Differentiable Manifolds
§20. The Lie Derivative and Interior Multiplication

Sichuan University, Fall 2022

The Lie Derivative

Reminder

Let X be a smooth vector field on a smooth manifold M. Then X defines a derivation on $C^\infty(M) = \Omega^0(M)$,

$$X : C^\infty(M) \longrightarrow C^\infty(M), \qquad f \longrightarrow Xf.$$

Question

Can we extend this derivation to a derivation on all $\Omega^*(M)$?

Solution (Lie)

Use the flow generated by X.

Reminder: Flows of Vector Fields

Reminder (Integral curves; see Section 14)

Suppose that X is a smooth vector field on M.

- An *integral curve* of X is any smooth curve $c : (a, b) \to M$ satisfying the equation,

$$\frac{d}{dt}c(t) = X_{c(t)} \qquad t \in (a, b).$$

- If the interval (a, b) contains 0 and $c(0) = p$, then we say that curve *starts at* p and p is its *initial point*.
- We say that an integral curve is *maximal* if it cannot be extended to an integral curve defined on a larger interval.

Reminder (see Theorem 14.7)

Given any $p \in M$, there is a unique maximal integral curve for X that starts at p.

Reminder: Flows of Vector Fields

Reminder (Fundamental Theorem on Flows; see Section 14 slides)

Suppose that X is a smooth vector field on M. Define

$$\Omega = \bigcup_{p \in M} I^{(p)} \times \{p\} \subset \mathbb{R} \times M,$$

where $I^{(p)}$ is the open interval around 0 on which is defined the maximal integral curve of X starting at p. Then:

(i) Ω is an open set in $\mathbb{R} \times M$ containing $\{0\} \times M$.

(ii) There is a smooth map $\varphi : \Omega \to M$, $(t, p) \to \varphi_t(p)$ (called the flow of X) such that, for every $p \in M$, the curve $I^{(p)} \ni t \to \varphi_t(p) \in M$ is the maximal integral curve of X starting at p.

Reminder: Flows of Vector Fields

Remarks (see Section 14 slides)

- For $t \in \mathbb{R}$, the set $M_t := \{p \in M;\ (t,p) \in \Omega\}$ is open in M.
- If $M_t \neq \emptyset$, then $\varphi_t : M_t \rightarrow M_{-t}$ is a diffeomorphism with inverse $\varphi_{-t} : M_{-t} \rightarrow M_t$.
- We have the 1-paramter group properties,

$$\varphi_0 = \mathbb{1}_M, \qquad \varphi_t \circ \varphi_s = \varphi_{t+s} \quad \text{on } M_s \cap M_{t+s}.$$

Remarks

- We say that the vector field X is *complete* when its flow is defined on $\mathbb{R} \times M$.
- In this case $\varphi_t : M \rightarrow M$ is a diffeomorphism for every $t \in \mathbb{R}$,

Remark

Let $p \in M$. As $c(t) = \varphi_t(p)$, $t \in I^{(p)}$, is an integral curve for X starting at p, this is a smooth curve in M such that $c(0) = p$ and

$$c'(0) = \frac{d}{dt}\varphi_t(p)\bigg|_{t=0} = X_{\varphi_t(p)}\bigg|_{t=0} = X_{\varphi_0(p)} = X_p.$$

Consequence (see Proposition 20.6)

Let $f \in C^\infty(M)$. For every $p \in M$, we have

$$(Xf)(p) = X_p f = \frac{d}{dt}\bigg|_{t=0} f(c(t)) = \frac{d}{dt}\bigg|_{t=0} f \circ \varphi_t(p).$$

As $f \circ \varphi_t = \varphi_t^* f$, we may rewrite this as

$$Xf = \frac{d}{dt}\bigg|_{t=0} \varphi_t^* f.$$

Remarks

- If $\omega \in \Omega^k(M)$, then we can make sense of $\varphi_t^*\omega$.
- However, we still need to make sense of $\frac{d}{dt}\Big|_{t=0}\varphi_t^*\omega$.
- To do this we need a little digression on smooth families in $\Omega^k(M)$.

Reminder

- By definition $\Omega^k(M)$ is the space of smooth sections of $\Lambda^k(T^*M)$.
- In particular, any smooth k-form is a smooth map from M to $\Lambda^k(T^*M)$.

Smooth Families of Differential Forms

Definition

Let I be an open interval in $\mathbb{R}$. A family $\{\omega_t\}_{t\in I}$ in $\Omega^k(M)$ is said to be smooth when the map $(t,p) \to (\omega_t)_p$ is smooth as a map from $I \times M$ to $\Lambda^k(T^*M)$.

Lemma

Let $(U, x^1, \ldots, x^n)$ be a chart for M and let $\{\omega_t\}_{t\in I}$ be a family in $\Omega^k(U)$. TFAE:

1. $\{\omega_t\}_{t\in I}$ *is a smooth family in* $\Omega^k(U)$.
2. *We may write* $\omega_t = \sum a_I(t,p)dx^I$, $t \in I$, *where the coefficients* $a_I(t,p)$ *are smooth functions on* $I \times U$.

Smooth Families of Differential Forms

Proposition

Let $\{\omega_t\}_{t\in I}$ *be a family in* $\Omega^k(M)$*. TFAE:*

1. $\{\omega_t\}_{t\in I}$ *is a smooth family in* $\Omega^k(M)$.
2. *For every* $p \in M$, *there is a chart* $(U, x^1, \ldots, x^n)$ *near* p *such that* $\omega_t = \sum a_I(t,p)dx^I$ *on* U, *where the coefficients* $a_I(t,p)$ *are smooth functions on* $I \times U$.
3. *For every chart* $(U, x^1, \ldots, x^n)$ *for* M *we may write* $\omega_t = \sum a_I(t,p)dx^I$ *on* U, *where the coefficients* $a_I(t,p)$ *are smooth functions on* $I \times U$.

Smooth Families of Differential Forms

Remark

Let $\{\omega_t\}_{t\in I}$ be a smooth family in $\Omega^k(M)$. Then, for every $p \in M$, the map $t \to (\omega_t)_p$ is smooth as a map from I to the vector space $\Lambda^k(T_p^*M)$.

Definition

Let $\{\omega_t\}_{t\in I}$ be a smooth family in $\Omega^k(M)$. For every $t_0 \in I$, the derivative $\frac{d}{dt}\Big|_{t=t_0}\omega_t$ is the k-form on M defined by

$$\left(\frac{d}{dt}\bigg|_{t=t_0}\omega_t\right)(p) = \frac{d}{dt}\bigg|_{t=t_0}(\omega_t)_p = \lim_{t\to t_0}\frac{(\omega_t)_p - (\omega_{t_0})_p}{t-t_0} \in \Lambda^k(T_p^*M), \quad p \in M.$$

Smooth Families of Differential Forms

Remark

Let $(U, x^1, \ldots, x^n)$ be a chart for M. On U write $\omega_t = \sum a_I(t,p)dx^I$ with $a_I(t,p) \in C^\infty(I \times U)$. Then, for every $p \in U$, we have

$$\left(\frac{d}{dt}\bigg|_{t=t_0} \omega_t\right)(p) = \sum_I \frac{\partial a_I}{\partial t}(t_0, p)dx^I.$$

In particular, $\frac{d}{dt}\big|_{t=t_0}\omega_t$ is smooth on U, since the coefficients $\partial_t a_I(t_0, p)$ are smooth functions on U.

Proposition

Let $\{\omega_t\}_{t\in I}$ be a smooth family in $\Omega^k(M)$. Then

$$\frac{d}{dt}\bigg|_{t=t_0} \omega_t \in \Omega^k(M) \qquad \forall t_0 \in I.$$

Smooth Families of Differential Forms

Proposition (Product Rule; Proposition 20.1)

Let $\{\omega_t\}$ and $\{\tau_t\}$ be smooth families in $\Omega^k(M)$ and $\Omega^\ell(M)$, respetively. Then $\{\omega_t \wedge \tau_t\}$ is a smooth family in $\Omega^{k+\ell}(M)$, and we have

$$\frac{d}{dt}(\omega_t \wedge \tau_t) = \left(\frac{d}{dt}\omega_t\right) \wedge \tau_t + \omega_t \wedge \left(\frac{d}{dt}\tau_t\right).$$

Proposition (Proposition 20.2)

Let $\{\omega_t\}$ be a smooth family in $\Omega^k(M)$. Then $\{d\omega_t\}$ is a smooth family in $\Omega^{k+1}(M)$, and we have

$$\frac{d}{dt}(d\omega_t) = d\left(\frac{d}{dt}\omega_t\right).$$

The Lie Derivative of Differential Forms

Facts

Let $\omega \in \Omega^k(M)$.

- If X is a complete vector field, then $\varphi_t : M \to M$ is a diffeomorphism for every $t \in \mathbb{R}$, and so we can form the pullback,
$$\left(\varphi_t^* \omega\right)_p = \left(\varphi_t\right)_{*,p}^* \left[\omega_{\varphi_t(p)}\right], \qquad p \in M.$$
- Here $(\varphi_t)_{*,p}^* : \Lambda^k(T^*_{\varphi_t(p)}M) \to \Lambda^k(T^*_p M)$ is the pullback by the differential $(\varphi_t)_{*,p} : T_pM \to T_{\varphi_t(p)}M$.
- In general $(\varphi_t^*\omega)_p = (\varphi_t)_{*,p}^*[\omega_{\varphi_t(p)}] \in \Lambda^k(T_p^*M)$ is defined for $(t,p) \in \Omega$ only.
- If $I \subset \mathbb{R}$ is an open interval and $U \subset M$ is an open such that $I \times U \subset \Omega$, then $\{(\varphi_t^*\omega)_{|U}\}_{t \in I}$ is a family in $\Omega^k(U)$.

The Lie Derivative of Differential Forms

Proposition

Let $\omega \in \Omega^k(M)$.

(i) *The map* $(t,p) \rightarrow (\varphi_t^*\omega)_p$ *is smooth as a map from* Ω *to* $\Lambda^k(T^*M)$.

(ii) *If* $I \subset \mathbb{R}$ *is an open interval and* $U \subset M$ *is an open such that* $I \times U \subset \Omega$, *then* $\{(\varphi_t^*\omega)_{|U}\}_{t\in I}$ *is a smooth family in* $\Omega^k(U)$.

Proof

- Let $(t_0, p) \in \Omega$ and let $(V, y^1, \ldots, y^n)$ be a local coordinates for M near $\varphi_{t_0}(p)$.
- As $\mathscr{V} = \{(t,q) \in \Omega;\ \varphi_t(q) \in V\}$ is an open set, there are an open interval $I \subset \mathbb{R}$ and an open $U \subset M$ such that $(t_0, p) \in I \times U \subset \mathscr{V}$. In particular, $\varphi_t(U) \subset V$ for all $t \in I$.
- Moreover, as $I \times U \subset \Omega$ we know that $\{(\varphi_t^*\omega)_{|U}\}_{t\in I}$ is a family in $\Omega^k(U)$.

Proof (Continued).

- We may assume that U is the domain of a chart $(x^1, \ldots, x^n)$ near p. Set $\varphi_t^j = y^j \circ \varphi_t$ and write $\omega = \sum b_J dy^J$ on V with $b_J \in C^\infty(V)$.
- By the local expression for pullbacks (see next slide), on U we have

$$\varphi_t^* \omega = \sum_{I,J} (b_J \circ \varphi_t) \frac{\partial\left(\varphi_t^{j_1}, \ldots \varphi_t^{j_k}\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)} dx^I, \quad t \in I.$$

- The coefficients of dx^I are smooth functions on $I \times U$, and so $\{(\varphi_t^* \omega)_{|U}\}_{t \in I}$ is a smooth family in $\Omega^k(U)$.
- Thus, the map $(t, q) \to (\varphi_t^* \omega)_q$ is smooth near (t_0, p) for every $(t_0, p) \in \Omega$, and hence is smooth on Ω.

This proves (i). The 2nd part (ii) follows from (i). □

The Lie Derivative of Differential Forms

Reminder (Local expression for pullback; see slides on Section 18)

Suppose that $F : N \to M$ is a smooth map. Let $(U, x^1, \ldots, x^m)$ be a chart for N and $(V, y^1, \ldots, y^n)$ a chart for M such that $U \subset F^{-1}(V)$. Set $F^j = y^j \circ F$. For any k-form $\omega = \sum b_J dy^J$ on V, we have

$$F^*\omega = \sum_{I,J} (b_J \circ F) \frac{\partial \left(F^{j_1}, \ldots F^{j_k}\right)}{\partial \left(x^{i_1}, \ldots, x^{i_k}\right)} dx^I \quad \text{on } U.$$

The Lie Derivative of Differential Forms

Remark

Let $p \in M$.

- As Ω is an open containing $(0, p)$, we always can find $\epsilon > 0$ and an open U of M containing p such that $(-\epsilon, \epsilon) \times U \subset \Omega$.
- By the previous proposition $\{(\varphi_t^* \omega)_{|U}\}_{|t|<\epsilon}$ is a smooth family in $\Omega^k(U)$.
- In particular, $\varphi_t^* \omega$ is a C^∞-family of smooth k-forms near p and $t = 0$.

The Lie Derivative of Differential Forms

Definition (Definition 20.5)

If $\omega \in \Omega^k(M)$, then its *Lie derivative* $\mathscr{L}_X\omega$ is the k-form on M defined by

$$\left(\mathscr{L}_X\omega\right)_p = \frac{d}{dt}\bigg|_{t=0} (\varphi_t^*\omega)_p, \qquad p \in M.$$

Remark (Proposition 20.6)

If $f \in C^\infty(M)$, then $\mathscr{L}_X f = Xf$.

The Lie Derivative of Differential Forms

Proposition

If $\omega \in \Omega^k(M)$, then $\mathscr{L}_X\omega$ is a smooth k-form on M.

Proof.

- Let $p \in M$. From the remark on slide 17 $\varphi_t^*\omega$ is a smooth family of smooth k-forms near p.
- Thus, by the proposition on slide 12 $\mathscr{L}_X\omega = \frac{d}{dt}\big|_{t=0}\varphi_t^*\omega$ is smooth near p.
- As this true for every $p \in M$, it follows that $\mathscr{L}_X\omega$ is a smooth k-form on M. □

Corollary

The Lie derivative $\mathscr{L}_X$ defines a degree 0 linear map,

$$\mathscr{L}_X : \Omega^*(M) \longrightarrow \Omega^*(M).$$

Lie Derivative of Vector Fields

Definition

If $F : N \to M$ is a diffeomorphism and Y is a vector field on M, then the *pullback* F^*X is the pushforward of Y by F^{-1}, i.e., $F^*Y = (F^{-1})_*Y$.

Remark

In other words, we have

$$\left(F^*Y\right)_p = \left(F^{-1}\right)_{*,F(p)}(Y_{F(p)}) \qquad \forall p \in M.$$

Lie Derivative of Vector Fields

Remarks

Let X and Y be a smooth vector fields and let $(t, p) \to \varphi_t(p)$ be flow of X.

- $\varphi_t : M_t \to M_{-t}$ is a diffeomorphism.
- We would like to define the Lie derivative $\mathscr{L}_X Y$ as $\frac{d}{dt}\big|_{t=0} \varphi_t^* Y$.
- Here $\varphi_t^{-1} = \varphi_{-t}$, and so we have

$$(\varphi_t^* Y)_p = (\varphi_t^{-1})_{*,\varphi_t(p)}\big(Y_{\varphi_t(p)}\big) = (\varphi_{-t})_{*,\varphi_t(p)}\big(Y_{\varphi_t(p)}\big).$$

  This makes sense for $(t, p) \in \Omega$.
- If $I \subset \mathbb{R}$ is an open interval and $U \subset M$ such that $I \times U \subset \Omega$, then $\{(\varphi_t^* Y)_{|U}\}_{t \in I}$ is a family in $\mathscr{X}(U)$.

Definition

Let $I \subset \mathbb{R}$ be an open interval. A family $\{Y_t\}_{t\in I}$ in $\mathscr{X}(M)$ is said to be smooth when the map $(t,p) \to (X_t)_p$ is smooth as a map from $I \times M$ to TM.

Proposition

Let $\{Y_t\}_{t\in I}$ be a family in $\mathscr{X}(M)$. TFAE:

1. $\{Y_t\}_{t\in I}$ *is a smooth family in* $\mathscr{X}(M)$.
2. *For every $p \in M$ there is a chart $(U, x^1, \ldots, x^n)$ near p such that $Y_t = \sum a^i(t,p)\partial/\partial x^i$ on U, where the coefficients $a^i(t,p)$ are smooth functions on $I \times U$.*
3. *For every chart $(U, x^1, \ldots, x^n)$ for M we may write $Y_t = \sum a^i(t,p)\partial/\partial x^i$ on U, where the coefficients $a^i(t,p)$ are smooth functions on $I \times U$.*

Lie Derivative of Vector Fields

Remark

Let $\{Y_t\}_{t\in I}$ be a smooth family in $\mathscr{X}(M)$. Then, for every $p \in M$, the map $t \to (Y_t)_p$ is smooth as a map from I to the vector space T_pM.

Definition

Let $\{Y_t\}_{t\in I}$ be a smooth family in $\mathscr{X}(M)$. For every $t_0 \in I$, the derivative $\frac{d}{dt}\Big|_{t=t_0} Y_t$ is the vector field on M defined by

$$\left(\frac{d}{dt}\Big|_{t=t_0} Y_t\right)(p) = \frac{d}{dt}\Big|_{t=t_0} \left(Y_t\right)_p = \lim_{t\to t_0} \frac{\left(Y_t\right)_p - \left(Y_{t_0}\right)_p}{t - t_0} \in T_pM, \quad p \in M.$$

Lie Derivative of Vector Fields

Remark

Let $(U, x^1, \dots, x^n)$ be a chart near for M. On U write $Y_t = \sum a^i(t,p)\partial/\partial x^i$ with $a^i(t,p) \in C^\infty(I \times U)$. Then, for every $p \in U$, we have

$$\left(\frac{d}{dt}\bigg|_{t=t_0} Y_t\right)(p) = \sum_I \frac{\partial a^i}{\partial t}(t_0, p)\frac{\partial}{\partial x^i}$$

In particular, $\frac{d}{dt}\big|_{t=t_0} Y_t$ is smooth on U, since the coefficients $\partial_t a^i(t_0, p)$ are smooth functions on U.

Proposition

Let $\{Y_t\}_{t\in I}$ be a smooth family in $\mathscr{X}(M)$. Then

$$\frac{d}{dt}\bigg|_{t=t_0} Y_t \in \mathscr{X}(M) \qquad \forall t_0 \in I.$$

Lie Derivative of Vector Fields

Proposition

Let $Y \in \mathscr{X}(M)$.

(i) *The map* $(t, p) \to (\varphi_t^* Y)_p$ *is smooth as a map from* Ω *to* TM.

(ii) *If* $I \subset \mathbb{R}$ *is an open interval and* $U \subset M$ *is an open such that* $I \times U \subset \Omega$, *then* $\{(\varphi_t^* Y)_{|U}\}_{t \in I}$ *is a smooth family in* $\mathscr{X}(U)$.

Definition (Definition 20.3)

If $Y \in \mathscr{X}(M)$, then its *Lie derivative* $\mathscr{L}_X Y$ is the vector field on M defined by

$$\left(\mathscr{L}_X Y\right)_p = \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^* Y)_p, \qquad p \in M.$$

Proposition (Theorem 20.4)

If $Y \in \mathscr{X}(M)$, *then* $\mathscr{L}_X Y = [X, Y]$.

Interior Multiplication

Definition (Interior multiplication)

Let V be a vector space. If β is a k-covector on V and $v \in V$, then the *interior multiplication* or *contraction* of β with v is the $(k-1)$-covector $\imath_v\beta$ defined as follows:

- If $k \geq 2$, then

$$\imath_v\beta(v_1, \ldots, v_{k-1}) = \beta(v, v_1, \ldots, v_{k-1}), \quad v_i \in V.$$

- If $k = 1$, then $\imath_v\beta = \beta(v)$.
- If $k = 0$, then $\imath_v\beta = 0$.

Interior Multiplication

Proposition (Proposition 20.7)

Let $\alpha^1, \dots, \alpha^k$ be 1-covectors (i.e., elements of V^*). Then

$$\imath_v(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k,$$

where $\widehat{\cdot}$ means omission.

Proposition (Proposition 20.8)

Let $v \in V$. The interior multiplication $\imath_v : A_*(V) \to A_{*-1}(V)$ satisfies the following properties:

1. $\imath_v \circ \imath_v = 0$.
2. *If $\beta \in A_k(V)$ and $\gamma \in A_\ell(V)$, then*
$$\imath_v(\beta \wedge \gamma) = (\imath_v \beta) \wedge \gamma + (-1)^k \beta \wedge (\imath_v \gamma).$$

In other words, $\imath_v$ is an antiderivation of degree -1 whose square is zero.

Interior Multiplication

Definition

Let M be a smooth manifold. If X is a vector field and ω is a k-form on M, then the interior product $\imath_X \omega$ is defined by

$$(\imath_X \omega)_p = \imath_{X_p} \omega_p, \qquad p \in M.$$

Remark

- If $k \geq 2$, then, for any vector fields $X_1, \ldots, X_{k-1}$ on M, we have
$$\imath_X \omega(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}).$$
- If $k = 1$, then $\imath_X \omega = \omega(X)$.
- If $k = 0$, then $\imath_X \omega = 0$.

Interior Multiplication

Reminder (Proposition 18.7)

Let ω be a k-form on M. Then TFAE:

1. ω is a smooth k-form.
2. For any smooth vector fields $X_1, \ldots, X_k$ on M, the function $\omega(X_1, \ldots, X_k)$ is smooth on M.

Interior Multiplication

Proposition

If X is a smooth vector field and ω is a smooth k-form on M, then $\imath_X \omega$ is a smooth form on M as well.

Proof.

- The case $k = 0$ is immediate, since in this case $\imath_X \omega = 0$.
- If $k \geq 2$, then for any smooth vector fields $X_1, \ldots, X_{k-1}$ on M we have

$$\imath_X \omega(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}) \in C^\infty(M).$$

- If $k = 1$, then $\imath_X \omega = \omega(X) \in C^\infty(M)$.

Proposition 18.7 then ensures us that $\imath_X \omega$ is a smooth form of degree $k - 1$ if $k \geq 1$. The proof is complete. □

Interior Multiplication

Corollary

If X is a smooth vector field on M, the interior product with X defines a degree -1 anti-derivation $\imath_X : \Omega^*(M) \to \Omega^{*-1}(M)$ such that $\imath_X \circ \imath_X = 0$.

Reminder

The space of smooth vector fields $\mathscr{X}(M)$ and the exterior algebra $\Omega^*(M)$ are modules over the ring $\mathscr{F} = C^\infty(M)$.

Proposition

The map $\mathscr{X}(M) \times \Omega^*(M) \to \Omega^{*-1}(M)$, $(X, \omega) \to \imath_X\omega$ is an $\mathscr{F}$-bilinear map. In particular,

$$\imath_{fX}\omega = \imath_X(f\omega) = f(\imath_X\omega).$$

Properties of the Lie Derivative

Theorem (Theorem 20.10)

Let X be a smooth vector field on M.

(i) *The Lie derivative $\mathscr{L}_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation. That is, it is a linear map such that*

$$\mathscr{L}_X(\omega \wedge \tau) = \big(\mathscr{L}_X \omega\big) \wedge \tau + \omega \wedge \big(\mathscr{L}_X \tau\big) \quad \forall \omega, \tau \in \Omega^*(M).$$

(ii) $d\mathscr{L}_X = \mathscr{L}_X d$.

(iii) Cartan homotopy formula: $\mathscr{L}_X = d\imath_X + \imath_X d$.

(iv) Product formula: *If $\omega \in \Omega^k(M)$ and $Y_1, \ldots, Y_k$ are smooth vector fields on M, then*

$$\begin{aligned}\mathscr{L}_X\big(\omega(Y_1, \ldots, Y_k)\big) =& \big(\mathscr{L}_X \omega\big)(Y_1, \ldots, Y_k) \\ &+ \sum_{i=1}^{k} \omega\big(Y_1, \ldots, \mathscr{L}_X Y_i, \ldots, Y_k\big).\end{aligned}$$

Global Formulas

The last part of Theorem 20.10 can be reformulated as follows:

Theorem (Theorem 20.12; Global formula for $\mathscr{L}_X$)

Let X be a smooth vector field on M and $\omega \in \Omega^k(M)$. Then, for any smooth vector fields $Y_1, \ldots, Y_{k-1}$ on M, we have

$$\begin{aligned}(\mathscr{L}_X\omega)(Y_1, \ldots, Y_k) =& X\big(\omega(Y_1, \ldots, Y_k)\big) \\ &- \sum_{i=1}^{k} \omega\big(Y_1, \ldots, [X, Y_i], \ldots, Y_k\big).\end{aligned}$$

Global Formulas

Proposition (Proposition 20.13)

Let $\omega \in \Omega^1(M)$. *Then, for any smooth vector fields* X *and* Y *on* M*, we have*

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega([X, Y]).$$

Theorem (Theorem 20.14; Global formula for the exterior derivative)

Let $\omega \in \Omega^k(M)$, $k \geq 1$. *Then, for any smooth vector fields* $Y_0, \ldots, Y_k$ *on* M*, we have*

$$d\omega(Y_0, \ldots, Y_k) = \sum_{i=1}^{k} (-1)^i Y_i\big(\omega(Y_0, \ldots, \widehat{Y_i}, \ldots, Y_k)\big) + \sum_{1 \leq i < j \leq k} (-1)^{i+j} \omega\big([Y_i, Y_j], Y_0, \ldots, \widehat{Y_i}, \ldots, \widehat{Y_j}, \ldots, Y_k\big).$$