Differentiable Manifolds
§19. The Exterior Derivative

Sichuan University, Fall 2022

Reminder: Exterior Derivative on an Open Set

Definition (Exterior derivative on open set)

Let U be an open subset of $\mathbb{R}^n$. The *exterior derivative* $d : \Omega^*(U) \to \Omega^*(U)$ is defined as follows:

- For $k = 0$ the exterior derivative of a 0-form (i.e., a C^∞ function) f on U is its differential, i.e., $df = \sum \frac{\partial f}{\partial x^i} dx^i$.
- For $k \geq 1$, the exterior derivative $\omega = \sum a_I dx^I \in \Omega^k(U)$ is

$$d\omega = \sum da_I \wedge dx^I = \sum_I \Big(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \Big) \wedge dx^I.$$

Remarks

- If $\omega \in \Omega^k(U)$, then $d\omega \in \Omega^{k+1}(U)$.
- In particular, $d\omega = 0$ for all $\omega \in \Omega^n(U)$.

Reminder: Exterior Derivative on an Open Set

Reminder (Graded Algebras)

An algebra A over a field $\mathbb{K}$ is called *graded* when it can be decomposed as

$$A = \bigoplus_{k=0}^{\infty} A^k,$$

where the A^k are subspaces such that the multiplication maps $A^k \times A^\ell$ to $A^{k+\ell}$.

Reminder (Antiderivation of a Graded Algebra; see Section 4)

Let $A = \oplus_{k=0}^{\infty} A^k$ be a graded algebra over a field $\mathbb{K}$.

- An *antiderivation* of A is any linear map $D : A \rightarrow A$ such that

$$D(ab) = (Da)b + (-1)^k aDb \qquad \text{for all } a \in A^k \text{ and } b \in A.$$

- We say that D has *degree* m when $D(A^k) \subset A^{k+m}$ for all k.

Reminder: Exterior Derivative on an Open Set

Reminder

$\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ is a graded algebra over $\mathbb{R}$.

Reminder (Proposition 4.7)

The exterior derivative $d : \Omega^*(U) \to \Omega^*(U)$ satisfies the following properties:

(i) It is an antiderivation of degree 1, i.e.,

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

(ii) $d^2 = 0$, i.e., $d(d\omega) = 0$ for all $\omega \in \Omega^*(U)$.

(iii) If $f \in C^\infty(U)$ and $X \in \mathscr{X}(U)$, then $(df)(X) = Xf$.

Reminder (Proposition 4.8)

The exterior derivative is the unique map $D : \Omega^*(U) \to \Omega^*(U)$ that satisfies the properties (i)–(iii) above.

Exterior Derivative on a Manifold

Reminder

Let M be a smooth manifold of dimension n. Then the exterior algebra of differential forms $\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ is a graded algebra.

Definition

An *exterior derivative* on a manifold M is a linear map $D : \Omega^*(M) \to \Omega^*(M)$ satisfying the following properties:

(i) It is an antiderivation of degree 1.

(ii) $D \circ D = 0$.

(iii) On $\Omega^0(M) = C^\infty(M)$ it agrees with the differential of functions, i.e., $Df = df$ for all $f \in C^\infty(M)$.

Theorem (Theorem 19.4)

There is a unique exterior derivative $d : \Omega^*(M) \to \Omega^*(M)$.

Construction of the Exterior Derivative

Reminder

Let $(U, x^1, \ldots, x^n)$ be a chart for M.

- $\{dx^I;\ I \in \mathscr{I}_{k,n}\}$ is a smooth frame of $\Omega^k(M)$ over U.
- Every smooth k-form ω on U can be uniquely written as $\omega = \sum a_I dx^I$ with a_I in $C^\infty(U)$.

Construction of the Exterior Derivative

Definition

Let $(U, x^1, \ldots, x^n)$ be a chart. Define $d_U : \Omega^*(U) \to \Omega^*(U)$ by

(i) If $f \in C^\infty(U) = \Omega^0(U)$, then

$$d_U f = df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

(ii) If $\omega = \sum a_I dx^I \in \Omega^k(U)$, $k \geq 1$, then

$$d\omega = \sum_I da_I \wedge dx^I.$$

In the same way as in the case of an open set of $\mathbb{R}^n$ we get:

Lemma

$d_U : \Omega^*(U) \to \Omega^*(U)$ is the unique exterior derivative on U.

Remark

- The proof of uniqueness in Tu's book lacks details.
- Tu's arguments require to show that if $D: \Omega^*(M) \rightarrow \Omega^*(M)$ is an exterior derivative, then
$$D(dx^I) = 0 \qquad \forall I \in \mathscr{I}_{k,n},\ k \geq 1.$$
This can be proved by induction on k.
- $\underline{k=1}$: $D(dx^i) = D \circ D(x^i) = 0$ since $D = d$ on $C^\infty(M)$.
- Assume the result for k. Let $I = (i_1, \ldots i_{k+1}) \in \mathscr{I}_{k+1,n}$ and set $J = (i_2, \ldots, i_{k+1}) \in \mathscr{I}_{k,n}$. We have
$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots dx^{i_{k+1}} = dx^{i_1} \wedge dx^J.$$
- As D is an antiderivation, we get
$$D(dx^I) = D(dx^{i_1} \wedge dx^J) = D(dx^{i_1}) \wedge dx^J - dx^{i_1} \wedge D(dx^J) = 0,$$
since $D(dx^{i_1}) = 0$ and $D(dx^J) = 0$.

Remark (Continued)

- Once it has been established that $D(dx^I) = 0$, it can be shown that $D = d_U$ as in Tu's book.
- Let $\omega = \sum a_I dx^I \in \Omega^k(U)$. As D is an antiderivation and agrees with the differential on functions, we get

$$\begin{aligned} D\omega = \sum D(a_I dx^I) &= \sum D(a_I) \wedge dx^I + \sum a_I D(dx^I) \\ &= \sum da^I \wedge dx^I \\ &= d_U \omega. \end{aligned}$$

- This shows that d_U is the only exterior derivative on U.

Construction of the Exterior Derivative

Facts

Let $\omega \in \Omega^k(M)$. Let $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ be charts for M near $p \in M$.

- Write $\omega = \sum a_I dx^I$ on U and $\omega = \sum b_I dy^I$ on V. Then on $U \cap V$ we have
$$\omega = \sum a_I dx^I = \sum b_I dy^I.$$
- The uniqueness of $d_{U\cap V}$ ensures that on $U \cap V$ we have
$$\sum da_I \wedge dx^I = d_{U\cap V}(\omega_{|U\cap V}) = \sum db_I \wedge dy^I.$$
- As $p \in U \cap V$, we get
$$d_U(\omega_{|U})_p = \sum \left(da_I \wedge dx^I\right)_p = \sum \left(db_I \wedge dy^I\right)_p = d_V(\omega_{|V})_p.$$

Consequence

$d_U(\omega_{|U})_p$ depends only on ω and p, not on U.

Definition

The map $d : \Omega^*(M) \to \Omega^*(M)$ is defined as follows: if $\omega \in \Omega^k(M)$ and $p \in M$, then

$$(d\omega)_p = d_U(\omega_{|U})_p,$$

where U is the domain of any chart near p.

Theorem (Theorem 19.4)

The map $d : \Omega^*(M) \to \Omega^*(M)$ is the unique exterior derivative on M.

Construction of the Exterior Derivative

Definition

$d : \Omega^*(M) \to \Omega^*(M)$ is called the *exterior derivative* of M.

Remark

Let $\omega \in \Omega^k(U)$ and $(U, x^1, \ldots, x^n)$ a chart for M.

- By definition $(d\omega)_p = d_U(\omega_{|U})_p$ for all $p \in U$. Thus,
$$(d\omega)_{|U} = d_U(\omega_{|U}).$$
- In particular, if $\omega = \sum a_I dx^I$ on U, then
$$\left(d\omega\right)_{|U} = d_U(\omega_{|U}) = \sum da^I \wedge dx^I \qquad \text{on } U.$$

Reminder (see slides on Section 18)

Let $F : N \to M$ be a smooth map.

- If ω is a k-form on M, then its pullback $F^*\omega$ is the k-form on N given by

$$\begin{aligned}(F^*\omega)_p(v_1, \ldots, v_k) &= \big((F_{*,p})^*\omega_p\big)(v_1, \ldots, v_k)\\ &= \omega_p\big(F_{*,p}v_1, \ldots, F_{*,p}v_k\big), \quad v_i \in T_pN.\end{aligned}$$

- If ω is a smooth form on M, then $F^*\omega$ is a smooth form on N.

Exterior Differentiation Under a Pullback

Exterior differentiation commutes with pullback. Namely, we have:

Proposition (Proposition 19.5)

Let $F : N \to M$ be a smooth map. If $\omega \in \Omega^k(M)$, then

$$F^*(d\omega) = d(F^*\omega).$$

Remark

- In Tu's book, Proposition 19.5 is used to show that smoothness of k-forms is preserved by pullback.
- This is not fully rigorous since in order to make sense Proposition 19.5 requires the smoothness of pullbacks of smooth forms.
- Anyway, smoothness of pullbacks of forms can be proved without using Proposition 19.5 (see slides on Section 18).

Restriction of k -Forms to Submanifolds

Reminder

Let S be an immersed submanifold in M.

- The inclusion $i : S \to M$ is an immersion, and so its differential $i_{*,p} : T_pS \to T_pM$ is an injection for every $p \in S$.
- This allows us to identify T_pS with a subspace of T_pM.
- We thus can restrict to S any k-covector $\omega_p \in \Lambda^k(T_p^*M)$; this defines a k-covector on T_pS, i.e., an element of $\Lambda^k(T_p^*S)$.

Definition

If ω is a k-form on M, its *restriction* to S, denoted $\omega_{|S}$, is the k-form on S defined by

$$\left(\omega_{|S}\right)_p(v_1, \ldots, v_k) = \omega_p(v_1, \ldots, v_k) \quad \text{for all } p \in S \text{ and } v_i \in T_pS.$$

Restriction of k-Forms to Submanifolds

In the same way as with 1-forms we have:

Proposition

Let S be an immersed submanifold in M. If $i : S \to M$ is the inclusion of S into M and ω is a k-form on M, then $\omega_{|S} = i^*\omega$.

As pullbacks by smooth maps preserve smoothness we get:

Corollary

Let S be an immersed submanifold in M. If ω is a smooth k-forms on M, then $\omega_{|S}$ is a smooth k-form on S.

Restriction of k-Forms to Submanifolds

Corollary

Let S be an immersed submanifold in M. If $\omega \in \Omega^k(M)$, then

$$(d\omega)_{|S} = d\left(\omega_{|S}\right).$$

Proof.

Let $i : S \to M$ be the inclusion of S into M. As exterior differentiation commutes with pullback by i, we get

$$(d\omega)_{|S} = i^*(d\omega) = d\left(i^*\omega\right) = d\left(\omega_{|S}\right).$$

The result is proved. $\square$

Remark

As $(d\omega)_{|S}$ and $d(\omega_{|S})$ agree, we simply write $d\omega_{|S}$ to mean either expression.

A Nowhere-Vanishing 1-Forms on $\mathbb{S}^1$

Example

- The unit circle $\mathbb{S}^1$ has equation $x^2 + y^2 = 1$. This a regular submanifold of $\mathbb{R}^2$. Thus,
$$\left[d(x^2 + y^2)\right]_{|\mathbb{S}^1} = d\left[(x^2 + y^2)_{|\mathbb{S}^1}\right] = d1 = 0.$$
- On $\mathbb{R}^2$ we also have
$$d(x^2+y^2) = \frac{\partial}{\partial x}(x^2+y^2)dx + \frac{\partial}{\partial y}(x^2+y^2)dy = 2xdx + 2ydy.$$
- Thus,
$$\left(xdx + ydy\right)_{|\mathbb{S}^1} = 0.$$
- In particular, on regions of $\mathbb{S}^1$ where $x \neq 0$ and $y \neq 0$, we have
$$\frac{dy}{x} = -\frac{dx}{y}.$$

A Nowhere-Vanishing 1-Forms on $\mathbb{S}^1$

Example (continued)

- Set $U_x = \{(x, y) \in \mathbb{S}^1; x \neq 0\}$ and $U_y = \{(x, y) \in \mathbb{S}^1; y \neq 0\}$. Let ω be the 1-form on $\mathbb{S}^1$ defined by
$$\omega = \frac{dy}{x} \quad \text{on } U_x, \qquad \omega = -\frac{dx}{y} \quad \text{on } U_y.$$
- This is well-defined since $\frac{dy}{x} = -\frac{dx}{y}$ on $U_x \cap U_y$.
- ω is a smooth 1-form, since on both U_x and U_y it is the restriction of a smooth 1-form on an open of $\mathbb{R}^2$.

Proposition (see Tu's book)

The 1-form ω is a nowhere-vanishing smooth 1-form on $\mathbb{S}^1$, i.e., $\omega_p \neq 0$ for all $p \in \mathbb{S}^1$.