Differentiable Manifolds
§14. Vector Fields

Sichuan University, Fall 2022

Smoothness of a Vector Field

Reminder (The Tangent bundle; see Section 12)

Suppose that M is a smooth manifold of dimension n.

- The tangent bundle $TM = \bigsqcup_{p \in M} T_pM$ is a smooth vector bundle of rank n over M with projection $\pi : TM \to M$ such that
$$\pi(p, v) = p \qquad \text{if } v \in T_pM,\ p \in M.$$
- Any chart (U, ϕ) for M defines a chart $(TU, \tilde{\phi})$ for TM, where $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ is given by

$$\tilde{\phi}(p, v) = (\phi(p), v^1, \ldots, v^n), \quad v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p \in T_pM,\ p \in M.$$

- This also defines a trivialization (TU, ψ) of TM over TU, where $\psi = (\phi^{-1} \times \mathbb{1}_{\mathbb{R}^n}) \circ \tilde{\phi} : TU \to U \times \mathbb{R}^n$ is given by

$$\psi(p, v) = (p, v^1, \ldots, v^n), \quad v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p \in T_pM,\ p \in M.$$

Smoothness of a Vector Field

Reminder (Sections and frames of a vector bundle; see Section 12)

Suppose $E \xrightarrow{\pi} M$ is a smooth vector bundle or rank r over M and U is an open set of M.

- A *section* of E over U is any map $s : U \to TM$ such that $\pi \circ s = \mathbb{1}_U$, i.e., $s(p) \in E_p$ for all $p \in U$.
- A *smooth frame* of E over U is given by smooth sections $s_1, \ldots, s_r$ over U such that $\{s_1(p), \ldots, s_r(p)\}$ is a basis of the fiber E_p for every $p \in U$.

Reminder (Proposition 12.2)

Let $\{s_1, \ldots, s_r\}$ be a C^∞ frame of E over U. A section $s = \sum c^i s_i$ of E over U is smooth if and only if $c^1, \ldots, c^r$ are smooth functions on U.

Smoothness of a Vector Field

Definition (Vector field)

A vector field is a section $X : M \to TM$. That is, it assigns to each $p \in M$ a tangent vector $X_p \in T_pM$.

Lemma (see slides on Section 12)

If $(U, x^1, \dots, x^n)$ is a chart for M, then $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ is a smooth frame of TM over U.

Specializing Proposition 12.12 to the smooth frame $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ then gives:

Lemma (Lemma 14.1)

Let $(U, x^1, \dots, x^n)$ be a chart for M. A vector field $X = \sum a^i \frac{\partial}{\partial x^i}$ on U is smooth if and only if the coefficients $a^1, \dots, a^n$ are smooth functions on U.

Smoothness of a Vector Field

Proposition

Let X be a vector field on M. TFAE:

(i) *X is a smooth vector field.*

(ii) *There is a C^∞ atlas of M such that, for each chart $(U, x^1, \ldots, x^n)$ of the atlas, the coefficients a^i of $X = \sum a^i \frac{\partial}{\partial x^i}$ relative to the frame $\left\{\frac{\partial}{\partial x^i}\right\}$ are C^∞ functions.*

(iii) *For every chart $(U, x^1, \ldots, x^n)$ of M, the coefficients a^i of $X = \sum a^i \frac{\partial}{\partial x^i}$ relative to the frame $\left\{\frac{\partial}{\partial x^i}\right\}$ are C^∞ functions.*

Remarks

1. It is immediate that (iii) $\Rightarrow$ (ii).
2. The implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) follow from the previous lemma.
3. The equivalence (i) $\Leftrightarrow$ (ii) holds for any C^∞ atlas of M. In the case of the maximal C^∞ atlas of M we obtain (i) $\Leftrightarrow$ (iii).

Smoothness of a Vector Field

Facts

Suppose that X is a vector field on M.

- Let $p \in M$. By definition T_pM is the space of point-derivation on the space of germs C_p^∞.
- Thus, the tangent vector $X_p \in T$ is a linear map $C_p^\infty \to \mathbb{R}$ such that

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g), \qquad f, g \in C_p^\infty.$$

Smoothness of a Vector Field

Definition

If X is a vector field on M and $f \in C^\infty(M)$, we define the function $Xf : M \to \mathbb{R}$ by

$$\big(Xf\big)(p) = X_p(f), \qquad p \in M.$$

Remark

$X_p(f)$ depends only on the germ of f at p. Thus,

$$f = g \text{ near } p \implies X_p(f) = X_p(g).$$

It follows that

$$f = g \text{ on an open } V \implies X(f) = X(g) \text{ on } V.$$

Example

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart for M. Denote by $(r^1, \dots, r^n)$ the coordinates on $\mathbb{R}^n$, so that $x^i = r^i \circ \phi$.

- Let $f \in C^\infty(U)$. For all $p \in U$, we have
$$\frac{\partial f}{\partial x^i}(p) = \left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} \left(f \circ \phi^{-1}\right) = \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial r^i}\left[\phi(p)\right].$$
- That is,
$$\frac{\partial f}{\partial x^i} = \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial r^i} \circ \phi \qquad \text{on } U.$$
It then follows that $\partial f / \partial x^i$ is C^∞ on U.
- For instance, for $f = x^j = r^j \circ \phi$ we get
$$\frac{\partial x^j}{\partial x^i} = \frac{\partial r^j}{\partial r^i} \circ \phi = \delta_i^j.$$

Smoothness of a Vector Field

Facts

Let $(U, x^1, \dots, x^n)$ be a chart for M and let $X = \sum a^i \partial/\partial x^i$ be a vector field on U.

- Let $f \in C^\infty(U)$. Then
$$Xf = \sum a^i \frac{\partial f}{\partial x^i}.$$
- We know from the previous example that $\partial f/\partial x^i \in C^\infty(U)$. Thus, if X is a C^∞ vector field, then the coefficients a^i are C^∞-functions on U, and hence Xf is C^∞ on U.
- For $f = x^j$ we get
$$X(x^j) = \sum_{1 \leq i \leq n} a^i \frac{\partial x^j}{\partial x^i} = \sum_{1 \leq i \leq n} a^i \delta_i^j = a^j.$$
- Thus, if $Xf \in C^\infty(U)$ for all $f \in C^\infty(U)$, then the coefficients $a^j = X(x^j)$ are C^∞, and hence the vector field X is C^∞ on U.

To sum up we have proved:

Lemma

Let (U, ϕ) be a chart for M and X a vector field on U. TFAE:

(i) *X is a smooth vector field on U.*

(ii) *$Xf \in C^\infty(U)$ for all $f \in C^\infty(U)$.*

More generally, we have:

Proposition (Proposition 14.3)

Let X be a vector field on M. TFAE:

(i) *X is a smooth vector field.*

(ii) *$Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$.*

Smoothness of a Vector Field

Proof of Proposition 4.3.

- If X is a smooth vector field and $f \in C^\infty(M)$, then Xf is C^∞ on every domain of chart, and hence is C^∞ on M.
- Conversely, suppose that $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$. Let $(U, x^1, \ldots, x^n)$ be a chart for M. Thus, $X = \sum a^i \partial/\partial x^i$ on U with $a^i = X(x^i)$.
- Let $p \in U$. By Proposition 13.2 there is $\tilde{x}^i \in C^\infty(M)$ such that $\tilde{x}^i = x^i$ near p, and then $a^i = X(x^i) = X(\tilde{x}^i)$ near p.
- As $X(\tilde{x}^i) \in C^\infty(M)$, it follows that the coefficients a^i are C^∞ near every $p \in U$, and hence are C^∞ on U.
- As this is true for every chart (U, ϕ) it follows from Proposition 14.2 that X is a smooth vector field on M.

The proof is complete. $\square$

Smoothness of a Vector Field

Reminder (Derivations of an algebra; see Section 2)

Let A be an algebra over a field $\mathbb{K}$. A *derivation* of A is any linear map $D : A \rightarrow A$ such that

$$D(ab) = (Da)b + a(Db) \qquad \text{for all } a, b \in A.$$

Corollary

Let X is a smooth vector field on M. Then $f \rightarrow Xf$ is a derivation of the algebra $C^{\infty}(M)$.

Remarks

- Conversely, it can be shown that every derivation of $C^{\infty}(M)$ arises from a smooth vector field (see Problem 19.12).
- We often identify a C^{∞} vector field X with the derivation $f \rightarrow Xf$.

Smoothness of a Vector Field

Proof of the corollary.

- As X is a smooth vector field $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$. As Xf depends linearly on f, it follows that $f \to Xf$ is a linear map from $C^\infty(M)$ to itself.
- Given $f, g \in C^\infty(M)$ and $p \in M$, as X_p is a point-derivation on C_p^∞, we have

  $$\begin{aligned} X(fg)(p) = X_p(fg) &= X_p(f)g(p) + f(p)X_p(g) \\ &= X(f)(p)g(p) + f(p)X(g)(p). \end{aligned}$$

  Thus, $X(fg) = (Xf)g + f(Xg)$, and so $f \to Xf$ is a derivation of $C^\infty(M)$.

The proof is complete. $\square$

Smoothness of a Vector Field

Reminder (Proposition 12.9)

If E is smooth vector bundle over E, then the set of C^∞ sections $\Gamma(E)$ is a module over $C^\infty(M)$ with respect to the scalar multiplication,

$$(fs)(p) = f(p)s(p), \quad f \in C^\infty(M),\ s \in \Gamma(E),\ p \in M.$$

Consequence

Let $\mathscr{X}(M)$ be the space of C^∞ vector fields on M. Then $\mathscr{X}(M)$ is a module over $C^\infty(M)$. If $f \in C^\infty$ and $X \in \mathscr{X}(M)$, then $fX \in \mathscr{X}(M)$ is given by

$$(fX)(p) = f(p)X_p, \qquad p \in M.$$

Smoothness of a Vector Field

Proposition (Proposition 14.4)

Let X be a smooth vector field on an open U in M. Given any $p \in U$, there is a smooth vector field $\widetilde{X}$ on M such that

$$\widetilde{X} = X \quad \text{near } p.$$

Proof of Proposition 14.4.

Let $\rho : M \to \mathbb{R}$ be a C^∞ bump function at p supported in U, and define $\widetilde{X} : M \to M$ by

$$\widetilde{X} = \rho X \quad \text{on } U, \qquad \widetilde{X} = 0 \quad \text{on } M \setminus U.$$

This defines a smooth section of TM over M. Namely:

- $\widetilde{X}$ agrees with the C^∞ vector field ρX on U, and hence it is C^∞ on U.
- $\widetilde{X} = 0$ on $M \setminus U$ and $U \setminus \mathrm{supp}(\rho)$, and hence is C^∞ on the open set $M \setminus \mathrm{supp}(\rho)$.

Thus, $\widetilde{X}$ is C^∞ on $U \cup (M \setminus \mathrm{supp}(\rho)) = M$. In addition, as $\rho = 1$ near p, we have

$$\widetilde{X} = \rho X = X \quad \text{near } p.$$

This proves the result. $\square$

The Lie Bracket

Facts

Suppose that X and Y are smooth vector fields on M.

- If $f \in C^\infty(M)$, then Yf and $X(Yf)$ are C^∞ functions on M.
- If $f = g$ near p, then $Yf = Yg$ near p. Thus, the germ of Yf at p depends only on the germ of f at p. We then have

$$X(Yf)(p) = X_p(Yf).$$

- It follows we get a linear map,

$$C_p^\infty \ni f \longrightarrow X_p(Yf) \in \mathbb{R}$$

The Lie Bracket

Definition

If X and Y are smooth vector fields on M, then their *Lie bracket at a point* $p \in M$ is the linear map $[X, Y]_p : C_p^\infty \to \mathbb{R}$ defined by

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf), \qquad f \in C_p^\infty.$$

Lemma

$[X, Y]_p \in T_pM$, i.e., $[X, Y]_p$ is a derivation at p.

Definition

If X and Y are smooth vector fields on M, then their *Lie bracket* is the vector field,

$$[X, Y] : M \longrightarrow TM, \qquad p \longrightarrow [X, Y]_p.$$

The Lie Bracket

Remark

If $f \in C^\infty(M)$ and $p \in M$, then

$$\big([X,Y]f\big)(p) = [X,Y]_p(f) = X_p(Yf) - Y_p(Xf)$$
$$= X(Yf)(p) - Y(Xf)(p).$$

Thus,

$$[X,Y]f = X(Yf) = Y(Xf) \in C^\infty(M).$$

As this is true for all $f \in C^\infty(M)$, we obtain:

Proposition (Proposition 14.10)

If X and Y are smooth vector fields on M, then $[X,Y]$ is a smooth vector field on M as well.

Remark

If we regard X, Y and $[X,Y]$ are derivations on $C^\infty(M)$, then

$$[X,Y] = X \circ Y - Y \circ X.$$

The Lie Bracket

Definition (Lie algebras)

A *Lie algebra over a field* $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ together with an alternating bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ satisfying Jacobi's identity,

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$

Remark

In general, a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ need not be an algebra, since the bracket $[\cdot,\cdot]$ may fail to be associative.

The Lie Bracket

Example

Any vector space V over $\mathbb{K}$ is a Lie algebra with respect to the zero bracket $[x, y] = 0$. Such a Lie algebra is called an *Abelian Lie algebra*.

Example

Any algebra A over $\mathbb{K}$ is a Lie algebra with respect to the bracket,

$$[x, y] = xy - yx, \qquad x, y \in \mathfrak{g}.$$

The Lie Bracket

Proposition (see Exercise 14.11)

The space $\mathscr{X}(M)$ of smooth vector fields on M is a Lie algebra over $\mathbb{R}$ with respect to the Lie bracket of vector fields.

Remark

Let A be an algebra over $\mathbb{K}$. Denote by $\mathrm{Der}(A)$ the space of derivations of A.

- If D_1 and D_2 are derivations of A, then

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$$

  is again a derivation of A.
- $(\mathrm{Der}(A), [\cdot, \cdot])$ is a Lie algebra.

Definition (Derivation of a Lie algebra)

A *derivation* of a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ such that

$$D\big([X, Y]\big) = [DX, Y] + [X, DY] \qquad \text{for all } X, Y \in \mathfrak{g}.$$

Example

Given X, define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}_X\, Y = [X, Y], \qquad Y \in \mathfrak{g}.$$

Then ad_X is a derivation of the Lie algebra $\mathfrak{g}$, called the adjoint endomorphism of X.

Remark

In fact, Jacobi's identity is equivalent to ad_X being a derivation for every $X \in \mathfrak{g}$.

Pushforwards and Related Vector Fields

Definition (Pushforward of tangent vector)

Let $F : N \to M$ be a C^∞ map between smooth manifolds. Given $p \in N$ and $X_p \in T_pN$, the tangent vector $F_*(X_p) \in T_{F(p)}M$ is called the *pushforward of* X *by* F *at* p.

Remarks

1. By the very definition of the differential $F_* : T_pN \to T_{F(p)}N$ we have
$$F_*(X_p)g = X_p(g \circ F) \qquad \text{for all } g \in C^\infty_{F(p)}(M).$$
2. In general, if X is a vector field on N there need not exist a vector field $\widetilde{X}$ on M such that $F_*(X_p) = \widetilde{X}_{F(p)}$ for all $p \in N$.
3. However, this is possible when F is a diffeomorphism, since

$$F_{*,p}(X_p) = \widetilde{X}_{F(p)}\ \forall p \in N \iff \widetilde{X}_q = F_{*,F^{-1}(q)}\big(X_{F^{-1}(q)}\big)\ \forall q \in M.$$

Pushforwards and Related Vector Fields

Definition (Pushforward of a vector field)

Suppose that $F : N \to M$ is a diffeomorphism. The *pushforward by* F of a vector field X on N is the vector field $F_* X$ on M defined by

$$(F_* X)_q = F_{*, F^{-1}(q)}\big(X_{F^{-1}(q)}\big), \qquad q \in M.$$

Remark

If $g \in C^\infty(M)$, then $(F_*X)g(q) = (F_*X)_q g$ is equal to

$$F_{*,F^{-1}(q)}\big(X_{F^{-1}(q)}\big)g = X_{F^{-1}(q)}(g \circ F) = X(g \circ F)\big(F^{-1}(q)\big).$$

Thus,

$$(F_*X)g = \big[X(g \circ F)\big] \circ F^{-1}.$$

In particular, if X is a smooth vector field, then $(F_*X)g \in C^\infty(M)$ for all $g \in C^\infty(M)$, and so F_*X is a smooth vector field.

Therefore, we have obtained:

Proposition

If $F : N \to M$ is a diffeomorphism and X is a smooth vector field on N, then the pushforward F_*X is a smooth smooth vector field on M.

Pushforwards and Related Vector Fields

Definition (Related vector fields)

Let $F : N \to M$ be a smooth map. We say that a vector field X on N and a vector field $\widetilde{X}$ on M are F*-related* if

$$F_{*,p}(X_p) = \widetilde{X}_{F(p)} \qquad \forall p \in N.$$

Example

- If F is a diffeomorphism, then X and F_*X are F-related.
- In fact, F_*X is the unique vector field on M that is F-related to X.

Pushforwards and Related Vector Fields

Remark

Let X be a vector field on N and $\widetilde{X}$ a vector field on M.

- X and $\widetilde{X}$ are F-related if and only if
$$F_{*,p}(X)g = \widetilde{X}_{F(p)}(g) \quad \forall g \in C^\infty(M)\ \forall p \in N.$$
- We have
$$F_{*,p}(X)g = X_p(g \circ F) = X(g \circ F)(p),$$
$$\widetilde{X}_{F(p)}(g) = \widetilde{X}g\big(F(p)\big) = \big(\widetilde{X}g\big) \circ F(p).$$
- Thus, X and $\widetilde{X}$ are F-related if and only if
$$X(g \circ F)(p) = \big(\widetilde{X}g\big) \circ F(p) \quad \forall g \in C^\infty(M)\ \forall p \in N.$$

Pushforwards and Related Vector Fields

From the previous remark we have the following result.

Proposition (Proposition 14.16)

Let $F : N \to M$ be a smooth map. Then a vector field X on N and a vector field $\tilde{X}$ on M are F-related if and only if

$$X(g \circ F) = \left(\widetilde{X} g\right) \circ F \quad \forall g \in C^\infty(M).$$

As an application of Proposition 14.16, we shall get:

Proposition (Proposition 14.17)

Let $F : N \to M$ be a smooth map. Suppose that X and Y are C^∞ vector fields on N that are F-related to smooth vector fields $\widetilde{X}$ and $\widetilde{Y}$ on M, respectively. Then the Lie bracket $[X, Y]$ is F-related to $[\widetilde{X}, \widetilde{Y}]$.

Proof of Proposition 14.17.

The proof is a consequence of Proposition 14.16.

- Let $g \in C^\infty(M)$. As X and Y are F-related to $\widetilde{X}$ and $\widetilde{Y}$, by Proposition 14.16 we have

$$Y(g \circ F) = \big(\widetilde{Y}g\big) \circ F,$$

$$X\big(Y(g \circ F)\big) = X\big((\widetilde{Y}g) \circ F\big) = \big(\widetilde{X}(\widetilde{Y}g)\big) \circ F = \big(\widetilde{X}\widetilde{Y}g\big) \circ F.$$

- Likewise, $Y\big(X(g \circ F)\big) = \big(\widetilde{Y}\widetilde{X}g\big) \circ F$. Thus,

$$\begin{aligned}[X, Y](g \circ F) &= X\big(Y(g \circ F)\big) - Y\big(X(g \circ F)\big)\\ &= \big(\widetilde{X}\widetilde{Y}g\big) \circ F - \big(\widetilde{Y}\widetilde{X}g\big) \circ F\\ &= \big([\widetilde{X}, \widetilde{Y}]g\big) \circ F.\end{aligned}$$

  As this holds for all $g \in C^\infty(M)$, it follows from Proposition 14.16 that $[X, Y]$ and $[\widetilde{X}, \widetilde{Y}]$ are F-related.

The proof is complete. $\square$

Corollary (see Problem 14.4)

Let $F : N \to M$ be a diffeormorphism. Given any smooth vector field X and Y on N we have

$$F_*([X, Y]) = [F_*X, F_*Y].$$

Proof.

- $F_*([X, Y])$ is the unique vector field on M that is F-related to $[X, Y]$ (see slide 27).
- As F_*X and F_*Y are F-related to X and Y, Proposition 14.17 ensures us that $[F_*X, F_*Y]$ is F-related to $[X, Y]$, and hence it agrees with $F_*([X, Y])$.

The proof is complete. □

Integral Curves and Flows

Definition

Suppose that X is a smooth vector field on M.

- An *integral curve* of X is any smooth curve $c : (a, b) \to M$ satisfying the equation,

$$\frac{d}{dt}c(t) = X_{c(t)} \qquad t \in (a, b).$$

- If the interval (a, b) contains 0 and $c(0) = p$, then we say that the curve *starts at* p and p is its *initial point*.
- We say that an integral curve is *maximal* if it cannot be extended to an integral curve defined on a larger interval.

Remark

In other words, an integral curve is a curve that is tangent to X at every point.

Integral Curves and Flows

Remark

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M and $c : I \to U$ a C^∞ curve in U. Set $\phi \circ c(t) = (c^1(t), \ldots, c^n(t))$, with $c^i = x^i \circ c$.

- As X is a C^∞ vector field, $X = \sum a^i \partial/\partial x^i$ on U with $a^i \in C^\infty(U)$. In particular,
$$X_{c(t)} = \sum a^i(c(t)) \frac{\partial}{\partial x^i}, \qquad t \in I.$$
- It is shown in Section 8 (see Propositions 8.11 and 8.15) that
$$\frac{dc}{dt} = \sum \dot{c}^i(t) \frac{\partial}{\partial x^i}\bigg|_{c(t)}, \qquad t \in I.$$
- Thus,
$$\frac{dc}{dt} = X_{c(t)} \Longleftrightarrow \dot{c}^i(t) = a^i(c(t)) \quad \text{for } i = 1, \ldots, n.$$
In other words the integral curve equation for X on U reduces to an ODE system for the components $c^1(t), \ldots, c^n(t)$.

Example

Let $X = -y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$.

- If $c(t) = (x(t), y(t))$, then

$$\frac{dc}{dt} = \big(\dot{x}(t), \dot{y}(t)\big), \qquad X_{c(t)} = -y(t)\partial/\partial x + x(t)\partial/\partial y.$$

- Thus,

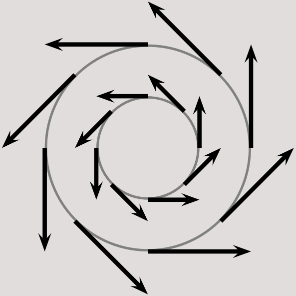
$$\frac{dc}{dt} = X_{c(t)} \Leftrightarrow \begin{cases} \dot{x}(t) = -y(t), \\ \dot{y}(t) = x(t). \end{cases} \Leftrightarrow \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- The solution of the ODE system is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

- The integral curves are circles about the origin.

Example (continued)



The integral curves of $X = -y\partial/\partial x + x\partial/\partial y$.

Using ODE theory we obtain the following existence and uniqueness theorem:

Theorem (see Theorem 4.7)

Let X be a smooth vector field on M. Given any $p \in M$, there is a unique maximal integral curve for X that starts at p.

Remarks

- In particular, any integral curve starting at p extends to a unique maximal integral curve.
- This implies that two integral curves with same initial point agree on their joint domain.

The following result is established in J.M. Lee's book.

Theorem (Fundamental Theorem on Flows)

Suppose that X is a smooth vector field on M. Define

$$\Omega = \bigcup_{p \in M} I^{(p)} \times \{p\} \subset \mathbb{R} \times M,$$

where $I^{(p)}$ is the open interval around 0 on which is defined the maximal integral curve of X starting at p. Then:

(i) *Ω is an open set in $\mathbb{R} \times M$ containing $\{0\} \times M$.*

(ii) *There is a smooth map $F : \Omega \to M$, $(t, p) \to F_t(p)$ (called <u>the flow of X</u>) such that, for every $p \in M$, the curve $I^{(p)} \ni t \to F_t(p) \in M$ is the maximal integral curve of X starting at p.*

Remarks

Let $p \in M$.

- We have $I^{(p)} = \{t \in \mathbb{R}; (t,p) \in \Omega\}$.
- As $F_t(p)$ is an integral curve for X starting at p, we have
$$\frac{d}{dt} F_t(p)\bigg|_{t=0} = X_{F_t(p)}\bigg|_{t=0} = X_{F_0(p)} = X_p.$$
Thus, we recover the vector field X from its flow.

Integral Curves and Flows

Remarks

- For $t \in \mathbb{R}$, the set $M_t = \{p \in M; (t,p) \in \Omega\}$ is open in M, since Ω is an open set. Thus, we may regard F_t as a smooth map $p \to F_t(p)$ from M_t to M.
- Here $M_0 = M$ and $F_0(p) = p$ for all $p \in M$. Thus, $F_0 = \mathbb{1}_M$.
- If $s \in I^{(p)}$, then $t \to F_t(F_s(p))$ and $t \to F_{t+s}(p)$ are maximal integral curves for X starting at $F_s(p)$, and so they agree.
- Thus, $I^{(F_s(p))} = I^{(p)} - s$, and we have

$$F_t \circ F_s = F_{t+s} \qquad \text{on } M_{s+t} \cap M_s = F_s^{-1}(M_t) \cap M_s.$$

- For $t = -s$ we get $F_s^{-1}(M_{-s}) \cap M_s = M_0 \cap M_s = M_s$, so that $F_s^{-1}(M_{-s}) = M_s$, and $F_{-s} \circ F_s = F_0 = \mathbb{1}_M$ on M_s.
- Likewise, $F_s \circ F_{-s} = \mathbb{1}_M$ on M_{-s}. Thus, $F_s : M_s \to M_{-s}$ is a diffeomorphism with inverse $F_{-s} : M_{-s} \to M_s$.

Integral Curves and Flows

Definition

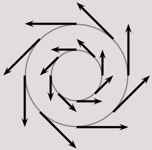
We say that X is a *complete vector field* when its the flow F is defined on all $\mathbb{R} \times M$. In this case we call F a *global flow*.

Remarks

- If F is a global flow defined on $\mathbb{R}$, then $M_t = M$ and F_t is a smooth map from M to itself for all $t \in \mathbb{R}$.
- We then have
$$F_0 = \mathbb{1}_M, \qquad F_s \circ F_t = F_{t+s} \quad \text{on } M \quad \forall s, t \in \mathbb{R}.$$
- For $s = -t$ we get $F_{-t} \circ F_t = F_t \circ F_{-t} = \mathbb{1}_M$, and hence $F_t : M \to M$ is a diffeomorphism with inverse F_{-t}.
- Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms of M. Then $t \to F_t$ is a morphism from the additive group $\mathbb{R}$ to $\mathrm{Diff}(M)$. It is called a *one-parameter group of diffeomorphisms*.

Example (see Tu's book)

Let X be the vector field $-y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$.



It has a global flow $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F_t(p) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad p = \begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R}.$$

That is, $F_t : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation of angle t about the origin. Note that

$$\left.\frac{d}{dt}\right|_{t=0} F_t(p) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Integral Curves and Flows

Remark (see Lee's book)

- It can be shown that every compactly supported smooth vector field is complete.
- In particular, every smooth vector field on a compact manifold is complete.

Remark

We will see in Section 16 that every left-invariant vector field on a Lie group is complete.