

Differentiable Manifolds

§17. Differentiable 1-Forms

Sichuan University, Fall 2022

Cotangent Space and Differential 1-Forms

Definition (Cotangent Space)

Let M be a smooth manifold.

- The *cotangent space* of M at p , denoted T_p^*M or $T_p^*(M)$, is the dual of the tangent space T_pM . That is,

$$T_p^*M = \text{Hom}(T_pM, \mathbb{R}).$$

- An element of T_p^*M is called a *covector* at p .

Remark

In other words, a covector at p is just a linear map $\omega : T_pM \rightarrow \mathbb{R}$.

Cotangent Space and Differential 1-Forms

Definition (Differential 1-Forms)

A *differential 1-form* (or a *1-form* or a *covector field*) is the assignment to each $p \in M$ of a covector $\omega_p \in T_p^*M$.

Remark

If ω is a 1-form and X is a vector field on M , then we denote by $\omega(X)$ the function on M defined by

$$\omega(X)(p) = \omega_p(X_p), \quad p \in M.$$

Cotangent Space and Differential 1-Forms

Definition (Differential of a Function)

Let $f \in C^\infty$. Its *differential* is the 1-form df on M defined by

$$(df)_p(X_p) = X_p f, \quad X_p \in T_p M, \quad p \in M.$$

Proposition (Proposition 17.2)

If $f : M \rightarrow \mathbb{R}$ is a C^∞ function and $p \in M$, then

$$f_{*,p}(X_p) = (df)_p(X_p) \frac{d}{dt} \Big|_{f(p)} \quad \forall X_p \in T_p M.$$

Thus, under the identification $T_{f(p)}\mathbb{R} \simeq \mathbb{R}$, we have

$$(df)_p = f_{*,p}.$$

Cotangent Space and Differential 1-Forms

Example

Let $(U, x^1, \dots, x^n)$ be a local chart near $p \in M$. Then

$$(dx^i)_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \frac{\partial x^i}{\partial x^j}(p) = \delta_j^i.$$

Therefore, we obtain:

Proposition (Proposition 17.3)

*Let $(U, x^1, \dots, x^n)$ be a local chart near $p \in M$. Then $\{(dx^1)_p, \dots, (dx^n)_p\}$ is the basis of T_p^*M which is dual to the basis $\{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p\}$ of T_pM .*

Cotangent Space and Differential 1-Forms

Consequences

- Every covector $\omega_p \in T_p^*M$, can be uniquely written as

$$\omega_p = \sum a_i (dx^i)_p, \quad \text{with } a^i = \omega_p \left(\frac{\partial}{\partial x^i} \Big|_p \right).$$

- Every 1-form on U can be uniquely written as

$$\omega = \sum a_i dx^i, \quad \text{with } a^i = \omega(\partial/\partial x^i).$$

- If $f \in C^\infty(M)$, then on U we have

$$df = \sum \frac{\partial f}{\partial x^i} dx^i, \quad \text{since } (df)(\partial/\partial x^i) = \frac{\partial f}{\partial x^i}.$$

The Cotangent Bundle

Definition

- The *cotangent bundle* is

$$T^*M = \bigsqcup_{p \in M} T_p^*M = \{(p, \omega); p \in M, \omega \in T_p^*M\}.$$

- The *canonical map* $\pi : T^*M \rightarrow M$ is given by

$$\pi((p, \omega)) = p, \quad p \in M, \quad \omega \in T_p^*M.$$

Remark

If U is an open set of M , then $T^*U = \bigsqcup_{p \in U} T_p^*M$.

The Cotangent Bundle

Facts

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart for M . Set $V = \phi(U)$.

- Every covector $\omega_p \in T_p^*M$, can be uniquely written as

$$\omega_p = \sum a_i (dx^i)_p, \quad \text{with } a^i = \omega_p \left(\frac{\partial}{\partial x^i} \Big|_p \right).$$

- We thus get a natural bijection $\tilde{\phi} : T^*U \rightarrow V \times \mathbb{R}^n$ such that, for all $p \in M$ and $\omega \in T_p^*M$, we have

$$\tilde{\phi}(p, \omega) = \left(x^1(p), \dots, x^n(p), \omega(\partial/\partial x^1|_p), \dots, \omega(\partial/\partial x^n|_p) \right).$$

Remark

In the same way as with the construction of the tangent bundle TM , the maps $\tilde{\phi}$ allow us to define a topology and a smooth structure on T^*M .

The Cotangent Bundle

Definition

Let (U, ϕ) be a chart for M and set $V = \phi(U)$. We endow T^*U with the topology such that

$$W \subset T^*U \text{ is open} \iff \tilde{\phi}(W) \text{ is open in } V \times \mathbb{R}^n.$$

Proposition

Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of M .

- Define

$$\mathcal{B} = \bigcup_{\alpha} \{W; W \text{ is an open in } T^*U_\alpha\}.$$

Then $\mathcal{B}$ is the basis for a unique topology on T^*M .

- The collection $\{(T^*U_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas on T^*M , and hence T^*M is a smooth manifold.
- $T^*M \xrightarrow{\pi} M$ is a smooth vector bundle over M .

Characterization of C^∞ 1-Forms

Remark

A 1-form on M is a section of the tangent bundle T^*M .

Definition

- We say that 1-form is C^∞ when it is C^∞ as a section of T^*M .
- We denote by $\Omega^1(M)$ the space of smooth 1-forms on M .

Remark

In other words, $\Omega^1(M)$ is the space of smooth sections of T^*M . In particular, this is a module over the ring $C^\infty(M)$.

Characterization of C^∞ 1-Forms

Example

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart for M . Set $V = \phi(U)$.

- Let $\tilde{\phi}: T^*U \rightarrow V \times \mathbb{R}$ be the corresponding chart of T^*M .
For all $p \in U$, we have

$$\begin{aligned}\tilde{\phi} \circ dx^i(p) &= \tilde{\phi}(p, (dx^i)_p) = (\phi(p), dx^i(\partial/\partial x^1), \dots, dx^i(\partial/\partial x^n)) \\ &= (\phi(p), \delta_1^i, \dots, \delta_n^i) \\ &= (\phi(p), e^i),\end{aligned}$$

where $(e^1, \dots, e^n)$ be the canonical basis of $\mathbb{R}^n$.

- Thus,
$$\tilde{\phi} \circ dx^i \circ \phi^{-1}(q) = (q, e^i) \quad \forall q \in V.$$

In particular, $\tilde{\phi} \circ dx^i \circ \phi^{-1}$ is a smooth map from V to $V \times \mathbb{R}^n$.

- It follows that dx^i is a smooth map from U to T^*U , and hence this is a smooth 1-form.

Characterization of C^∞ 1-Forms

Consequence

If $(U, x^1, \dots, x^n)$ is a chart for M , then $\{dx^1, \dots, dx^n\}$ is a C^∞ -frame of T^*U over U .

Reminder (Proposition 12.2)

Let $\{s_1, \dots, s_r\}$ be a C^∞ frame of a vector bundle E over U . A section $s = \sum c^i s_i$ of E over U is smooth if and only if $c^1, \dots, c^r$ are smooth functions on U .

We immediately obtain:

Lemma (Lemma 17.5)

Let $(U, x^1, \dots, x^n)$ be a chart for M . A 1-form $\omega = \sum a_i dx^i$ on U is smooth if and only if the coefficients $a_1, \dots, a_n$ are smooth functions on U .

Characterization of C^∞ 1-Forms

In the same way as in Section 14, from the previous lemma we obtain:

Proposition (Proposition 17.6)

Let ω be a 1-form on M . Then TFAE:

- ① ω is a smooth 1-form.
- ② M has an atlas such that, for every chart $(U, x^1, \dots, x^n)$ of this atlas, we may write $\omega = \sum a_i dx^i$, where the coefficients a^i are smooth functions on U .
- ③ For every chart $(U, x^1, \dots, x^n)$ of M , we may write $\omega = \sum a_i dx^i$, where the coefficients a^i are smooth functions on U .

Corollary (Corollary 17.7)

If $f \in C^\infty(M)$, then its differential df is a smooth 1-form.

Characterization of C^∞ 1-Forms

Proposition (Proposition 17.8)

Let ω be a 1-form on M . If X is a vector field on M and f is a function on M , then

$$\omega(fX) = f\omega(X).$$

Proposition (Proposition 17.9)

Let ω be a 1-form on M . Then TFAE:

- ① ω is a smooth 1-form.
- ② For every smooth vector field X on M the function $\omega(X)$ is smooth on M .

Characterization of C^∞ 1-Forms

Reminder (see Section 14)

The space $\mathcal{X}(M)$ of smooth vector fields on M is a module over the ring $C^\infty(M)$.

Corollary

Every smooth 1-form ω on M defines a $C^\infty(M)$ -module homomorphism,

$$\omega : \mathcal{X}(M) \longrightarrow C^\infty(M), \quad X \longrightarrow \omega(X).$$

Pullbacks of 1-Forms

Reminder (Pullback by a linear map)

Let $f : V \rightarrow W$ be a linear maps between vector spaces.

- By duality we have a linear map,

$$f^* : W^* \longrightarrow V^*, \quad \varphi \longrightarrow \varphi \circ f.$$

- If $\varphi \in W^*$, we call $f^*\varphi = \varphi \circ f$ the pullback of φ by f .

Consequence

Let $F : N \rightarrow M$ be a smooth map and let $p \in N$.

- The differential $F_{*,p} : T_p N \rightarrow T_{F(p)} M$ is a linear map.
- We thus get a pullback map $(F_{*,p})^* : T_{F(p)}^* M \rightarrow T_p^* N$.

Pullbacks of 1-Forms

Definition (Pullback of 1-forms)

Let $F : N \rightarrow M$ be a smooth map. If ω is a 1-form on M , then its *pullback* $F^*\omega$ is the 1-form on N defined by

$$(F^*\omega)_p = (F_{*,p})^* \omega_{F(p)} = \omega_{F(p)} \circ F_{*,p}, \quad p \in N.$$

Remark

In other words, for all $X_p \in T_p N$, we have

$$(F^*\omega)_p(X_p) = \omega_{F(p)} \circ F_{*,p}(X_p) = \omega_{F(p)}(F_{*,p}(X_p)).$$

Remark

Smooth functions can be pullbacked as well: if $h \in C^\infty(M)$, then $F^*h = h \circ F$.

Pullbacks of 1-Forms

Proposition (Proposition 17.10)

Let $F : N \rightarrow M$ be a smooth map. If $h \in C^\infty(M)$, then

$$F^*(dh) = d(F^*h).$$

Proposition (Proposition 17.11)

Let $F : N \rightarrow M$ be a smooth map. If $\omega, \tau \in \Omega^1(M)$ and $g \in C^\infty(M)$, then

$$F^*(\omega + \tau) = F^*\omega + F^*\tau,$$

$$F^*(g\omega) = (F^*g)(F^*\omega).$$

Proposition (Proposition 17.12)

Let $F : N \rightarrow M$ be a smooth map. If ω is a smooth 1-form on M , then its pullback $F^*\omega$ is a smooth 1-form as well.

Restriction to an Immersed Submanifold

Facts

Let S be an immersed submanifold in M .

- The inclusion $i : S \rightarrow M$ is an immersion, and so its differential $i_{*,p} : T_p S \rightarrow T_p M$ is an injection for every $p \in S$.
- This allows us to identify $T_p S$ with a subspace of $T_p M$.

Definition

If ω is a 1-form on M , its *restriction* to S , denoted $\omega|_S$, is the 1-form on S defined by

$$(\omega|_S)_p(v) = \omega_p(v) \quad \text{for all } p \in S \text{ and } v \in T_p S.$$

Proposition

If $i : S \rightarrow M$ is the inclusion of S into M and ω is a 1-form on M , then $\omega|_S = i^*\omega$.