^{\mathbb{Z}}(M,\mathcal{E}).$$

- If $P \in \Psi^{-n}(M,\mathcal{E})$, then

$$\mathrm{Res}(P) = \int_{S^*M} \mathrm{tr}_{\mathcal{E}}\left[\sigma_{-n}(P)(x,\xi)\right] dx đ\xi.$$

Proposition

1. *The NC residue is a trace on the algebra* $\Psi^{\mathbb{Z}}(M,\mathcal{E})$.
2. *It vanishes on differential operators and* Ψ*DOs of order* $\leq -(n+1)$, *including smoothing operators.*
3. *In particular,*

$$P_1 - P_2 \in \Psi^{-(n+1)}(M,\mathcal{E}) \Longrightarrow \mathrm{Res}(P_1) = \mathrm{Res}(P_2).$$

Connes' Trace Theorem

Proposition

If $P \in \Psi^{-m}(M, \mathcal{E})$, $m > 0$, *then* P *is in the weak Schatten class* $\mathcal{L}^{nm^{-1},\infty}$. *That is,* P *is an infinitesimal of order* $\geq mn^{-1}$.

Corollary

Every $P \in \Psi^{-n}(M, \mathcal{E})$ *is the weak trace class* $\mathcal{L}^{1,\infty}$, *i.e., it's an infinitesimal of order* ≥ 1.

Connes' Trace Theorem

Theorem (Connes' Trace Theorem)

If $P \in \Psi^{-n}(M, \mathcal{E})$, *then* P *is strongly measurable, and we have*

$$\int\!\!\!\!\!- P = \frac{1}{n} \operatorname{Res}(P).$$

Remark

1. Connes (CMP '88) established measurability and derived the trace formula.
2. Lord-Potapov-Sukochev (Adv. Math. '13) obtained strong measurability.

Proof.

Step 1: $P \in \Psi^{-(n+1)}(M, \mathcal{E})$.

- In this case $\operatorname{Res}(P) = 0$.
- Moreover, P is in the weak Schatten class $\mathcal{L}^{p,\infty}$, with $p = n(n+1)^{-1} < 1$.
- Thus, P is in $\mathcal{L}^1 \subset (\mathcal{L}^{1,\infty})_0$.
- It follows that P is strongly measurable, and we have

$$\int\!\!\!\!\!- P = 0 = \frac{1}{n} \operatorname{Res}(P).$$

□

Proof.

Step 2: $P = Q^{-1}$, where $Q \in \Psi^n(M, \mathcal{E})$ is positive-elliptic.

- In this case we saw that P is strongly measurable, and we have
$$⨍ P = \frac{1}{n} \int_{S^*M} \operatorname{tr}_{\mathcal{E}} \left[\sigma_n(Q)(x,\xi)^{-1}\right] dx đ\xi.$$
- Here $\sigma_n(Q)(x,\xi)^{-1} = \sigma_{-n}(Q^{-1})(x,\xi) = \sigma_{-n}(P)(x,\xi)$.
- Thus,
$$⨍ P = \frac{1}{n} \int_{S^*M} \operatorname{tr}_{\mathcal{E}} \left[\sigma_{-n}(P)(x,\xi)\right] dx đ\xi = \frac{1}{n} \operatorname{Res}(P).$$

□

Connes' Trace Theorem

Proof.

Step 3: $\sigma_{-n}(P)(x,\xi) > 0$.

- Let $Q \in \Psi^n(M,\mathcal{E})$ be pos.-ellipt. w/ $\sigma_n(Q) = \sigma_{-n}(P)(x,\xi)^{-1}$.
- Here Q^{-1} is an operator in $\Psi^{-n}(M,\mathcal{E})$ with

$$\sigma_{-n}\left(Q^{-1}\right)(x,\xi) = \sigma_n\left(Q\right)(x,\xi)^{-1} = \sigma_{-n}(P)(x,\xi).$$

- This ensures that $R := P - Q^{-1}$ is in $\Psi^{-(n+1)}(M,\mathcal{E})$.
- In particular $\operatorname{Res}(P) = \operatorname{Res}(Q^{-1})$.
- By Step 2 Q^{-1} is strongly meas. and $⨍ Q^{-1} = \frac{1}{n}\operatorname{Res}(Q^{-1})$.
- As $R \in \Psi^{-(n+1)}(M,\mathcal{E})$, by Step 1 it is strongly measurable and $⨍ R = 0$.
- It follows that $P = Q^{-1} + R$ is strongly measurable, and

$$⨍ P = ⨍ Q^{-1} + ⨍ R = \frac{1}{n}\operatorname{Res}\left(Q^{-1}\right) = \frac{1}{n}\operatorname{Res}(P).$$

□

Connes' Trace Theorem

Proof.

Step 4: $P^* = P$.

- Here $\sigma_{-n}(P)(x,\xi)$ is Hermitian for all $(x,\xi) \in T^*M\backslash 0$.
- Bearing in mind that S^*M is compact, set

$$c := \sup\left\{\|\sigma_{-n}(P)(x,\xi)\|_x; (x,\xi) \in S^*M\right\} < \infty,$$

  where $\|\cdot\|_x$ is the norm of $\mathrm{End}(\mathcal{E}_x)$.
- For $(x,\xi) \in T^*M\backslash 0$, we then have

$$\begin{aligned}\sigma_{-n}(P)(x,\xi) &\leq \|\sigma_{-n}(P)(x,\xi)\|_x \\ &\leq |\xi|_g^{-n} \left\|\sigma_{-n}(P)(x, |\xi|_g^{-n}\xi)\right\|_x \\ &\leq c|\xi|_g^{-n}.\end{aligned}$$

□

Proof.

- Let $P_1 \in \Psi^{-n}(M, \mathcal{E})$ have principal symbol
$$\sigma_{-n}(P_1)(x, \xi) = (c + \epsilon)|\xi|_g \, \mathrm{id}_{\mathcal{E}_x}, \quad \epsilon > 0.$$
- Set $P_2 = P_1 - P$. Then $P_2 \in \Psi^{-n}(M, \mathcal{E})$, and
$$\begin{aligned} \sigma_{-n}(P_2)(x, \xi) &= \sigma_{-n}(P_1)(x, \xi) - \sigma_{-n}(P)(x, \xi) \\ &\geq (c + \epsilon)|\xi|_g^{-n} - c|\xi|_g^{-n} \\ &\geq \epsilon|\xi|_g^{-n} > 0. \end{aligned}$$
- As $\sigma_{-n}(P_j)(x, \xi)$, Step 3 ensures that each P_j is strongly measurable and $⨍ P_j = \frac{1}{n}\mathrm{Res}(P_j)$.
- It then follows that $P = P_1 - P_2$ is strongly measurable, and
$$⨍ P = ⨍ P_1 - ⨍ P_2 = \frac{1}{n}\mathrm{Res}(P_1) - \frac{1}{n}\mathrm{Res}(P_2) = \frac{1}{n}\mathrm{Res}(P).$$

□

Proof.

Step 5: General case $P \in \Psi^{-n}(M, \mathcal{E})$.

- Put $P = \Re P + i\Im P$, with

$$P_1 = \Re P = \frac{1}{2}\left(P + P^*\right), \qquad P_2 = \Im P = \frac{1}{2i}\left(P - P^*\right).$$

- Here P_1 and P_2 are selfadjoint operators in $\Psi^{-n}(M, \mathcal{E})$.
- By Step 4 each operator P_j is is strongly measurable and $⨍ P_j = \frac{1}{n}\operatorname{Res}(P_j)$.
- It then follows that $P = P_1 + iP_2$ is strongly measurable, and

$$⨍ P = ⨍ P_1 + i ⨍ P_2 = \frac{1}{n}\operatorname{Res}(P_1) + \frac{i}{n}\operatorname{Res}(P_2) = \frac{1}{n}\operatorname{Res}(P).$$

□

Connes' Trace Theorem

Consequence

- The NC integral makes sense for ΨDOs of order $\leq -n$
- The NC residue, however, makes for all integer order ΨDOs.
- Therefore, the equality between the NC integral and the NC residue enables us to extend the NC integral to all ΨDOs.
- This includes ΨDOs that are not infinitesimals or are not even bounded.

Definition

For any $P \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$ we set

$$\int\!\!\!\!\!- P := \frac{1}{n} \operatorname{Res}(P) = \frac{1}{n} \int_M \operatorname{tr}_{\mathcal{E}}[c_P(x)].$$

Connes' Integration Formula

Theorem (Connes' Integration Formula)

For every $f \in C^\infty(M)$, the operators $f\Delta_g^{-n/2}$ and $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ are both strongly measurable, and we have

$$\fint f\Delta_g^{-\frac{n}{2}} = \fint \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M f(x)\nu(g)(x),$$

where we have set $c(n) = \frac{1}{n}(2\pi)^{-n}|\mathbb{S}^{n-1}| = (2\pi)^{-n}|\mathbb{B}^n|$.

Remarks

1. We saw in Chapter 7 that the result for $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ continues to hold for all $f \in L\log L(M)$.
2. The result for $f\Delta_g^{-n/2}$ continues to holds for functions f such that $|f|^2 \in L\log L(M)$, including functions in $L^p(M)$, $p > 2$.

Proof.

- $\Delta_g^{-n/2}$ is ΨDOs of order $-n$ with $\sigma_{-n}(\Delta_g^{-n/2})(x,\xi) = |\xi|_g^{-n}$.
- As $f \in C^\infty(M)$, this is a ΨDO of order 0.
- So $f\Delta_g^{-n/2} \in \Psi^{-n}(M)$ and $\sigma_{-n}(f\Delta_g^{-n/2})(x,\xi) = f(x)|\xi|_g^{-n}$.
- Thus, by Connes' trace theorem $f\Delta_g^{-n/2}$ is strongly measurable, and we have

$$\begin{aligned}
\int\!\!\!\!\!- f\Delta_g^{-\frac{n}{2}} = \frac{1}{n}\operatorname{Res}\left(\Delta_g^{-\frac{n}{2}}\right) &= \frac{1}{n}\int_{S^*M} \sigma_{-n}\left(f\Delta_g^{-\frac{n}{2}}\right)(x,\xi)\,dx\,đ\xi \\
&= \frac{1}{n}\int_{S^*M} f(x)|\xi|_g^{-n}\,dx\,đ\xi \\
&= \frac{1}{n}(2\pi)^{-n}|\mathbb{S}^{n-1}|\int_M f(x)\nu(g)(x).
\end{aligned}$$

- The result for $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ is proved similarly.

□

Lower Dimensional Volumes

Consequence

- Connes' integration formula shows that the NC integral recaptures the Riemannian volume density. Namely,

$$\int f\nu(g)(x) = c(n)^{-1} ⨍ f\Delta_g^{-\frac{n}{2}} \qquad \forall f \in C^\infty(M).$$

- We regard $c(n)^{-1}\Delta_g^{n/2}$ as the NC volume element.
- The volume element is ds^n, where ds is the length element.
- Thus, ds as the n-th root of the volume element.

Lower Dimensional Volumes

Definition

The NC length element of (M^n, g) is the operator,

$$ds := \left(c(n)^{-1}\Delta_g^{-\frac{n}{2}}\right)^{\frac{1}{n}} = c(n)^{-\frac{1}{n}}\Delta_g^{-\frac{1}{2}}.$$

Remark

ds is a ΨDO of order -1.

Lower Dimensional Volumes

Facts

- For $k = 1, \ldots, n$ the k-th dimensional volume is meant to be the integral of ds^k.
- Here ds^k is a ΨDO of order $-k$.
- The NC integral has been extended to all ΨDOs.
- This enables us to define k-dimensional volumes for all $k = 1, \ldots, n-1$.

Definition

For $k = 1, \ldots, n$, the k-th dimensional volume of (M^n, g) is

$$\mathrm{Vol}_g^{(k)}(M) := \fint ds^k = c(n)^{-\frac{k}{n}} \fint \Delta_g^{-\frac{k}{2}}.$$

In particular, the length and area of (M, g) are

$$\mathrm{Length}_g(M) := \fint ds = c(n)^{-\frac{1}{n}} \fint \Delta_g^{-\frac{1}{2}},$$

$$\mathrm{Area}_g(M) := \fint ds^2 = c(n)^{-\frac{2}{n}} \fint \Delta_g^{-1}.$$

Proposition

1. *If* k *and* n *have opposite parities (i.e.,* $n-k$ *is odd), then* $\mathrm{Vol}_g^{(k)}(M)=0$.
2. *If* $k=n-2$*, then*
$$\mathrm{Vol}_g^{(n-2)}(M)=c(n,2)\int_M \kappa_g(x)\nu(g)(x),$$
where $\kappa_g(x)$ *is the scalar curvature of* (M,g).
3. *In general, we have*
$$\mathrm{Vol}_g^{(n-k)}(M)=c(n,k)\int_M I_g^{(k)}(x)\nu(g)(x),$$
where $I_g^{(k)}(x)$ *is a universal polynomial in the curvature tensor and its covariant derivatives.*

Lower Dimensional Volumes

Remark

- The definition of the k-th dimensional volumes involved noncommutative geometry.
- However, the formulas in the previous slide provide purely differential-geometric expressions for the k-th dimensional volumes.

Lower Dimensional Volumes

Remark

- The functional $g \to \int_M \kappa_g(x)\nu(g)(x)$ is called the Einstein-Hilbert action.
- It accounts for the contribution of gravity forces in the Standard Model from Theoretical Physics.
- We have
$$\begin{aligned}\int_M \kappa_g(x)\nu(g)(x) &= c(n,2)^{-1} ⨍ \Delta_g^{-n+2}\\ &= \frac{1}{n}c(n,2)^{-1}\operatorname{Res}\left(\Delta_g^{-n+2}\right)\\ &= \frac{2}{n}c(n,2)^{-1}\operatorname{Res}_{z=\frac{n}{2}-1}\operatorname{Tr}\left[\Delta_g^{-z}\right].\end{aligned}$$
- This yields a spectral theoretic interpretation of the Einstein-Hilbert action.
- This an important ingredient in the spectral action formalism of Connes-Chamseddine.