# Noncommutative Geometry
# Chapter 8:
# Pseudodifferential Operators

Sichuan University, Fall 2022

## Additional References


- Shubin, M.A.: *Pseudodifferential operators and spectral theory*, 2nd edition. Springer, Berlin, 2001.
- Taylor, M.E.: *Pseudodifferential operators*. Princeton University Press, Princeton, NJ, 1981.

- Notes to be posted on the course's website (draft version).

# Differential Operators

**Setup**

$U \subset \mathbb{R}^n$ is an open set.

**Notation**

- If $\alpha \in \mathbb{N}_0^n$, then $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
- $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 1, \ldots, n$.
- If $\alpha \in \mathbb{N}_0^n$, then $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$.

# Differential Operators

**Definition**

A differential operator $P : C^\infty(U) \to C^\infty(U)$ of order $m$ is of the form,

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \qquad a_\alpha(x) \in C^\infty(U).$$

**Example**

Laplace operator $\Delta := -(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2) = D_{x_1}^2 + \cdots + D_{x_n}^2$.

# Differential Operators

## Definition

Let $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a differential operator.

- Its symbol is
$$p(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \qquad (x,\xi) \in U \times \mathbb{R}^n.$$
- The principal part is the $m$-th degree part,
$$p(x,\xi) := \sum_{|\alpha| = m} a_\alpha \xi^\alpha, \qquad (x,\xi) \in U \times \mathbb{R}^n.$$

## Example

For the Laplace operator $\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2$, we have
$$p(x,\xi) = p_2(x,\xi) = \xi_1^2 + \cdots + \xi_n^2 = |\xi|^2.$$

## Remark

If $P$ has symbol $p(x,\xi)$, we often write $P = p(x,D)$.

## Notation

- If $u \in L^1(\mathbb{R}^n)$, then its Fourier transform is

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x) dx, \qquad x \in \mathbb{R}^n.$$

- Its inverse Fourier transform is

$$\check{u}(\xi) = \int e^{ix\cdot\xi} u(\xi) đ\xi, \qquad đ\xi := (2\pi)^{-n} d\xi.$$

## Remark

If $u$ is in the Schwartz's class $\mathcal{S}(\mathbb{R}^n)$, then

$$(D_x^\alpha u)^\alpha = \xi^\alpha \hat{u}.$$

**Fact**

Let $P = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha$ be a differential operator on $U$. If $p(x,\xi)$ is the symbol of $P$, then

$$Pu(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi) đ\xi \qquad \forall u \in C_c^\infty(U).$$

**Proof.**

- As $(D_x^\alpha u)^\wedge = \xi^\alpha \hat{u}$, we have
$$D_x^\alpha u = \left((D_x^\alpha u)^\wedge\right)^\vee = (\xi^\alpha \hat{u})^\vee = \int e^{ix\cdot\xi}\xi^\alpha \hat{u}(\xi) đ\xi.$$
- Thus,
$$\begin{aligned} Pu = \sum a_\alpha(x) D_x^\alpha u &= \sum a_\alpha(x) \int e^{ix\cdot\xi}\xi^\alpha \hat{u}(\xi) đ\xi \\ &= \int e^{ix\cdot\xi}\left(\sum a_\alpha(x)\xi^\alpha\right)\hat{u}(\xi) đ\xi \\ &= \int e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi) đ\xi. \end{aligned}$$

□

**Definition (Homogeneous Symbols)**

$S_m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, consists of functions $p(x,\xi)$ in $C^\infty(U \times (\mathbb{R}^n \setminus 0))$ such that

$$p(x, \lambda\xi) = \lambda^m p(x,\xi) \qquad \forall (x,\xi) \in U \times (\mathbb{R}^n \setminus 0)\ \forall \lambda > 0.$$

**Definition (Classical Symbols)**

$S^m(U \times \mathbb{R}^n)$, $m \in \mathbb{C}$, consists of functions $p(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ with an expansion,

$$p(x,\xi) \sim \sum_{j \geq 0} p_{m-j}(x,\xi), \qquad p_{m-j} \in S_{m-j}(U \times \mathbb{R}^n),$$

Here $\sim$ means that, for any integer $N$, compact $K \subset U$ and multi-orders $\alpha$ and $\beta$, there is $C_{NK\alpha\beta} > 0$ s.t., for all $x \in K$ and all $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$, we have

$$|\partial_x^\alpha \partial_\xi^\beta (p - \sum_{j<N} p_{m-j})(x,\xi)| \leq C_{NK\alpha\beta} |\xi|^{\Re m - N - |\beta|}.$$

**Remark**

- $p_{m-j}(x,\xi)$ is called the symbol of degree $m-j$.
- $p_m(x,\xi)$ is called the principal symbol.

**Remark**

The homogeneous symbols $p_{m-j}(x,\xi)$ are uniquely determined by $p(x,\xi)$, since we have

$$p_m(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-m} p(x,\lambda\xi), \qquad \xi \neq 0,$$

$$p_{m-j}(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-m+j}\Big(p(x,\lambda\xi) - \sum_{k<j} \lambda^{m-k} p_{m-k}(x,\xi)\Big), \quad j \geq 1.$$

**Example**

Let $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a differential operator of order $m$.

- Its symbol is
$$p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha.$$
- We have
$$p(x,\xi) = \sum_{0 \leq j \leq m} p_{m-j}(x,\xi), \quad p_{m-j}(x,\xi) := \sum_{|\alpha| = m-j} a_\alpha(x)\xi^\alpha.$$
- Here $p_{m-j}(x,\lambda\xi) = \lambda^{m-j} p_{m-j}(x,\xi)$, and so $p_{m-j}(x,\xi) \in S_{m-j}(U \times \mathbb{R}^n)$
- It then follows that $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$.

**Example**

- Set $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$ (Russian bracket).
- For any $s \in \mathbb{R}$, the binomial formula implies that

$$\langle\xi\rangle^s = |\xi|^s (1+|\xi|^{-2})^{\frac{s}{2}} \sim \sum_{j\geq 0} \binom{\frac{s}{2}}{j} |\xi|^{s-2j}.$$

- It follows that $\langle\xi\rangle^s$ is a classical symbol of order $s$ whose principal symbol is $|\xi|^s$.

**Definition (Standard Symbols)**

$\mathbb{S}^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, consists of functions $p(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ such that, for any compact $K \subset U$ and multi-orders $\alpha$ and $\beta$ in, there exists $C_{K\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{K\alpha\beta}(1+|\xi|)^{m-|\beta|} \qquad \forall (x,\xi) \in K \times \mathbb{R}^n.$$

**Remark**

$S^m(U \times \mathbb{R}^n) \subset \mathbb{S}^{\Re m}(U \times \mathbb{R}^n)$.

**Remark**

$\mathbb{S}^m(U \times \mathbb{R}^n)$ is a Fréchet space with respect to the locally convex topology generated by the seminorms given by the best constants $C_{K\alpha\beta}$ above.

**Definition (Smoothing Symbols)**

$\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$ consists of function $p(x, \xi) \in C^\infty(U \times \mathbb{R}^n)$ such that, for any integer $N$, any compact $K \subset U$ and any multi-orders $\alpha$ and $\beta$, there exists $C_{NK\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{NK\alpha\beta}(1 + |\xi|)^{-N} \qquad \forall (x, \xi) \in K \times \mathbb{R}^n.$$

**Remark**

$\mathbb{S}^{-\infty}(U \times \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} \mathbb{S}^m(U \times \mathbb{R}^n)$.

**Remark**

Let $p(x, \xi) \in S^m(U \times \mathbb{R}^n)$, $p(x, \xi) \sim \sum p_{m-j}(x, \xi)$. Then

$$\begin{aligned} p(x, \xi) \in \mathbb{S}^{-\infty}(U \times \mathbb{R}^n) &\iff p(x, \xi) \sim 0 \\ &\iff p_{m-j}(x, \xi) = 0 \quad \forall j \geq 0. \end{aligned}$$

**Lemma (Borel Lemma for Symbols)**

*For* $j = 0, 1, \ldots$, *let* $p_{m-j}(x,\xi) \in S_{m-j}(U \times \mathbb{R}^n)$. *Then:*

(i) *There is a symbol* $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$ *such that*

$$p(x,\xi) \sim \sum_{j \geq 0} p_{m-j}(x,\xi).$$

(ii) $p(x,\xi)$ *is unique modulo* $\mathbb{S}^{-\infty}(U \times \mathbb{R}^n)$.

# Pseudodifferential Operators

**Definition**

If $p(x, \xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, then $p(x, D) : C_c^\infty(U) \to C^\infty(U)$ is the linear operator given by

$$p(x, D)u(x) = \int e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi) đ\xi, \quad u \in C_c^\infty(U).$$

**Remark**

It can be shown that:

- $p(x, D) : C_c^\infty(U) \to C^\infty(U)$ is a continuous linear operator.
- $\mathbb{S}^m(U \times \mathbb{R}^n) \ni p(x, \xi) \to p(x, D) \in \mathcal{L}(C_c^\infty(U), C^\infty(U))$ is a continuous linear map.

# Pseudodifferential Operators

## Notation

- $\mathcal{D}'(U) =$ space of distributions on $U$.
- $\mathcal{E}'(U) =$ space of compactly supported distributions.

## Reminder

$\mathcal{D}'(U)$ and $\mathcal{E}'(U)$ are the respective topological duals of $C_c^\infty(U)$ and $C^\infty(U)$.

## Definition (Smoothing Operators)

- An operator $R : C_c^\infty(U) \to C^\infty(U)$ is called smoothing if it is given by a kernel $k_R(x,y) \in C^\infty(U \times U)$, i.e.,
$$Ru(x) = \int_U k_R(x,y)u(y)dy, \qquad u \in C_c^\infty(U).$$
- The space of smoothing operators is denoted $\Psi^{-\infty}(U)$.

**Proposition**

*Let $R : C_c^\infty(U) \to C^\infty(U)$ be a continuous linear operator. TFAE:*

(i) *$R$ is smoothing.*

(ii) *It uniquely extends to a continuous operator $\mathcal{E}'(U) \to C^\infty(U)$.*

**Example**

If $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$, then

$$p(x,D) \in \Psi^{-\infty}(U) \Longleftrightarrow p(x,\xi) \in \mathbb{S}^{-\infty}(U \times \mathbb{R}^n).$$

# Pseudodifferential Operators

## Definition (Pseudodifferential Operators (ΨDOs))

$\Psi^m(U)$, $m \in \mathbb{C}$, consists of linear operators $P : C_c^\infty(U) \to C^\infty(U)$ of the form,

$$P = p(x, D) + R,$$

with $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$, $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$, and $R \in \Psi^{-\infty}(U)$.

## Remarks

- The symbol $p_m(x,\xi)$ is called the principal symbol of $P$.
- The homogeneous symbols $p_{m-j}(x,\xi)$ depends only on $P$.
- In particular,

$$\begin{aligned} P \in \Psi^{-\infty}(U) &\Longleftrightarrow p(x,\xi) \in \mathbb{S}^{-\infty}(U \times \mathbb{R}^n) \\ &\Longleftrightarrow p(x,\xi) \sim 0 \\ &\Longleftrightarrow p_{m-j}(x,\xi) = 0 \quad \forall j \geq 0. \end{aligned}$$

**Example**

With $U = \mathbb{R}^n$ and let $\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2$ be the Laplace operator.

- This is a differential operator with symbol $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$.
- Thus,
$$(\Delta u)(x) = \int e^{ix\cdot\xi}|\xi|^2 \hat{u}(\xi) đ\xi.$$
- Therefore, if $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the Fourier transform, then
$$\Delta = U^{-1} M_{|\xi|^2} U,$$
where $M_{|\xi|^2}$ is the multiplication by $|\xi|^2$.
- This is the spectral theorem for $\Delta$, since $U$ is a unitary isomorphism.

**Example**

- Set $\Lambda = (1+\Delta)^{1/2}$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.
- Given any $s \in \mathbb{R}$, the Borel functional calculus for $\Delta$ gives

$$\Lambda^s = (1+\Delta)^{\frac{s}{2}} = U^{-1} M_{(1+|\xi|^2)^{\frac{s}{2}}} U = U^{-1} M_{\langle\xi\rangle^s} U.$$

- That is,

$$\Lambda^s u(x) = \int e^{ix\cdot\xi} \langle\xi\rangle^s \hat{u}(\xi) đ\xi.$$

- We saw that $\langle\xi\rangle^s \in S^s(\mathbb{R}^n \times \mathbb{R}^n)$ and has $|\xi|^s$ as principal symbol.
- Therefore, $\Lambda^s$ is an operator in $\Psi^s(\mathbb{R}^n)$ whose principal symbol is $|\xi|^s$.

# Schwartz Kernels of ΨDOs

**Theorem (Schwartz's Kernel Theorem)**

*Let* $P : C_c^\infty(U) \to \mathcal{D}'(U)$ *be a linear operator. TFAE:*

(i) $P$ *is continuous.*

(ii) *There is* $k_P(x, y) \in \mathcal{D}'(U \times U)$ *such that*

$$\langle Pu, v\rangle = \langle k_P(x, y), u(x)v(y)\rangle \qquad \forall u, v \in C_c^\infty(U).$$

**Remark**

If $P$ is continuous, then $k_P(x, y)$ is unique and is called the (Schwartz) kernel of $P$.

# Schwartz Kernels of ΨDOs

**Remark**

Let $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$ with $m < -n$.

- For any compact $K \subset U$,
$$|p(x,\xi)| \leq C_K(1+|\xi|)^m, \quad (x,\xi) \in K \times \mathbb{R}^n.$$
- As $m < -n$, the function $(1+|\xi|)^m$ is in $L^1(\mathbb{R}^n)$, and so we may define
$$\check{p}_{\xi \to y}(x,y) := \int e^{ix\cdot y} p(x,\xi) đ\xi \in C(K \times \mathbb{R}^n).$$

Therefore, we obtain:

**Lemma**

*If $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$, $m < -n$, then*
$$\check{p}_{\xi \to y}(x,y) := \int e^{ix\cdot y} p(x,\xi) đ\xi \in C(U \times \mathbb{R}^n).$$

**Lemma**

*Let* $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$ *with* $m < -n$, *and set* $P = p(x,D)$. *Then the the Schwartz kernel of* $P$ *is*

$$k_P(x,y) = \check{p}_{\xi\to y}(x, x-y).$$

**Proof.**

If $u \in C_c^\infty(U)$, then

$$\begin{aligned} Pu(x) &= \int e^{ix\cdot y} p(x,\xi)\hat{u}(\xi) đ\xi \\ &= \int e^{ix\cdot y} p(x,\xi) \left( \int e^{-iy\cdot\xi} u(y)dy \right) đ\xi \\ &= \int \left( \int e^{i(x-y)\cdot\xi} p(x,\xi) đ\xi \right) u(y)dy \\ &= \int \check{p}_{\xi\to y}(x,x-y)u(y)dy. \end{aligned}$$

This gives the result. □

**Remark**

- In general, if $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$, then the partial (inverse) Fourier transform $\check{p}_{\xi\to y}(x,y)$ makes a distribution.
- Namely, if $v \in C_c^\infty(\mathbb{R}^n)$, then

$$\langle \check{p}_{\xi\to y}(x,y), v(y)\rangle := \langle p(x,\xi), \check{v}(\xi)\rangle = \int p(x,\xi)\check{v}(\xi).$$

**Lemma**

*Let $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$ with $m < -n$, and set $P = p(x,D)$. Then the the Schwartz kernel of $P$ is*

$$k_P(x,y) = \check{p}_{\xi\to y}(x, x-y).$$

*More precisely, for all $u \in C_c^\infty(U)$,*

$$Pu(x) = \langle \check{p}_{\xi\to y}(x, x-y), u(y)\rangle = \langle \check{p}_{\xi\to y}(x,y), u(x-y)\rangle$$

**Lemma**

*Let* $p(x,\xi) \in \mathbb{S}^m(U \times \mathbb{R}^n)$, $m \in \mathbb{R}$. *Then* $\check{p}_{\xi\to y}(x,y)$ *is* $C^\infty$ *on* $U \times (\mathbb{R}^n \backslash 0)$.

**Proof.**

- We know that if $m < -n$, then $\check{p}_{\xi\to y}(x,y) \in C(U \times \mathbb{R}^n)$.
- In general as $y^\alpha \check{p}_{\xi\to y}(x,y) = (D_\xi^\alpha p)^\vee_{\xi\to y}$, we get

$$|y|^2 \check{p}_{\xi\to y} = \sum y_j^2 \check{p}_{\xi\to y} = \sum (D_{\xi_j}^2 p)^\vee_{\xi\to y} = (\Delta_\xi p)^\vee_{\xi\to y}.$$

- More generally, for all $N \geq 1$,

$$|y|^{2N} \check{p}_{\xi\to y}(x,y) = (\Delta_\xi^N p)^\vee_{\xi\to y}(x,y).$$

- Note that $\Delta_\xi^N p(x,\xi) \in \mathbb{S}^{m-2N}(U \times \mathbb{R}^n)$.
- Thus, if $N$ is so that $m - 2N < -n$, then $\Delta_\xi^N p(x,\xi)$ has order $< -n$, and hence $(\Delta_\xi^N p)^\vee_{\xi\to y}(x,y) \in C(U \times \mathbb{R}^n)$. □

**Proof.**

- If $N$ is such that $m-2N<-n$, i.e., $2N>m+n$, then
$$|y|^{2N}\check{p}_{\xi\to y}(x,y)=(\Delta_\xi^N p)^\vee_{\xi\to y}(x,y)\in C(U\times\mathbb{R}^n).$$
- This implies that
$$\check{p}_{\xi\to y}(x,y)\in C(U\times(\mathbb{R}^n\backslash 0)).$$
- More generally,
$$D_x^\alpha D_\xi^\beta\check{p}_{\xi\to y}(x,y)=(\xi^\beta D_x^\alpha p)(x,y)\in C(U\times(\mathbb{R}^n\backslash 0)).$$
- It follows that $\check{p}_{\xi\to y}(x,y)$ is $C^\infty$ on $U\times(\mathbb{R}^n\backslash 0)$. □

# Schwartz Kernels of ΨDOs

**Notation**

$\Gamma = \{(x,x);\ x \in U\}$ (diagonal of $U \times U$).

**Proposition**

*Let $P \in \Psi^m(U)$, $m \in \mathbb{C}$, have Schwartz kernel $k_P(x,y)$.*

1. *$k_P(x,y)$ is $C^\infty$ on $(U \times U)\setminus\Gamma$.*
2. *If $\Re m < -n$, then $k_P(x,y) \in C(U \times U)$.*

**Proof.**

- Let $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$ and $R \in \Psi^{-\infty}(U)$ be such that
$$P = p(x,D) + R.$$
- The Schwartz kernel of $p(x,D)$ is $\check{p}_{\xi\to y}(x, x-y)$.
- Let $k_R(x,y) \in C^\infty(U \times U)$ be the kernel of $R$. Then
$$k_P(x,y) = \check{p}_{\xi\to y}(x, x-y) + k_R(x,y).$$
- As $\check{p}_{\xi\to y}(x,y)$ is $C^\infty$ on $U \times (\mathbb{R}^n \backslash 0)$, we see that $k_P(x,y)$ is $C^\infty$ on $(U \times U)\backslash\Gamma$.
- If $\Re m < -n$, then $\check{p}_{\xi\to y}(x,y) \in C(U \times (\mathbb{R}^n \backslash 0))$, and hence $k_P(x,y) \in C(U \times U)$.

The proof is complete. □

# Properly Supported ΨDOs

## Definition

A linear operator $P : C_c^\infty(U) \to C^\infty(U)$ is called properly supported if, for every compact $K \subset U$, there are compact sets $K_1 \subset U$ and $K_2 \subset U$ such that

$$\operatorname{supp} u \subset K \implies \operatorname{supp} Pu \subset K_1,$$
$$K_2 \cap \operatorname{supp} u = \emptyset \implies K \cap \operatorname{supp} Pu = \emptyset.$$

## Proposition

*A (continuous) linear operator $P : C_c^\infty(U) \to C^\infty(U)$ is properly supported if and only if it gives rise to continuous operators,*

$$P : C_c^\infty(U) \longrightarrow C_c^\infty(U) \qquad \textit{and} \qquad P : C^\infty(U) \longrightarrow C^\infty(U)$$

# Properly Supported ΨDOs

**Proposition**

*Let* $P \in \Psi^m(U)$, $m \in \mathbb{C}$.

1. *If* $P$ *is properly supported, then there is* $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$ *such that*
$$P = p(x,D).$$
2. *We always can choose the symbol* $p(x,\xi)$ *of* $P$ *so that* $p(x,D)$ *is properly supported.*

**Remark**

In Part 1 we have $p(x,\xi) = e^{-ix\cdot\xi}P(e_\xi)$, with $e_\xi(x) := e^{ix\cdot\xi}$.

# Properly Supported ΨDOs

**Corollary**

*Any* $P \in \Psi^m(U)$ *can be put in the form,*

$$P = Q + R,$$

*with* $Q \in \Psi^m(U)$ *properly supported and* $R \in \Psi^{-\infty}(U)$.

# Composition of ΨDOs

## Remark

Let $P: C_c^\infty(U) \to C^\infty(U)$ and $Q: C_c^\infty(U) \to C^\infty(U)$ be ΨDOs.

- If $P$ is properly supported, then it uniquely extends to a continuous operator $P: C^\infty(U) \to C^\infty(U)$.
- Thus, $PQ$ makes sense as the composition,

$$PQ: C_c^\infty(U) \xrightarrow{Q} C^\infty(U) \xrightarrow{P} C^\infty(U).$$

- If $Q$ is properly supported, then it induces a continuous operator $Q: C_c^\infty(U) \to C_c^\infty(U)$.
- In this case, $PQ$ makes sense as the composition,

$$PQ: C_c^\infty(U) \xrightarrow{Q} C_c^\infty(U) \xrightarrow{P} C^\infty(U).$$

## Conclusion

If $P$ or $Q$ is properly supported, then the composition $PQ$ makes sense as an operator $PQ: C_c^\infty(U) \to C^\infty(U)$.

# Composition of ΨDOs

## Proposition

*Assume that:*

- $P \in \Psi^m(U)$ *has symbol* $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$.
- $Q \in \Psi^{m'}(U)$ *has symbol* $q(x,\xi) \sim \sum q_{m'-j}(x,\xi)$.
- $P$ *or* $Q$ *is properly supported.*

*Then:*

1. $PQ \in \Psi^{m+m'}(U)$.
2. *If* $r(x,\xi) \sim \sum_{j\geq 0} r_{m+m'-j}(x,\xi)$ *is the symbol of* $PQ$, *then*

$$r(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi).$$

3. *In particular,*

$$r_{m+m'}(x,\xi) = p_m(x,\xi) q_{m'}(x,\xi),$$

$$r_{m+m'-j}(x,\xi) = \sum_{|\alpha|+k+l=j} \frac{1}{\alpha}\partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{m'-l}(x,\xi), \quad j \geq 1.$$

**Remark**

In particular, the principal symbol of $PQ$ is the product,

$$p_m(x,\xi)q_{m'}(x,\xi).$$

**Definition**

If $P : C_c^\infty(U) \to C^\infty(U)$ is a continuous linear operator, its transpose $P^t : \mathcal{E}'(U) \to \mathcal{D}'(U)$ is given by

$$(*) \qquad \left\langle P^t u, v \right\rangle = \langle u, Pv \rangle , \qquad u \in \mathcal{E}'(U), \ v \in C_c^\infty(U).$$

**Remark**

If $P^t$ induces an operator $P^t : C_c^\infty(U) \to C^\infty(U)$, then $(*)$ is equivalent to

$$\int (Pu)(x)v(x)dx = \int u(P^t v)(x)dx \qquad \forall u, v \in C_c^\infty(U).$$

**Example**

If $P = \sum a_\alpha(x) D_x^\alpha$ is a differential operator, then

$$P^t u = \sum (-1)^{|\alpha|} D_x^\alpha (a_\alpha u), \qquad u \in C_c^\infty(U).$$

In particular, $P^t$ is a differential operator.

**Proof.**

Integration by parts:

$$\begin{aligned} \int Pu(x)v(x) &= \sum \int a_\alpha(x) D_x^\alpha u(x) v(x) dx \\ &= \sum (-1)^{|\alpha|} \int u(x) D_x^\alpha (a_\alpha(x) v(x)) dx \\ &= \int u(x) \left( \sum (-1)^{|\alpha|} D_x^\alpha (a_\alpha(x) v(x)) \right) dx. \end{aligned}$$

□

**Proposition**

*Let* $P \in \Psi^m(U)$ *have symbol* $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$.

1. *Its transpose* $P^t$ *is an operator in* $\Psi^m(U)$, *i.e., it induces an operator* $P^t : C_c^\infty(U) \to C^\infty(U)$ *which is in* $\Psi^m(U)$.
2. *If* $p^t(x,\xi) \sim \sum p^t_{m-j}(x,\xi)$ *be the symbol of* $P^t$, *then*

$$p^t(x,\xi) \sim \sum \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha D_\xi^\alpha p)(x,-\xi).$$

3. *In particular,*

$$p^t_m(x,\xi) = p_m(x,-\xi),$$

$$p^t_{m-j}(x,\xi) = \sum_{|\alpha|+k=j} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha D_\xi^\alpha p_{m-k})(x,-\xi), \qquad j \geq 1.$$

**Corollary**

*Every* $P \in \Psi^m(U)$ *uniquely extends to a continuous operator,*

$$P : \mathcal{E}'(U) \longrightarrow \mathcal{D}'(U).$$

**Proof.**

- As $P^t \in \Psi^m(U)$, its transpose is a continuous linear operator,
$$(P^t)^t : \mathcal{E}'(U) \longrightarrow \mathcal{D}'(U).$$
- For all $u, v \in \mathbb{C}_c^\infty(U)$, we have
$$\left\langle (P^t)^t u, v \right\rangle = \left\langle u, P^t v \right\rangle = \langle Pu, v \rangle .$$
- Thus $(P^t)^t = P$ on $C_c^\infty(U)$, i.e., $(P^t)^t$ extends $P$ to a continuous operator from $\mathcal{E}'(U)$ to $\mathcal{D}'(U)$.
- The density of $C_c^\infty(U)$ in $\mathcal{E}'(U)$ ensures this extension is unique.

□

**Remark**

If $P$ is properly supported, we actually get operators,

$$P : \mathcal{E}'(U) \longrightarrow \mathcal{E}'(U), \qquad P : \mathcal{D}'(U) \longrightarrow \mathcal{D}'(U).$$

# Transposes and Adjoints

## Reminder

The inner product of $L^2(U)$ is given by

$$\langle u|v\rangle = \int_U \overline{u(x)}v(x)dx = \langle \overline{u}, v\rangle\,, \quad u, v \in L^2(U).$$

## Definition

A formal adjoint of an operator $P : C_c^\infty(U) \to C^\infty(U)$ is an operator $P^* : C_c^\infty(U) \to C^\infty(U)$ such that

$$(*) \qquad \langle Pu|v\rangle = \langle u|P^*v\rangle \quad \forall u, v \in C_c^\infty(U).$$

## Remarks

1. If $P$ admits a formal adjoint, then it is unique.
2. Here $(*)$ is equivalent to

$$\int \overline{Pu(x)}v(x) = \int \overline{u(x)}P^*v(x)dx \quad \forall u, v \in C_c^\infty(U).$$

**Remark**

Assume $P^t$ induces an operator $P^t : C_c^\infty(U) \rightarrow C^\infty(U)$.

- Let $u, v \in C_c^\infty(U)$. We have

$$\begin{aligned} \int \overline{Pu(x)}v(x)dx &= \overline{\int Pu(x)\overline{v(x)}dx} \\ &= \overline{\int u(x)(P^t\bar{v})(x)dx} \\ &= \int \overline{u(x)(P^t\bar{v})(x)}dx \end{aligned}$$

- Thus, $P$ has a formal adjoint given by

$$P^*v(x) = \overline{(P^t\bar{v})(x)}, \qquad v \in C_c^\infty(U).$$

**Example**

If $P = \sum a_\alpha(x) D_x^\alpha$ is a differential operator, then

$$P^* u = \overline{P^t \overline{u}} = \sum (-1)^{|\alpha|} \overline{D_x^\alpha (a_\alpha \overline{u})}$$
$$= \sum (-1)^{|\alpha|} D_x^\alpha (\overline{a_\alpha} u).$$

In particular $P^*$ is a differential operator.

**Example**

- For $P = D_x^\alpha$, we get
$$(D_x^\alpha)^* = D_x^\alpha.$$
- For $P = D_{x_1}^2 + \cdots + D_{x_n}^2$ we get
$$\Delta^* = \Delta.$$

## Proposition

*Let* $P \in \Psi^m(U)$ *have symbol* $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$.

1. *The formal adjoint* $P^*$ *of* $P$ *exists and is an operator in* $\Psi^{\overline{m}}(U)$.
2. *If* $p^*(x,\xi) \sim \sum p^*_{\overline{m}-j}(x,\xi)$ *is the symbol of* $P^*$, *then*
$$p^*(x,\xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}.$$
3. *In particular,*
$$p^*_{\overline{m}}(x,\xi) = \overline{p_m(x,\xi)},$$
$$p^*_{\overline{m}-j}(x,\xi) = \sum_{|\alpha|+k=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{p_{m-k}(x,\xi)}, \quad j \geq 1.$$

**Definition**

Let $P \in \Psi^m(U)$ have principal symbol $p_m(x,\xi)$. We say that $P$ is elliptic if

$$p_m(x,\xi) \neq 0 \qquad \forall (x,\xi) \in U \times (\mathbb{R}^n \backslash 0).$$

**Example**

Let $\Delta = D_{x_1}^2 + \cdots + D_{x_n}^2$ be the Laplace operator.

- It has (principal) symbol,

$$\xi_1^2 + \cdots + \xi_n^2 = |\xi|^2 \neq 0 \qquad \text{for } \xi \neq 0.$$

- Thus, $\Delta$ is elliptic.

# Ellipticity and Parametrices

**Theorem**

*Let* $P \in \Psi^m(U)$ *have symbol* $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$. *TFAE:*

(i) $P$ *is elliptic.*

(ii) *There exists a properly supported operator* $Q \in \Psi^{-m}(U)$ *which is a two-sided parametrix for* $P$, *i.e.,*

$$PQ = QP = 1 \qquad \mod \Psi^{-\infty}(U).$$

**Proof of (ii) $\Rightarrow$ (i).**

- Suppose there is $Q \in \Psi^{-m}(U)$ such that
$$PQ = QP = 1 \mod \Psi^{-\infty}(U).$$
- Therefore, the symbol of $PQ$ is $1$, and so $PQ$ has $1$ as principal symbol.
- If $q_{-m}(x,\xi)$ is the principal symbol of $Q$, then we also know that the principal symbol of $PQ$ is $p_m(x,\xi)q_{-m}(x,\xi)$.
- Thus, $p_m(x,\xi)q_{-m}(x,\xi) = 1$ for $\xi \neq 0$.
- This implies that $p_m(x,\xi) \neq 0$ for $\xi \neq 0$, i.e., $P$ is elliptic. □

**Proof of (i) $\Rightarrow$ (ii).**

- Suppose that $P$ is elliptic, i.e., $p_m(x,\xi) \neq 0$ for $\xi \neq 0$.
- Let $Q \in \Psi^{-m}(U)$ have symbol $q(x,\xi) \sim \sum q_{-m-j}(x,\xi)$.
- Assume $Q$ is properly supported, and let $r(x,\xi) \sim \sum r_{-j}(x,\xi)$ be the symbol of $PQ$.
- We have

$$
\begin{aligned}
PQ = 1 \mod \Psi^{-\infty}(U) &\Longleftrightarrow r(x,\xi) = 1 \mod \mathbb{S}^{-\infty}(U \times \mathbb{R}^n) \\
&\Longleftrightarrow r(x,\xi) \sim 1 \\
&\Longleftrightarrow \begin{cases} r_0(x,\xi) = 1, \\ r_{-j}(x,\xi) = 0, \quad j \geq 1. \end{cases}
\end{aligned}
$$

□

# Ellipticity and Parametrices

**Proof of (i) ⇒ (ii).**

- Recall that

$$r_0(x,\xi) = p_m(x,\xi)q_{-m}(x,\xi),$$

$$r_{-j}(x,\xi) = \sum_{|\alpha|+k+l=j} \frac{1}{\alpha}\partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi), \quad j \geq 1.$$

- Therefore, the equation $PQ = 1 \mod \Psi^{-\infty}(U)$ is equivalent to the system of equations,

$$(*) \begin{cases} p_m(x,\xi)q_{-m}(x,\xi) = 1, \\ \displaystyle\sum_{|\alpha|+k+l=j} \tfrac{1}{\alpha}\partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi) = 0, \quad j \geq 1. \end{cases}$$

- The first equation has solution,

$$q_{-m}(x,\xi) = p_m(x,\xi)^{-1}.$$

□

# Ellipticity and Parametrices

**Proof of (i) ⇒ (ii).**

- Moreover, we have

$$\sum_{|\alpha|+k+l=j} \frac{1}{\alpha} \partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi) = 0$$

$$\Leftrightarrow p_m(x,\xi) q_{-m-j}(x,\xi) + \sum_{\substack{|\alpha|+k+l=j \\ l<j}} \frac{1}{\alpha} \partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi) = 0$$

$$\Leftrightarrow q_{-m-j}(x,\xi) = -p_m(x,\xi)^{-1} \sum_{\substack{|\alpha|+k+l=j \\ l<j}} \frac{1}{\alpha} \partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi).$$

- Thus, if $p_m(x,\xi) \neq 0$ (i.e., $P$ is elliptic), then the system $(*)$ has a unique solution. □

**Proof of (i) $\Rightarrow$ (ii).**

- Let $q_{-m-j}(x,\xi)$, $j=0,1,\ldots$, be the solutions of $(*)$.
- We always can find a properly supported ΨDO $Q \in \Psi^{-m}(U)$ with symbol $q(x,\xi) \sim \sum q_{-m-j}(x,\xi)$.
- As the $q_{-m-j}(x,\xi)$ are solutions of $(*)$, we have
$$PQ = 1 \mod \Psi^{-\infty}(U).$$
- We similarly can construct $Q' \in \Psi^{-m}(U)$ such that
$$Q'P = 1 \mod \Psi^{-\infty}(U).$$
- These two equations imply that
$$Q' = Q'\cdot 1 = Q'(PQ) = (Q'P)Q = 1\cdot Q = Q \mod \Psi^{-\infty}(U).$$
- Thus,
$$QP = Q'P = 1 \mod \Psi^{-\infty}(U).$$
- That is, $Q$ is a two-sided parametrix.

□

# Ellipticity and Parametrices

## Remarks

- The proof shows that if $Q$ and $Q'$ are two parametrices, then $Q' = Q \mod \Psi^{-\infty}(U)$.
- In particular, $Q$ and $Q'$ have same homogeneous symbols.
- This implies that if $Q \in \Psi^{-m}(U)$ is a parametrix, then it has symbol $q(x,\xi) \sim \sum q_{-m-j}(x,\xi)$, where

$$q_{-m}(x,\xi) = -p_m(x,\xi)^{-1},$$

$$q_{-m-j}(x,\xi) = -p_m(x,\xi)^{-1} \sum_{\substack{|\alpha|+k+l=j \\ l<j}} \frac{1}{\alpha} \partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi).$$

**Corollary**

*Let* $P \in \Psi^m(U)$ *be elliptic. Given any* $u \in \mathcal{E}'(U)$, *we have*

$$Pu \in C^\infty(U) \Longrightarrow u \in C_c^\infty(U).$$

## Ellipticity and Parametrices

**Proof.**

Let $u \in \mathcal{E}'(U)$ be such that $Pu \in C^\infty(U)$.

- As $P$ is elliptic, there are $Q \in \Psi^{-m}(U)$ and $R \in \Psi^{-\infty}(U)$ with $Q$ properly supported such that
$$QP = 1 - R.$$
- Thus,
$$u = Q(Pu) + Ru.$$
- As $R$ is smoothing, it maps $\mathcal{E}'(U)$ to $C^\infty(U)$, and so $Ru \in C^\infty(U)$.
- As $Pu$ is in $C^\infty(U)$, we see that $Q(Pu)$ is in $C^\infty(U)$ as well.
- It then follows that $u = Q(Pu) + Ru \in C^\infty(U)$.

□

# Ellipticity and Parametrices

**Remark**

We actually have a local version:

- If $P$ is elliptic, then, given any $u \in \mathcal{E}'(U)$ and $x_0 \in U$, we have

$$Pu \text{ is } C^\infty \text{ near } x_0 \Longrightarrow u \text{ is } C^\infty \text{ near } x_0.$$

- Here $v \in \mathcal{D}'(M)$ is $C^\infty$ near $x_0$, if there is an open neigborhood $V$ of $x_0$ and $w \in C^\infty(V)$ such that $v_{|V} = w$.

# Sobolev Spaces

**Definition**

$W_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, consists of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \in L^2(\mathbb{R}^n).$$

**Proposition**

$W_2^s(\mathbb{R}^n)$ *is a Hilbert space with respect to the norm,*

$$\|u\|_{(s)} = \left( \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 đ\xi \right)^{\frac{1}{2}}$$

**Remark**

Let $k \in \mathbb{N}_0$.

- We have

$$D_x^\alpha u \in L^2(\mathbb{R}^n) \Longleftrightarrow \left(D_x^\alpha u\right)^\wedge \in L^2(\mathbb{R}^n) \Longleftrightarrow \xi^\alpha \hat{u} \in L^2(\mathbb{R}^n).$$

- Using this it can be shown that

$$\begin{aligned} u \in W_2^k(\mathbb{R}^n) &\Longleftrightarrow \left(1+|\xi|^2\right)^{\frac{k}{2}} \hat{u} \in L^2(\mathbb{R}^n) \\ &\Longleftrightarrow \xi^\alpha \hat{u} \in L^2(\mathbb{R}^n) \text{ for } |\alpha| \leq k \\ &\Longleftrightarrow D_x^\alpha u \in L^2(\mathbb{R}^n) \text{ for } |\alpha| \leq k. \end{aligned}$$

- Thus,

$$W_2^k(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n);\ D_x^\alpha u \in L^2(\mathbb{R}^n) \text{ for } |\alpha| \leq k\right\}.$$

**Remark**

- If $\Lambda^s = (1+\Delta)^{s/2}$, then we saw that
$$\Lambda^s u = \int e^{ix\cdot\xi}(1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)đ\xi, \quad u \in C_c^\infty(\mathbb{R}^n).$$
- That is,
$$(\Lambda^s u)^\wedge = (1+|\xi|^2)^{\frac{s}{2}}\hat{u}.$$
- Thus,
$$\|u\|_{(s)}^2 = \int (1+|\xi|^2)^s|\hat{u}(\xi)|^2 đ\xi = \int \left|(\Lambda^s u)^\wedge(\xi)\right|^2 đ\xi = \|\Lambda^s u\|_{L^2}^2.$$
- This gives an isometric isomorphism,
$$\Lambda^s : W_2^s(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$
- More generally, we have isometric isomorphisms,
$$\Lambda^s : W_2^{s+t}(\mathbb{R}^n) \to W_2^t(\mathbb{R}^n), \qquad t \in \mathbb{R}.$$

**Remark (Riesz Isomorphism)**

- We have a natural pairing,

$$(u, v) = \int u(x)v(x)dx, \quad u, v \in L^2(\mathbb{R}^n).$$

- This pairing is non-degenerate, and so this yields an (isometric) isomorphism,

$$L^2(\mathbb{R}^n) \simeq L^2(\mathbb{R}^n)'.$$

**Proposition**

*The above pairing uniquely extends to a non-degenerate continuous bilinear pairing,*

$$(\cdot, \cdot) : W_2^{-s}(\mathbb{R}^n) \times W_2^s(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

*In particular, we get an (isometric) isomorphism,*

$$W_2^{-s}(\mathbb{R}^n) \simeq W_2^s(\mathbb{R}^n)'.$$

**Sketch of Proof.**

- If $u, v \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}(u, v) &= \int \hat{u}(\xi)\hat{v}(\xi)đ\xi \\ &= \int (1+|\xi|^2)^{-\frac{s}{2}}\hat{u}(\xi)(1+|\xi|^2)^{\frac{s}{2}}\hat{v}(\xi)đ\xi \\ &= \int \left(\Lambda^{-s}u\right)^\wedge(\xi)\left(\Lambda^{s}v\right)^\wedge(\xi)đ\xi \\ &= \int (\Lambda^{-s}u)(x)(\Lambda^{s}v)(x)dx \\ &= \left(\Lambda^{-s}u, \Lambda^{s}v\right).\end{aligned}$$

- As $\Lambda^{\pm s} : W_2^{\pm s}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isometric isomorphisms the result follows. □

**Theorem (Sobolev Embedding Theorems)**

1. *For* $s' > s$ *the inclusion* $W_2^{s'}(\mathbb{R}^n) \hookrightarrow W_2^s(\mathbb{R}^n)$ *is compact.*
2. *If* $s > k + n/2$, *then we have a continuous embedding,*
$$W_2^s(\mathbb{R}^n) \hookrightarrow C^k(\mathbb{R}^n).$$

**Corollary**

1. *We have (continuous) inclusions,*
$$\mathcal{S}(\mathbb{R}^n) \subset \bigcap_{s\in\mathbb{R}} W_2^s(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n).$$
2. *By duality this gives inclusion,*
$$\mathcal{E}'(U) \subset \bigcup_{s\in\mathbb{R}} W_2^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$
*In particular, every* $u \in \mathcal{E}'(\mathbb{R}^n)$ *is in some* $W_2^s(\mathbb{R}^n)$.

**Setup**

$U \subset \mathbb{R}^n$ open subset.

**Remark**

- Any $u \in \mathcal{E}'(U)$ can be regarded as a distribution on $\mathbb{R}^n$.
- Namely, if $\chi \in C_c^\infty(U)$ is such that $\varphi = 1$ near $\operatorname{supp} U$, then

$$\langle u, v \rangle := \langle u, \chi v \rangle\,, \qquad v \in C_c^\infty(\mathbb{R}^n).$$

**Definition**

$W^s_{2,\mathrm{loc}}(U)$ consists of all $u \in \mathcal{D}'(U)$ such that

$$\varphi u \in W^s_2(\mathbb{R}^n) \qquad \forall \varphi \in C^\infty_c(U).$$

**Remark**

$W^0_{2,\mathrm{loc}}(U) = L^2_{\mathrm{loc}}(U)$.

**Remark**

The topology of $W^s_{2,\mathrm{loc}}(U)$ is the locally convex topology generated by the semi-norms,

$$W^s_{2,\mathrm{loc}}(U) \ni u \longrightarrow \|\varphi u\|_{(s)}, \qquad \varphi \in C^\infty_c(U).$$

**Definition**

$W^s_{2,c}(U)$ consists of all $u \in \mathcal{E}'(U)$ that are in $W^s_{2,c}(\mathbb{R}^n)$.

**Remark**

$W^0_{2,c}(U) = L^2_c(U)$.

**Remark**

We have

$$W^s_{2,c}(U) = W^s_2(\mathbb{R}^n) \cap \mathcal{E}'(U) = W^s_{2,\mathrm{loc}}(U) \cap \mathcal{E}'(U)$$

**Remark**

Let $K \subset U$ be compact.

- $W^s_{2,K}(U)$ consists of all $u \in W^s_{2,c}(U)$ s.t. $\operatorname{supp} u \subset K$.
- We regard it as a subspace of $W^s_2(\mathbb{R}^n)$.
- As such this is a closed subspace, and so this a Hilbert space with respect to the $\| \cdot \|_{(s)}$-norm.

**Definition**

The topology of $W^s_{2,c}(U)$ is the weakest locally convex topology with respect to which the inclusions $W^s_{2,K}(U) \hookrightarrow W^s_{2,c}(U)$ are continuous.

**Remark**

We have duality isomorphisms,

$$W_{2,\mathrm{loc}}^{-s}(U) \simeq W_{2,c}^{s}(U)', \qquad W_{2,c}^{-s}(U) \simeq W_{2,\mathrm{loc}}^{s}(U)'.$$

**Remark**

- If $s > k + n/2$, then we have continuous embeddings,

$$C_c^\infty(U) \subset W_{2,c}^s(U) \subset C_c^k(U), \quad C^\infty(U) \subset W_{2,\mathrm{loc}}^s(U) \subset C^k(U).$$

- It follows that

$$C_c^\infty(U) = \bigcap W_{2,c}^s(U), \qquad C^\infty(U) = \bigcap W_{2,\mathrm{loc}}^s(U).$$

- By duality we get

$$\mathcal{E}'(U) = \bigcup W_{2,c}^s(U), \qquad \mathcal{D}'(U) = \bigcup W_{2,\mathrm{loc}}^s(U).$$

**Theorem**

*Let* $P \in \Psi^m(U)$*, and set* $a = \Re(m)$*.*

1. *For every* $s \in \mathbb{R}$*, the operator* $P$ *uniquely extend to a continuous linear operator,*
$$P : W^s_{2,c}(U) \longrightarrow W^{s-a}_{2,\text{loc}}(U).$$
2. *In particular, for* $a = 0$ *we get a continuous linear operator,*
$$P : L^2_c(U) \longrightarrow L^2_{\text{loc}}(U).$$

**Remark**

If in addition $P$ is properly supported, then we get continuous linear operators,

$$P : W^s_{2,c}(U) \longrightarrow W^{s-a}_{2,c}(U), \qquad P : W^s_{2,\text{loc}}(U) \longrightarrow W^{s-a}_{2,\text{loc}}(U).$$

**Theorem (Elliptic Regularity Theorem)**

*Let* $P \in \Psi^m(U)$ *be elliptic, and set* $a = \Re(m)$. *If* $u \in \mathcal{E}'(U)$, *then*

$$Pu \in W^s_{2,\mathrm{loc}}(U) \Longrightarrow u \in W^{s+a}_{2,c}(U).$$

**Proof.**

Let $u \in \mathcal{E}'(U)$ be such that $Pu \in W^{s}_{2,\mathrm{loc}}(U)$.

- As $P$ is elliptic, there are $Q \in \Psi^{-m}(U)$ and $R \in \Psi^{-\infty}(U)$ with $Q$ properly supported such that
$$QP = 1 - R.$$
- Thus,
$$u = Q(Pu) + Ru.$$
- As $R$ is smoothing, it maps $u$ to $C^{\infty}(U) \subset W^{s+a}_{2,c}(U)$.
- As $Q$ has order $-m$ and is properly supported it extends to an operator $Q : W^{s}_{2,\mathrm{loc}}(U) \to W^{s+a}_{2,\mathrm{loc}}(U)$.
- As $Pu \in W^{s}_{2,\mathrm{loc}}(U)$, we see that $Q(Pu) \in W^{s+a}_{2,\mathrm{loc}}(U)$.
- It follows that $u = Q(Pu) + Ru$ is in $W^{s+a}_{2,\mathrm{loc}}(U)$.

□

# Action of Diffeomorphisms

**Setup**

$\phi : U \to V$ is diffeomorphism from $U$ onto an open $V \subset \mathbb{R}^n$.

**Remark**

- The pushforward map $\phi_* : C^\infty(U) \to C^\infty(V)$ and the pullback map $\phi^* : C^\infty(V) \to C^\infty(U)$ are given by
$$\phi_* u = u \circ \phi^{-1}, \qquad u \in C^\infty(U),$$
$$\phi^* v = v \circ \phi, \qquad v \in C^\infty(V)$$
- If $P : C_c^\infty(U) \to C^\infty(U)$, then $\phi_* P : C_c^\infty(V) \to C^\infty(V)$ is given by
$$(\phi_* P)v = \phi_* \left(P(\phi^* v)\right) = [P(v \circ \phi)] \circ \phi^{-1}, \qquad v \in C_c^\infty(V).$$

# Action of Diffeomorphisms

**Theorem**

*Let* $P \in \Psi^m(U)$ *have principal symbol* $p_m(x,\xi)$.

1. $\phi_* P$ *is an operator in* $\Psi^m(V)$.
2. *Its principal symbol is*

$$p_m^{\phi}(x,\xi) := p_m\left(\phi^{-1}(x), (\phi^{-1})'(x)^t \xi\right), \quad (x,\xi) \in V \times \mathbb{R}^n \backslash 0.$$

**Remark**

Let $p(x,\xi) \sim \sum p_{m-j}(x,\xi)$ and $p^{\phi}(x,\xi) \sim p^{\phi}_{m-j}(x,\xi)$ be the full symbols of $P$ and $\phi_* P$

- It can be shown that there are functions $a_{\alpha\beta}(x) \in C^{\infty}(V)$, $2|\beta| \leq |\alpha|$ depending only on $\phi$ such that

$$p^{\phi}(x,\xi) \sim \sum_{2|\beta|\leq|\alpha|} \frac{1}{\alpha!} a_{\alpha\beta}(x)\xi^{\beta}(D^{\alpha}_{\xi} p)(\phi^{-1}(x),(\phi^{-1})'(x)^t\xi).$$

- In particular,

$$p^{\phi}_{m-j}(x,\xi) = \sum_{\substack{k+|\alpha|-|\beta|=j \\ 2|\beta|\leq|\alpha|}} \frac{1}{\alpha!} a_{\alpha\beta}(x)\xi^{\beta}(D^{\alpha}_{\xi} p)(\phi^{-1}(x),(\phi^{-1})'(x)^t\xi).$$

**Remark**

Let $V \subset U \subset \mathbb{R}^n$ be open sets and $P : C_c^\infty(U) \to C^\infty(U)$ a linear operator.

- The restriction $P_{|V} : C_c^\infty(V) \to C^\infty(V)$ is defined by
$$P_{|V} u = (Pu)_{|V}, \qquad u \in C_c^\infty(V).$$
- If $P \in \Psi^m(U)$, then $P_{|V} \in \Psi^m(V)$.

# ΨDOs on Manifolds

### Setup

$M^n$ = smooth manifold.

### Convention

A (local) chart $\kappa : U \to V$ is a $C^\infty$-diffeomorphism, where $U \subset M$ and $V \subset \mathbb{R}^n$ are open sets.

### Remark

If $u$ is a function on $M$, then TFAE:

(i) $u$ is $C^\infty$-function.

(ii) $u \circ \kappa^{-1} \in C^\infty(V)$ for every chart $\kappa : U \to V$.

(iii) For every $x_0 \in M$, there is a chart $\kappa : U \to V$ near $x_0$ such that $u \circ \kappa^{-1} \in C^\infty(V)$.

**Remark**

If $P : C_c^\infty(M) \to C^\infty(M)$ is a linear operator and $\kappa : U \to V$ is a chart, then $\kappa_*(P_{|U}) : C_c^\infty(V) \to C^\infty(V)$ is defined by

$$\kappa_*(P_{|U})u = [P(u \circ \kappa)] \circ \kappa^{-1}, \qquad u \in C_c^\infty(V).$$

**Definition**

$\Psi^m(U)$, $m \in \mathbb{C}$, consists of continuous linear operators $P : C_c^\infty(M) \to C^\infty(M)$ such that there is a chart

$$\kappa_*(P_{|U}) \in \Psi^m(V) \quad \text{for every chart } \kappa : U \to V.$$

**Remark**

The above definition is consistent.

- For $j = 1, 2$ let $\kappa_j : U_j \to V_j$ be a chart, and set $\phi = \kappa_2 \circ \kappa_1^{-1}$. We have

$$(\kappa_2)_*(P_{|U_1 \cap U_2}) = (\kappa_2 \circ \kappa_1^{-1})_* \left[(\kappa_1)_*(P_{|U_1 \cap U_2})\right] = \phi_*[(\kappa_1)_*(P_{|U_1 \cap U_2})].$$

- Thus, by the diffeomorphism invariance of ΨDOs, we get

$$(\kappa_1)_*(P_{|U_1 \cap U_2}) \in \Psi^m(\kappa_1(U_1 \cap U_2)) \\ \Longleftrightarrow (\kappa_2)_*(P_{|U_1 \cap U_2}) \in \Psi^m(\kappa_2(U_1 \cap U_2)).$$

In particular, using this we obtain:

**Proposition**

*Let $P : C_c^\infty(M) \to C^\infty(M)$ be a continuous linear operator. TFAE:*

1. $P \in \Psi^m(M)$.
2. *The Schwartz kernel of $P$ is smooth off the diagonal, and, for every $x_0 \in M$, there is a chart $\kappa : U \to V$ near $x_0$ such that $\kappa_*(P_{|U}) \in \Psi^m(V)$.*

## Remark

Let $P \in \Psi^m(M)$. For $j = 1, 2$, let $\kappa_j : U_j \to V_j$ be a chart, and set $\phi = \kappa_2 \circ \kappa_1^{-1}$.

- We know that $(\kappa_1)_*(P_{|U_1 \cap U_2})$ and $(\kappa_2)_*(P_{|U_1 \cap U_2})$ are ΨDOs such that
$$(\kappa_2)_*(P_{|U_1 \cap U_2}) = \phi_*[(\kappa_1)_*(P_{|U_1 \cap U_2})].$$
- Let $p_m^{\kappa_j}(x, \xi)$ be the principal symbol of $(\kappa_j)_*(P_{|U_1 \cap U_2})$. We have
$$p_m^{\kappa_2}(x, \xi) = p_m^{\kappa_1}\left(\phi^{-1}(x), (\phi^{-1})'(x)^t \xi\right).$$
- This is a change of local coordinate formula for functions on $T^*M \setminus 0$.

**Remark**

- If $\kappa : U \rightarrow V$ is a chart, then a chart for $TU^* = T^*M_{|U}$ is given by
$$T^*M_{|U} \ni (x,\xi) \longrightarrow (\kappa(x), (\kappa'(x)^{-1})^t\xi) \in V \times \mathbb{R}^n.$$
- Thus, a function $v$ on $T^*M$ is smooth if and only if

$$v\left(\kappa^{-1}(x), (\kappa^{-1})'(x)^t\xi\right) \in C^\infty(V\times\mathbb{R}^n) \quad \text{for every chart } \kappa : U \rightarrow V.$$

Therefore, we obtain:

**Proposition**

*If $P \in \Psi^m(M)$, then there is a unique function $p_m(x,\xi) \in C^\infty(T^*M \setminus 0)$ such that:*

- $p_m(x,\lambda\xi) = \lambda^m p_m(x,\xi)$ *for all* $(x,\xi) \in T^*M \setminus 0$.
- *For every chart* $\kappa : U \to V$ *we have*

$$p_m(x,\xi) = p_m^\kappa\left(\kappa(x), (\kappa'(x)^{-1})^t\xi\right) \quad \forall (x,\xi) \in T^*U \setminus 0.$$

**Reminder**

A continuous operator $P : C_c^\infty(M) \to C^\infty(M)$ is properly supported if, for every compact $K \subset M$, there are compact sets $K_1 \subset U$ and $K_2 \subset U$ such that

$$\operatorname{supp} u \subset K \implies \operatorname{supp} Pu \subset K_1,$$
$$K_2 \cap \operatorname{supp} u = \emptyset \implies K \cap \operatorname{supp} Pu = \emptyset.$$

**Remarks**

1. If $M$ is compact, then every continuous operator $P : C_c^\infty(M) \to C^\infty(M)$ is automatically properly supported.
2. A continuous operator $P : C_c^\infty(M) \to C^\infty(M)$ is properly supported if only if it gives rise to continuous operators,

$$P : C_c^\infty(M) \longrightarrow C_c^\infty(M) \quad \text{and} \quad P : C^\infty(M) \longrightarrow C^\infty(M).$$

## Proposition

*Assume that*

- $P \in \Psi^m(M)$ *has principal symbol* $p_m(x,\xi)$.
- $Q \in \Psi^{m'}(M)$ *has principal symbol* $q_{m'}(x,\xi)$.
- $P$ *or* $Q$ *is properly supported.*

*Then:*

1. *The composition* $PQ$ *is an operator in* $\Psi^{m+m'}(M)$.
2. *Its principal symbol is* $p_m(x,\xi)q_{m'}(x,\xi)$.

## Remark

- If $M$ is compact, then the properly supported assumption is superfluous, since $P$ and $Q$ are automatically properly supported.
- In fact, $P$ and $Q$ both are operators from $C^\infty(M)$ to itself, and so the composition $PQ$ is always well defined.

**Proposition**

*Let* $P \in \Psi^m(M)$.

1. $P$ *uniquely extend to a continuous operator,*

$$P : \mathcal{E}'(M) \longrightarrow \mathcal{D}'(M).$$

2. *If in addition* $P$ *is properly supported, then it gives rise to continuous operators,*

$$P : \mathcal{E}'(M) \longrightarrow \mathcal{E}'(M), \qquad P : \mathcal{D}'(M) \longrightarrow \mathcal{D}'(M).$$

**Setup**

- $\mu =$ smooth measure on $M$ (e.g., Riemannian measure associated with a Riemannian metric).
- $L^2_\mu(M)$ has inner product,

$$\langle u|v\rangle = \int_M \overline{u(x)}v(x)d\mu(x), \quad u, v \in L^2_\mu(M).$$

**Remark**

If $M$ is compact, then as a topological vector space $L^2_\mu(M)$ does not depend on the choice of $\mu$ (all the norms are equivalent to each other).

**Definition**

A formal adjoint of an operator $P : C_c^\infty(M) \to C^\infty(M)$ is any operator $P^* : C_c^\infty(M) \to C^\infty(M)$ such that

$$\langle Pu|v\rangle = \langle u|P^*v\rangle \qquad \forall u, v \in C^\infty(M).$$

**Remark**

If a formal adjoint exists, then it is unique.

**Proposition**

*Let $P \in \Psi^m(M)$ has principal symbol $p_m(x,\xi)$.*

1. *$P$ has a formal adjoint $P^* \in \Psi^{\overline{m}}(M)$.*
2. *The principal symbol of $P^*$ is $\overline{p_m(x,\xi)}$.*

**Definition**

Let $P \in \Psi^m(M)$ have principal symbol $p_m(x,\xi)$. We see that $P$ is elliptic if

$$p_m(x,\xi) \neq 0 \qquad \forall (x,\xi) \in T^*M \setminus 0.$$

**Proposition**

*Let $P \in \Psi^m(M)$ have principal symbol $p_m(x,\xi)$. TFAE:*

(i) *$P$ is elliptic.*

(ii) *It admits a properly supported parametrix $Q \in \Psi^{-m}(M)$, i.e.,*

$$PQ = QP = 1 \qquad \text{mod } \Psi^{-\infty}(M).$$

*Moreover, if (ii) holds, then the principal symbol of $Q$ is $p_m(x,\xi)^{-1}$.*

**Corollary**

*If* $P \in \Psi^m(M)$ *is elliptic, then, given any* $u \in \mathcal{E}'(M)$,

$$Pu \in C^\infty(M) \Longrightarrow u \in C_c^\infty(M).$$

**Remark**

If $\kappa : U \to V$ is a chart and $u \in \mathcal{D}'(U)$, then $\kappa_* u \in \mathcal{D}'(V)$ is given by

$$\langle \kappa_* u, v \rangle = \left\langle u, v \circ \kappa^{-1} \right\rangle, \qquad v \in C_c^\infty(V).$$

**Definition**

$W^s_{2,\mathrm{loc}}(M)$, $s \in \mathbb{R}$, consists of all $u \in \mathcal{D}'(M)$ such that

$$\kappa_*(u_{|U}) \in W^s_{2,\mathrm{loc}}(V) \quad \text{for every chart } \kappa : U \to V.$$

**Remark**

In other words $u \in W^s_{2,\mathrm{loc}}(M)$ if for every chart $\kappa : U \to V$ and $\varphi \in C_c^\infty(U)$, we have

$$\kappa_*(\varphi u) \in W^s_2(\mathbb{R}^n).$$

**Remark**

We equip $W^s_{2,\mathrm{loc}}(M)$ with the locally convex topology generated by the semi-norms,

$$W^s_{2,\mathrm{loc}}(M) \ni u \longrightarrow \|\kappa_*(\varphi u)\|_{(s)},$$

where $\kappa : U \to V$ ranges over all charts and $\varphi$ ranges over $C^\infty_c(U)$.

**Definition**

$W^s_{2,c}(M)$ consists of all $u \in W^s_{2,\mathrm{loc}}(M)$ that are compact support.

**Remark**

$W^s_{2,c}(M) = W^s_{2,\mathrm{loc}}(M) \cap \mathcal{E}'(M)$.

**Remark**

Let $K \subset M$ be compact

- $W^s_{2,K}(M)$ consists of all $u \in W^s_{2,c}(M)$ whose support is contained in $K$.
- Let $(\varphi_i)_{i \in I}$ a $C^\infty$-partition of unity subordinated to a locally finite cover $(U_i)_{i\in I}$ of domains of charts $\kappa_i : U_i \to V_i$.
- $J := \{i \in I; K \cap U_i \neq \emptyset\}$ is finite.
- $W^s_{2,K}(M)$ is a Hilbert space with respect to the norm,

$$\|u\|_{s,K} := \left( \sum_{i \in J} \|(\kappa_i)_*(\varphi_i u)\|^2_{(s)} \right)^{\frac{1}{2}}, \quad u \in W^s_{2,K}(M).$$

- The topology does not depend on the choice of the partition of unity.

**Remark**

The topology of $W^s_{2,c}(M)$ is the weakest locally convex topology with respect to which the inclusions $W^s_{2,K}(M) \hookrightarrow W^s_{2,c}(M)$ are continuous, as $K$ ranges over all compact sets of $M$.

**Remark**

Assume $M$ is compact.

- In this case $W^s_{2,c}(M) = W^s_{2,\text{loc}}(M)$, so we simply use the notation $W^s_2(M)$.
- $W^s_2(M)$ is a Hilbert space with respect to the norm,

$$\|u\|_s := \left( \sum_i \|(\kappa_i)_*(\varphi_i u)\|^2_{(s)} \right)^{\frac{1}{2}}, \quad u \in W^s_2(M),$$

  where $(\varphi_i)$ is a finite $C^\infty$-partition of unity as above.

**Remark**

If $\mu$ is any smooth measure on $M$, then $W^0_2 = L^2_\mu(M)$ with equivalent norms.

# Sobolev Spaces on Manifolds

**Remark**

- If $s > k + n/2$, then we have continuous embeddings,

$$C_c^\infty(M) \subset W^s_{2,c}(M) \subset C_c^k(M), \quad C^\infty(M) \subset W^s_{2,\text{loc}}(M) \subset C^k(M).$$

- It follows that

$$C_c^\infty(M) = \bigcap W^s_{2,c}(M), \qquad C^\infty(M) = \bigcap W^s_{2,\text{loc}}(M).$$

- By duality we get

$$\mathcal{E}'(M) = \bigcup W^s_{2,c}(M), \qquad \mathcal{D}'(M) = \bigcup W^s_{2,\text{loc}}(M).$$

**Theorem (Sobolev Embedding Theorem)**

*Assume $M$ is compact. If $s' > s$, then the inclusion $W_2^{s'}(M) \hookrightarrow W_2^s(M)$ is compact.*

# ΨDOs on Manifolds

## Theorem

*Let* $P \in \Psi^m(M)$*, and set* $a = \Re(m)$*.*

1. *For every* $s \in \mathbb{R}$*, the operator* $P$ *uniquely extend to a continuous linear operator,*
$$P : W^s_{2,c}(M) \longrightarrow W^{s-a}_{2,\text{loc}}(M).$$
2. *In particular, for* $a = 0$ *we get a continuous linear operator,*
$$P : L^2_c(M) \longrightarrow L^2_{\text{loc}}(M).$$

## Remark

If in addition $P$ is properly supported, then we get continuous linear operators,

$$P : W^s_{2,c}(M) \longrightarrow W^{s-a}_{2,c}(M), \qquad P : W^s_{2,\text{loc}}(M) \longrightarrow W^{s-a}_{2,\text{loc}}(M).$$

**Corollary (Elliptic Regularity Theorem)**

*Let* $P \in \Psi^m(M)$ *be elliptic, and set* $a = \Re m$. *For any* $u \in \mathcal{E}'(M)$,

$$Pu \in W^s_{2,\text{loc}}(M) \Longrightarrow u \in W^{s+a}_{2,c}(M).$$

## Setup

- $M^n$ = compact manifold.
- $\mu$ = smooth measure on $M$.

## Theorem

*Let $P \in \Psi^m(M)$, and set $a = \Re(m)$.*

1. *For every $s \in \mathbb{R}$, the operator $P$ uniquely extend to a continuous linear operator,*
$$P : W_2^s(M) \longrightarrow W_2^{s-a}(M).$$
2. *In particular, for $a = 0$ we get a bounded operator,*
$$P : L^2_\mu(M) \longrightarrow L^2_\mu(M).$$

**Proposition**

*Let* $P \in \Psi^m(M)$. *If with* $\Re(m) < 0$, *then* $P : L^2_\mu(M) \to L^2_\mu(M)$ *is a compact operator.*

**Proof.**

Set $s = -\Re m > 0$.

- We know that $P$ extends to a bounded operator from $L^2_\mu(M) = W^0_2(M)$ to $W^s_2(M)$.
- As $s > 0$, the inclusion $W^s_2(M) \hookrightarrow L^2_\mu(M)$ is compact.
- Here $P : L^2_\mu(M) \to L^2_\mu(M)$ agrees with the composition,

$$L^2_\mu(M) \xrightarrow{P} W^s_2(M) \hookrightarrow L^2_\mu(M).$$

- As the 2nd arrow is compact, it follows that $P$ is compact. □

**Proposition**

*Let $P \in \Psi^m(M)$, $m > 0$, be elliptic. Then $P$ with domain $W^m(M)$ is a closed operator on $L^2_\mu(M)$.*

**Proof.**

We need to prove the closedness of the graph,

$$G(P) = \{(u, Pu);\ u \in W^m(M)\} \subset L^2_\mu(M) \oplus L^2_\mu(M).$$

- Let $G(P) \ni (u_\ell, Pu_\ell) \to (u, v)$ in $L^2_\mu(M) \oplus L^2_\mu(M)$, i.e., $W^m_2(M) \ni u_\ell \to u$ and $Pu_\ell \to v$ in $L^2_\mu(M)$.
- As $P : L^2_\mu(M) \to W_2^{-m}(M)$ is continuous, we see that $Pu_\ell \to Pu$ in $W_2^{-m}(M)$.
- Thus $Pu = v \in L^2_\mu(M)$.
- By elliptic regularity $u \in W_2^m(M)$, and so $(u, v) = (u, Pu)$ is contained in $G(P)$.
- Thus, $G(P)$ is closed, i.e., $P$ is a closed operator. □

**Setup**

- $M$ = compact manifold with smooth measure $\mu$.
- $P \in \Psi^m(M)$ is elliptic with $m > 0$.

**Fact**

*$C^\infty(M)$ is dense in $W_2^s(M)$ for every $s \in R$.*

**Remark**

This implies that $P : W_2^m(M) \to L^2(M)$ is the closure of $P : C^\infty(M) \to C^\infty(M)$.

As $P$ is closed we may define its spectrum.

**Definition**

$\mathrm{Sp}(P)$ consists of all $\lambda \in \mathbb{C}$ such that $P - \lambda : W_2^m(M) \to L^2(M)$ is not a bijection.

**Remark**

- As $P - \lambda : W_2^m(M) \to L^2(M)$ is continuous, this is a bijection if and only if it is an isomorphism.
- In particular, $(P - \lambda)^{-1} : L^2(M) \to W_2^m(M)$ is continuous for all $\lambda \in \mathbb{C} \setminus \mathrm{Sp}(P)$.

**Proposition**

*There are two possibilities:*

(i) $\mathrm{Sp}(P) = \mathbb{C}$, *or*

(ii) $\mathrm{Sp}(P)$ *is an unbounded discrete set consisting of eigenvalues with finite multiplicity.*

**Proof.**

Assume there is $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 \notin \mathrm{Sp}(P)$.

- In this case $P - \lambda_0 : W_2^m(M) \to L^2(M)$ is an isomorphism.
- In particular, $(P - \lambda_0)^{-1} : L^2(M) \to W_2^m(M)$ is continuous.
- As the inclusion of $W_2^m(M) \hookrightarrow L^2(M)$ is compact, we see that $(P - \lambda_0)^{-1} : L^2(M) \to L^2(M)$ is a compact operator.
- Thus, $\mathrm{Sp}(P)$ consists of isolated eigenvalues with finite multiplicity clustering at $0$.

□

**Proof.**

- If $\lambda \neq \lambda_0$, then
$$P - \lambda = -(P - \lambda_0)\left((P - \lambda_0)^{-1} - (\lambda - \lambda_0)^{-1}\right)(\lambda - \lambda_0).$$
- Thus,
$$P - \lambda \text{ is invertible} \Longleftrightarrow (P - \lambda_0)^{-1} - (\lambda - \lambda_0)^{-1} \text{ is invertible}.$$
- That is,
$$\lambda \in \operatorname{Sp}(P) \Longleftrightarrow \left(\lambda \neq \lambda_0 \text{ and } (\lambda - \lambda_0)^{-1} \in \operatorname{Sp}\left((P - \lambda_0)^{-1}\right)\right).$$
- It follows that $\operatorname{Sp}(P)$ is unbounded and consists of isolated eigenvalues with finite multiplicity. □

**Remark**

- If $\operatorname{Sp}(P) \neq \mathbb{C}$, then $\operatorname{Sp}(P)$ consists of eigenvalues only.
- Thus,

$$P - \lambda \text{ is invertible} \Longleftrightarrow \ker(P - \lambda) = \{0\}.$$

**Proposition**

*If $\lambda \in \operatorname{Sp}(P)$, then $\ker(P-\lambda) \subset C^\infty(M)$.*

**Proof.**

- If $N \geq 1$, then $P^N \in \Psi^{mN}(M)$.
- If $p_m(x,\xi)$ is the principal symbol of $P$, then $p_m(x,\xi)^N \neq 0$ is the principal symbol of $P^N$, and hence $P^N$ is elliptic.
- The elliptic regularity theorem then ensures that
$$P^N v \in L^2 \Longrightarrow v \in W_2^{mN}(M).$$
- If $u \in \ker(P-\lambda)$, then $P^N u = \lambda^N u \in L^2(M)$.
- Thus $u \in W_2^{mN}(M)$ for all $N \geq 1$. That is,
$$u \in \bigcap_{N\geq 1} W_2^{mN}(M) = \bigcap_{s\in\mathbb{R}} W_2^{s}(M) = C^\infty(M).$$
- This shows that $\ker(P-\lambda) \subset C^\infty(M)$.

□

**Lemma**

*Assume* $\mathrm{Sp}(P) \neq \mathbb{C}$. *If* $P$ *is invertible, i.e.,* $\ker P = 0$, *then* $P^{-1} \in \Psi^{-m}(M)$.

**Proof.**

- As $P$ is elliptic there are $Q \in \Psi^{-m}(M)$ and $R_1, R_2 \in \Psi^{-\infty}(M)$ such that
$$QP = 1 - R_1, \qquad PQ = 1 - R_2.$$
- We then have
$$\begin{aligned} Q &= QPP^{-1} = (1 - R_1)P^{-1} = P^{-1} - R_1 P^{-1}, \\ Q &= P^{-1}PQ = P^{-1}(1 - R_2) = P^{-1} - P^{-1}R_2. \end{aligned}$$
- Thus,
$$P^{-1} = Q - R_1 P^{-1} = Q - P^{-1}R_2.$$

□

**Proof.**

- Here $Q : C^\infty(M) \to C^\infty(M)$ and $P^{-1} : L^2(M) \to L^2(M)$ are continuous.
- As $R_1$ is smoothing, it extends to a continuous operator $R_1 : \mathcal{E}'(M) \to C^\infty(M)$.
- Therefore $R_1 P$ maps continuously $C^\infty(M)$ to itself.
- The equality $P^{-1} = Q - R_1 P^{-1}$ then ensures that $P^{-1}$ induces a continuous operator $P^{-1} : C^\infty(M) \to C^\infty(M)$.
- As with $R_1$, the operator $R_2$ is smoothing, and so it extends to a continuous operator $R_2 : \mathcal{E}'(M) \to C^\infty(M)$.
- Thus, the composition $P^{-1}R_2 : \mathcal{E}'(M) \to C^\infty(M)$ is continuous, and so this is is smoothing operator.
- The equality $P^{-1} = Q - P^{-1}R_2$ then shows that $P^{-1}$ and $Q$ agree up to a smoothing operator.
- Thus, $P^{-1}$ is a ΨDO of same order as $Q$, i.e., $P^{-1} \in \Psi^{-m}(M)$.

□

**Remark**

- $P^{-1}$ is a parametrix for $P$.
- Therefore, it has the same symbol as any parametrix for $P$.
- In particular, if $p_m(x, \xi)$ is the principal symbol of $P$, then its principal symbol is $p_m(x, \xi)^{-1}$.

**Proposition**

*TFAE:*

1. $P$ *is formally selfadjoint (i.e., it agrees with its formal adjoint).*
2. $P$ *is selfadjoint on* $L^2_\mu(M)$ *as an operator with domain* $W_2^m(M)$.

Proof.

- If $P$ is selfadjoint, then it is formally selfadjoint, since we have

$$(*)\qquad \langle Pu|v\rangle = \langle u|Pv\rangle \qquad \forall u,v\in C^\infty(M).$$

- Assume $P$ is formally selfadjoint. Let us denote by $\overline{P}$ the operator $P$ with domain $W_2^m(M)$ and by $\overline{P}^*$ its adjoint.
- By definition the graph of $\overline{P}^*$ is

$$G(\overline{P}^*) = \left\{(u,v);\ \langle u|Pw\rangle = \langle v|w\rangle\ \forall w\in W_2^m(M)\right\}.$$

- We want to show that $G(\overline{P}) = G(\overline{P}^*)$.
- As $P: W_2^m(M)\rightarrow L^2(M)$ is continuous and $C^\infty(M)$ the equality $(*)$ implies that

$$\langle Pu|v\rangle = \langle u|Pv\rangle \qquad \forall u,v\in W_2^m(M).$$

- This shows that $G(\overline{P})\subset G(\overline{P}^*)$.

□

Proof.

- Let $(u,v) \in G(\overline{P}^*)$. Thus,
$$(**) \qquad \langle u|Pw\rangle = \langle v|w\rangle \quad \forall w \in W_2^m(M).$$
- If we regard $u$ as a distribution, then, given any $w \in C^\infty(M)$,
$$\langle Pu, w\rangle = \left\langle u, P^t u\right\rangle.$$
- The relation between formal adjoint and transpose and the formal selfadjointness of $P$ imply that
$$P^t w = \overline{P^*(\overline{w})} = \overline{P(\overline{w})}.$$
- Combining this with $(**)$ we get
$$\langle Pu, w\rangle = \left\langle u, \overline{P(\overline{w})}\right\rangle = \overline{\left\langle u|P\overline{w}\right\rangle} = \overline{\langle v|\overline{w}\rangle} = \langle v, w\rangle.$$
- This shows that $Pu = v \in L^2(M)$
- By elliptic regularity this implies that $u \in W_2^m(M)$, and so $(u,v) = (u,Pu) \in G(\overline{P})$. Thus, $G(\overline{P}^*) \subset G(\overline{P})$. □

**Remark**

If $P$ is formally selfadjoint, we often just say it is selfadjoint, since it is selfadjoint as an operator with $W_2^m(M)$ as domain.

**Proposition**

*Assume $P$ is (formally) selfadjoint. Then:*

- *Its spectrum is an unbounded discrete set of isolated real eigenvalues with finite multiplicity.*
- *It admits an orthonormal eigenbasis consisting of $C^\infty$-functions.*

**Proof.**

- As $P$ is essentially selfadjoint, it is selfadjoint, and so $\operatorname{Sp}(P) \subset \mathbb{R} \subsetneq \mathbb{C}$.
- Thus, $\operatorname{Sp}(P)$ is an unbounded discrete set of eigenvalues with finite multiplicity. □

**Proof.**

Let $\lambda_0 \in \mathbb{R} \setminus \mathrm{Sp}(P)$.

- $(P-\lambda_0)^{-1}$ is selfadjoint and compact, and so it admits an orthonormal eigenbasis $(e_j)_{j\geq 0}$.
- Recall that

$$P-\lambda = -(P-\lambda_0)\left((P-\lambda_0)^{-1} - (\lambda-\lambda_0)^{-1}\right)(\lambda-\lambda_0).$$

- Thus,

$$\ker(P-\lambda) = \ker\left[(P-\lambda_0)^{-1} - (\lambda-\lambda_0)^{-1}\right].$$

- Therefore, the $e_j$ are eigenfunctions of $P$.
- As $\ker(P-\lambda) \subset C^\infty(M)$ all the eigenfunctions $e_j$ are $C^\infty$. □

**Definition**

A selfadjoint operator $A : \mathrm{dom}(A) \to L^2_\mu(M)$ is bounded from below, if there is $c \in \mathbb{R}$ such that $A \geq -c$, i.e.,

$$\langle Au|u\rangle \geq -c\|u\|^2 \qquad \forall u \in \mathrm{dom}(A).$$

**Remark**

If $A$ is selfadjoint, then $A$ is bounded from below if and only $\mathrm{Sp}(A) \subset [-c, \infty)$ for some $c \in \mathbb{R}$.

**Proposition**

*If $P$ is selfadjoint and its principal symbol $p_m(x, \xi)$ is $> 0$, then $P$ is bounded from below.*

**Proof.**

The proof is based on the following:

**Claim**

*Let $N \geq m$, then there is $Q \in \Psi^{m/2}(M)$ such that*

$$P = Q^*Q \qquad \mod \Psi^{m-N}(M).$$

- Set $R = P - Q^*Q \in \Psi^{m-N}(M)$. As $m - N < 0$, the operator $R$ is bounded.
- Thus, given any $u \in W_2^m(M)$, we have

$$\langle Ru|u\rangle \geq -|\langle Ru|u\rangle| \geq -\|R\|\|u\|^2.$$

- As $P = Q^*Q + R$, we get

$$\langle Pu|u\rangle = \langle Q^*Qu|u\rangle + \langle Ru|u\rangle \geq \langle Qu|Qu\rangle - \|R\|\|u\|^2 \geq -\|R\|\|u\|^2.$$

- Thus, $P$ is bounded from below. □

**Proof of the claim.**

Set $q_{m/2}(x,\xi) := \sqrt{p_m(x,\xi)}$.

- Let $Q_0 \in \Psi^{m/2}(M)$ have $q_{m/2}(x,\xi)$ as principal symbol.
- The operator $Q_0^* Q_0$ is in $\Psi^m(M)$ and its principal symbol is $q_{m/2}(x,\xi)^* q_{m/2}(x,\xi) = p_m(x,\xi)$.
- Thus, $P$ and $Q_0^* Q_0$ have the same principal symbol, and hence
$$P = Q_0^* Q_0 \qquad \text{mod } \Psi^{m-1}(M).$$
- By induction we can construct $Q_j \in \Psi^{m/2-j}(M)$, $j = 1, 2, \ldots$, such that
$$P = (Q_0 + \cdots + Q_j)^*(Q_0 + \cdots + Q_j) \quad \text{mod } \Psi^{m-j}(M).$$
- We get the claim by taking $j = N$.

□

### Remark

- If $P$ is selfadjoint and a positive principal symbol, it is bounded from below.
- In particular, its spectrum is bounded from below.
- Therefore, $\text{Sp}(P)$ is a bounded from below discrete set of real eigenvalues with finite multiplicity.
- We thus can arrange $\text{Sp}(P)$ as a non-decreasing sequence,

$$\lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

**Remark**

- We have a natural action of $\mathbb{R}_+^*$ on $(T^*M)\setminus 0$ given by

$$\lambda \cdot (x,\xi) = (x, \lambda\xi), \qquad (x,\xi) \in T^*M,\ \lambda > 0.$$

- The cosphere bundle $S^*M$ is the sphere-bundle,

$$S^*M = \left[T^*M \setminus 0\right]/\mathbb{R}_+^*.$$

- The Liouville measure $dxd\xi$ of $T^*M$ descends to $S^*M$.
- If $g$ is any Riemannian metric, then

$$S^*M \simeq S_g^*M := \left\{\xi \in T^*M;\ |\xi|_g = 1\right\},$$

where $|\xi|_g^2 = \sum \xi_i g^{ij} \xi_j$ is the Hermitian metric on $T^*M$.

**Theorem (Weyl's Law)**

*Suppose that $P$ is selfadjoint and has a positive principal symbol $p_m(x,\xi)$. As $j \to \infty$, we have*

$$\lambda_j(P) \sim j^{\frac{m}{n}} \left( \frac{1}{n} \int_{S^*M} p_m(x,\xi)^{-\frac{n}{m}} dx đ\xi \right)^{-\frac{m}{n}}.$$

### Setup

- $(M^n, g)$ = compact Riemannian manifold.
- $\nu(g) = \sqrt{g(x)}dx$ = Riemannian measure.
- $d : C^\infty(M) \to C^\infty(M, T^*M)$ = de Rham differential.

### Notation

- If $g(x) = g_{ij}(x)dx^i \otimes dx^j$ and $g(x)^{-1} = (g^{ij}(x))$, then the metric defined by $g$ on $T^*M$ is given by

$$(\xi|\eta)_g = \sum \xi_i g^{ij}\eta_j, \quad \xi = \sum \xi_i dx^i, \ \eta = \sum \eta_i dx^i,$$
$$|\xi|_g = \sqrt{(\xi|\eta)_g} = \sqrt{\sum \xi_i g^{ij}(x)\xi_j}.$$

- The inner product on $1$-forms is then given by

$$\langle \xi|\eta\rangle_g = \int_M (\xi(x)|\eta(x))_g \, d\nu_g(x), \quad \xi, \eta \in C^\infty(M, T^*M).$$

**Definition**

The Laplacian $\Delta_g : C^\infty(M) \to C^\infty(M)$ is defined by

$$\langle \Delta_g u | u \rangle = \langle du | du \rangle \,, \qquad \forall u \in C^\infty(M).$$

**Proposition**

*In local coordinates,*

$$\Delta_g u = \frac{-1}{\sqrt{\det(g(x))}} \sum \partial_i \left( g^{ij}(x) \sqrt{\det(g(x))} \partial_j u \right) .$$

*In particular, $\Delta_g$ is a 2nd order differential operator whose principal symbol is*

$$p_2(x, \xi) = \sum \xi_i g^{ij}(x) \xi_j = |\xi|_g^2.$$

**Proposition**

1. *The Laplacian* $\Delta_g$ *is elliptic.*
2. $\Delta_g$ *is selfadjoint and has non-negative spectrum.*
3. *We have*

$$\ker \Delta_g = H^0(M),$$

*where* $H^0(M, \mathbb{R})$ *is the degree 0 de Rham homology space.*

**Remarks**

1. $H^0(M)$ consists of functions that are constant on each connected component of $M$.
2. It follows that we can arrange the spectrum of $P$ as a non-decreasing sequence,

$$0 = \lambda_0(\Delta_g) \leq \lambda_1(\Delta_g) \leq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

**Reminder**

The volume of $(M, g)$ is

$$\mathrm{Vol}_g(M) := \int_M d\nu_g(x).$$

**Theorem**

*As $j \to \infty$, we have*

$$\lambda_j(\Delta_g) \sim j^{\frac{2}{n}} \big(c(n)\,\mathrm{Vol}_g(M)\big)^{-\frac{2}{n}}, \quad c(n) := (2\pi)^{-n}|\mathbb{B}^n|.$$

Proof.

- Here $\Delta_g$ is a selfadjoint 2nd order elliptic differential operator with principal symbol $p_2(x,\xi) = |\xi|_g^2$.
- Thus, by the Weyl's law for elliptic ΨDOs we have

$$\lambda_j(\Delta_g) \sim j^{\frac{2}{n}} \left( \frac{1}{n} \int_{S^*M} |\xi|_g^{-n} dx đ\xi \right)^{-\frac{2}{n}}.$$

- We have

$$\begin{aligned} \frac{1}{n} \int_{S^*M} |\xi|_g^{-n} dx đ\xi &= \frac{1}{n} \int_{|\xi|_g=1} dx đ\xi \\ &= \frac{1}{n} \int_M (2\pi)^n |\mathbb{S}^{n-1}| \sqrt{\det(g(x))} dx \\ &= (2\pi)^n \frac{1}{n} |\mathbb{S}^{n-1}| \int d\nu_g(x) \\ &= (2\pi)^n |\mathbb{B}^n| \operatorname{Vol}_g(M). \end{aligned}$$

- This gives the result.

□

# Principal Symbol Map

## Definition

$S_m(T^*M)$, $m \in \mathbb{C}$, consists of functions $\sigma \in C^\infty(T^*M \setminus 0)$ such that

$$\sigma(x, \lambda\xi) = \lambda^m \sigma(x, \xi), \quad (x, \xi) \in T^*M \setminus 0, \ \lambda > 0.$$

## Remark

A have a well-defined principal symbol map,

$$\sigma_m : \Psi^m(M) \longrightarrow S_m(T^*M).$$

## Lemma

*For every* $\sigma \in S_m(T^*M)$ *there is* $P \in \Psi^m(M)$ *s.t.* $\sigma_m(P) = \sigma$.

**Proof.**

Let $\kappa : U \to V$ be a chart.

- Define
$$p_m(x,\xi) = \sigma(\kappa^{-1}(x), \kappa'(\kappa^{-1}(x))^t\xi), \quad (x,\xi) \in V \times (\mathbb{R}^n \setminus 0).$$
- This defines a homogeneoys symbol in $S_m(V \times \mathbb{R}^n)$.
- Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi = 1$ near $\xi = 0$, and set $p(x,\xi) = (1-\chi(\xi))p_m(x,\xi)$.
- Then $p(x,\xi) \in S^m(V \times \mathbb{R}^n)$ and $\tilde{p}_m(x,\xi) \sim p_m(x,\xi)$.
- Thus, $p(x,D) \in \Psi^m(V)$ and has $p_m(x,\xi)$ as principal symbol.
- Therefore, $P := \kappa_* P \in \Psi^m(U)$ and its principal symbol is
$$p_m\left(\kappa(x), (\kappa'(x)^{-1})^t\xi\right) = \sigma(x,\xi), \qquad (x,\xi) \in T^*U \setminus 0.$$
- This gives the result for domains of charts.

□

Proof.

- Let $(\varphi_i)_{i\in I}$ be a $C^\infty$-partition of unity subordinate to an open covering $(U_i)_{i\in I}$ by domains of charts.
- By the first part of the proof, for each $i$ there is $P_i \in \Psi^m(U_i)$ whose principal symbol is $\sigma(x,\xi)$.
- For each $i$ let $\psi_i \in C_c^\infty(U_i)$ be such that $\psi_i = 1$ near $\operatorname{supp}(\varphi_i)$, and set
$$P = \sum \varphi_i P \psi_i.$$
- Then $P$ is an operator in $\Psi^m(M)$ whose principal symbol is
$$\sum \varphi_i(x)\sigma_m(P_i)(x,\xi)\psi_i(x) = \sum \varphi_i(x)\sigma(x,\xi) = \sigma(x,\xi).$$
- This proves the result.

□

# Principal Symbol Map

The previous lemma implies the following result.

**Proposition**

*The principal symbol map gives rise to an exact sequence,*

$$0 \longrightarrow \Psi^{m-1}(M) \hookrightarrow \Psi^{m}(M) \xrightarrow{\sigma_m} S_m(T^*M) \longrightarrow 0.$$

# Densities on Manifolds

## Definition

A smooth (positive) measure on $M$ is a Borel measure $\mu$ on $M$ such that, for every chart $\kappa : U \to V$, we can write

$$\kappa_*(\mu_{|U}) = \rho_\kappa(x)|dx|, \qquad 0 < \rho_\kappa \in C^\infty(V),$$

where $|dx|$ is the Lebesgue measure on $V$.

## Remark

If $\kappa_1 : U_1 \to V_1$ is another chart and $\phi = \kappa \circ \kappa_1^{-1}$, then

$$(*) \qquad \rho_{\kappa_1}(x) = |\det(\phi'(x))|\rho_\kappa(\phi(x)), \quad x \in \kappa_1(U \cap U_1),$$

### Facts

Assume that for every chart $\kappa : U \to V$ we are given $\rho_\kappa \in C^\infty(V)$ in such a way to have $(*)$.

- Given $x \in M$, let $\kappa : U \to V$ be a chart around $x$.
- For any $X \in \Lambda^n T_x M$ we have

$$\kappa_* X = \lambda_\kappa(X) \partial_1 \wedge \cdots \wedge \partial_n.$$

- Define $\rho(x) : \Lambda^n T_x M \to \mathbb{C}$ by

$$\rho(x)(X) = \rho_\kappa(\kappa(x)) |\lambda_\kappa(X)|, \qquad X \in \Lambda^n T_x M.$$

## Facts

- Let $\kappa_1 : U_1 \to V_1$ be another chart near $x$; set $\phi = \kappa \circ \kappa_1^{-1}$.
- If $X \in \Lambda^n(T_x M)$, then

$$\kappa_* X = \phi_* (\kappa_1)_* X = \det(\phi'(\kappa_1(x)))\lambda_{\kappa_1}(X)\partial_1 \wedge \cdots \wedge \partial_n.$$

- Thus $\lambda_\kappa(X) = \det(\phi'(\kappa_1(x)))\lambda_{\kappa_1}(X)$, and so

$$\rho(x)(X) = \rho_\kappa(\kappa(x))|\det(\phi'(\kappa_1(x)))||\lambda_{\kappa_1}(X)|$$

- Thanks to $(*)$ we have

$$\begin{aligned}|\det(\phi'(\kappa_1(x)))|\rho_\kappa(\kappa(x)) &= |\det(\phi'(\kappa_1(x)))|\rho_\kappa[\phi(\kappa_1(x))] \\ &= \rho_{\kappa_1}(\kappa_1(x)).\end{aligned}$$

- Thus,

$$\rho(x)(X) = \rho_{\kappa_1}(\kappa_1(x))|\lambda_{\kappa_1}(X)|.$$

- This shows that $\rho(x)$ does not depend on the choice of the chart $\kappa$.

In fact, we get a section $x \to \rho(x)$ of the following vector bundle:

**Definition**

The density bundle of $M$ is

$$|\Lambda|(M) := \bigsqcup_{x \in M} \{\rho : \Lambda^n T_x M \to \mathbb{C},\ \rho(\lambda X) = |\lambda| \rho(X)\}.$$

**Remarks**

- $|\Lambda|(M)$ is a smooth line bundle, since every chart $\kappa : U \to V$ defines a trivialization,

$$|\Lambda|(U) \longrightarrow U \times \mathbb{C}, \qquad (x, \rho) \longrightarrow (x, \rho\,(\kappa^*(\partial_1 \wedge \cdots \wedge \partial_n))).$$

- A section of $|\Lambda|(M)$ is called a density.
- A smooth density is therefore a smooth section of $|\Lambda|(M)$.

### Example

Assume $M$ is orientable, and let $\omega$ be a nowhere vanishing $n$-form

- Here $\omega$ is a (nowhere vanishing) smooth section of $\Lambda^n T^* M = (\Lambda^n TM)^n$.
- Thus, we get a smooth density $|\omega|$ given by

$$\rho(x)X = |\langle \omega(x), X \rangle|, \qquad x \in M,\ X \in \Lambda^n T_x M.$$

## Example (Riemmanian Density)

Let $g$ be a Riemannian metric on $M$.

- If $\kappa : U \to V$ is chart, then $\kappa_* g(x)$, $x \in V$, is a smooth family of positive-definite matrices such that
$$\langle g_\kappa(x)X|X\rangle = g(\kappa(x))(\kappa^* X, \kappa^* X), \qquad X \in T_x V = \mathbb{R}^n.$$
- If $\kappa_1 : U_1 \to V_1$ is another chart, and $\phi = \kappa \circ \kappa_1^{-1}$, then
$$g_{\kappa_1}(x) = \phi'(x)^t g_\kappa(\phi(x))\phi'(x), \qquad x \in \kappa_1(U \cap U_1).$$
- Define
$$\nu(g)_\kappa(x) := \sqrt{\det(g_\kappa(x))} \in C^\infty(V).$$
- We have
$$\nu(g)_{\kappa_1}(x) = |\det(\phi'(x))|\nu(g)_\kappa(\phi(x)).$$
- Therefore, this defines a smooth (positive) density $\nu(g)$.
- It is called the Riemannian density (associated with $g$).

**Remark**

- In both examples we get a nowhere-vanishing density.
- That is, a global frame of the line bundle $|\Lambda|(M)$.
- Therefore, the line bundle $|\Lambda|(M)$ is always trivializable.

**Remark**

Let $V \in \mathbb{R}^n$ be an open set.

- We have a natural nowhere-vanishing smooth density,
$$|dx| := |dx^1 \wedge \cdots \wedge dx^n|.$$
- This yields a canonical identification,
$$C^\infty(V, |\Lambda|(V)) \simeq C^\infty(V), \qquad \rho(x) \longrightarrow \rho(x)(\partial_1 \wedge \cdots \wedge \partial_n).$$

More generally:

**Facts**

Let $\rho(x) \in C^\infty(M, |\Lambda(M))$ be a smooth density on $M$.

- For every chart $\kappa : U \to V$ we may write

$$\kappa_*(\rho_{|U}) = \rho_\kappa(x)|dx|, \quad \rho_\kappa(x) := \kappa_*(\rho_{|U})(\partial_1 \wedge \cdots \wedge \partial_n).$$

- Here $\rho_\kappa(x) \in C^\infty(V)$.
- If $\kappa_1 : U_1 \to V_1$ is another chart, and $\phi = \kappa \circ \kappa_1^{-1}$, then

$$\rho_{\kappa_1}(x) = |\det(\phi'(x))|\rho_\kappa(\phi(x)), \quad x \in \kappa_1(U \cap U_1),$$

These properties allow us to define the integrals of densities in an intrinsic fashion.

**Fact**

Let $\rho$ be a $C^\infty$ (or even $C^0$) density on $M$ with compact support in the domain a chart $\kappa : U \to V$.

- Define
$$\int_M \rho(x) := \int_V \rho_\kappa(x)dx.$$
- If If $\kappa_1 : U_1 \to V_1$ is another chart such that $\operatorname{supp} \rho \subset U_1$, and $\phi = \kappa \circ \kappa_1^{-1}$, then
$$\int_{V_1} \rho_{\kappa_1}(x)dx = \int_{V_1} |\det(\phi'(x))|\rho_\kappa(\phi(x))dx = \int_V \rho_\kappa(x)dx.$$
- This shows that $\int \rho(x)$ does not depend on the choice of the chart $\kappa$.

This leads to the following definition.

**Definition**

If $\rho(x)$ be a continuous density on $M$ with compact support, then

$$\int_M \rho(x) := \sum \int_M \varphi_i(x)\rho(x),$$

where $(\varphi_i)$ is any $C^\infty$-partition of unity subordinate to an open covering by domains of charts.

**Remark**

The r.h.s. above does not depend on the choice of the partition of unity.

**Remark**

We get a natural pairing between densities and functions.

- If $f(x)$ is continuous function and $\rho(x)$ is a continuous density such that $f$ or $\rho$ has compact, then we define
$$\langle \rho, f\rangle = \int_M f(x)\rho(x).$$
- This is a continuous bilinear pairings,
$$C^0(M, |\Lambda|(M)) \times C_c(M) \rightarrow \mathbb{C}, \quad C_c^0(M, |\Lambda|(M)) \times C(M) \rightarrow \mathbb{C}.$$
- In particular, any continuous density defines a (signed) Radon measure on $M$.

**Example**

If $g$ is a Riemannian metric, then the Riemannian measure is the measure defined by the Riemannian density $\nu(g)$.

**Definition**

$$C^{-\infty}(M) := C_c^{\infty}\left(M, |\Lambda|(M)\right)',$$
$$C_c^{-\infty}(M) := \left\{u \in C^{-\infty}(M);\ \text{supp}\, u \text{ compact}\right\}.$$

**Remark**

$C_c^{-\infty}(M) = C^{\infty}\left(M, |\Lambda|(M)\right)'$.

**Proposition**

*We have continuous embeddings with dense ranges,*

$$C^{\infty}(M) \hookrightarrow C^{-\infty}(M), \qquad C^{\infty}(M, |\Lambda|(M)) \hookrightarrow \mathcal{D}'(M).$$

### Remark

We say that $u \in \mathcal{D}'(M)$ is smooth, if it is given by a smooth density.

### Remark

- Any nowhere-vanishing smooth density $\rho$ defines a trivialization of $|\Lambda|(M)$.
- This yields an identification $C^\infty(M) \simeq C^\infty(M, |\Lambda|(M))$, $f \to f\rho$.
- This then yields an embedding $C^\infty(M) \hookrightarrow \mathcal{D}'(M)$.
- This depends on the choice of $\rho$, so this is not canonical.
- On an open $V \subset \mathbb{R}^n$ we take $\rho = |dx|$. This yields the usual embedding of $C^\infty(V)$ into $\mathcal{D}'(V)$.

# Densities on Manifolds

With the previous embeddings in mind, we get:

**Proposition**

*Any* $P \in \Psi^m(M)$, $m \in \mathbb{C}$, *uniquely extends to a continuous operator,*

$$P : C_c^{-\infty}(M) \longrightarrow C^{-\infty}(M).$$

*If* $P$ *is properly supported, then we further have operators,*

$$P : C_c^{-\infty}(M) \longrightarrow C_c^{-\infty}(M), \quad P : C^{-\infty}(M) \longrightarrow C^{-\infty}(M).$$

**Remark**

- A nowhere-vanishing smooth density $\rho$ also gives to identifications $C^{-\infty}(M) \simeq \mathcal{D}'(M)$ and $C_c^{-\infty}(M) \simeq \mathcal{E}'(M)$.
- This allows us to recover the results on actions of ΨDOs on distributions on manifolds.
- This depends on the choice of $\rho$ though.

## Facts

- If $P : C_c^\infty(M) \to C^\infty(M)$ is a continuous linear operator, then it has a Schwartz kernel,
$$k_P(x,y) \in C^\infty(M)\hat{\otimes}\mathcal{D}'(M) = C^\infty(M, \mathcal{D}'(M)).$$
- Namely,
$$Pu(x) = \langle k_P(x,y), u(y)\rangle\,, \qquad u \in C_c^\infty(M).$$
- The kernel is $C^\infty$ precisely if
$$k_P(x,y) \in C^\infty(M)\hat{\otimes}C^\infty(M, |\Lambda|(M)) = C^\infty\left(M \times M, \pi_2^*|\Lambda|(M)\right),$$
where $\pi_2 : M \times M \to M$ is the projection onto the 2nd factor.

### Facts (Continued)

- If $k_P(x,y)$ is $C^\infty$, then $P$ uniquely extends to a continuous operator,
$$P : C_c^{-\infty}(M) \longrightarrow C^\infty(M).$$
- Namely, if $u \in C_c^{-\infty}(M)$, then
$$Pu = \langle u(y), k_P(x,y) \rangle \, .$$

### Proposition

*TFAE:*

(i) $P$ *has a smooth Schwartz kernel.*

(ii) $P$ *uniquely extends to a cont. op.* $P : C_c^{-\infty}(M) \rightarrow C^\infty(M)$.

### Remark

The operators with smooth Schwartz kernels are called smoothing operators. The space of smoothing operators is $\Psi^{-\infty}(M)$.

**Notation**

$\Gamma = \{(x, x);\ x \in M\}$ diagonal of $M \times M$.

**Proposition**

*Let $P \in \Psi^m(M)$ has Schwartz kernel $k_P(x, y)$.*

1. $k_P(x, y)$ *is* $C^\infty$ *on* $(M \times M) \setminus \Gamma$.
2. *If* $\Re m < -n$, *then* $k_P(x, y)$ *is* $C^0$ *on* $M \times M$.

**Proof.**

- The result is true for ΨDOs on open sets $V \subset \mathbb{R}^n$.
- For ΨDOs on $M$ is enough to prove the result on the domain of a chart $\kappa : U \to V$.
- If $P \in \Psi^m(M)$, then $\kappa_*(P_{|U}) \in \Psi^m(V)$.
- On $U \times U$, we have

$$k_P(x,y) = k_{P_{|U}}(x,y) = (\kappa^{-1})_* \left[ k_{\kappa_*(P_{|U})}(x,y) \right].$$

- As $k_{\kappa_*(P_{|U})}(x,y)$ is $C^\infty$ off the diagonal, we see that $k_P(x,y)$ is smooth on $(U \times U) \setminus \Gamma$.
- If $\Re m < -n$, then $k_{\kappa_*(P_{|U})}(x,y)$ is $C^0$ on $V \times V$, and so $k_P(x,y)$ is $C^0$ on $U \times U$.

□

**Remark**

Let $P \in \Psi^m(M)$, $\Re m < -n$.

- Its Schwartz kernel $k_P(x,y)$ is in $C^0(M \times M, \pi_2^*|\Lambda|(M))$.
- Thus,

$$k_P(x,x) \in C^0\left(M, |\Lambda|(M)\right).$$

**Proposition**

*If* $P \in \Psi^m(M)$, $\Re m < -n$, *then*

$$k_P(x,x) \in C^\infty\left(M, |\Lambda|(M)\right).$$

*That is,* $k_P(x,y)$ *is a smooth density on* $M$.

**Proof.**

Let $U \subset \mathbb{R}^n$ be open and $P \in \Psi^m(U)$, $\Re m < -n$.

- Let $p(x,\xi) \in S^m(U \times \mathbb{R}^n)$ and $R \in \Psi^{-\infty}(U)$ be such that
$$P = p(x, D) + R.$$
- We then have
$$k_P(x,y) = \check{p}_{\xi\rightarrow y}(x, x-y) + k_R(x,y).$$
- Thus,
$$k_P(x,x) = \check{p}_{\xi\rightarrow y}(x, 0) + k_R(x,x).$$
- Here $k_R(x,x) \in C^\infty(U)$, and
$$\check{p}_{\xi\rightarrow y}(x,0) = \int p(x,\xi)đ\xi.$$

□

**Proof.**

- For any compact $K \subset U$, we have

$$|D_x^\alpha p(x,\xi)| \leq C_{K\alpha}(1+|\xi|)^{\Re m} \quad \forall (x,\xi) \in K \times \mathbb{R}^n.$$

- Here $\Re m < -n$, and hence $(1+|\xi|)^{\Re m} \in L^1(\mathbb{R}^n)$.
- This ensures that

$$\check{p}_{\xi\rightarrow y}(x,0) = \int p(x,\xi) đ\xi \in C^\infty(U).$$

- Thus,

$$k_P(x,x) = \check{p}_{\xi\rightarrow y}(x,0) + k_R(x,x) \in C^\infty(U).$$

□

**Proof.**

Let $P \in \Psi^m(M)$, $\Re m < -n$.

- Let $\kappa : U \to V$ be a chart, and set $P_\kappa = \kappa_*(P_{|U}) \in \Psi^m(V)$.
- On $U \times U$ we have
$$k_P(x,y) = \kappa^* \left[ k_{P_\kappa}(x,y) \right].$$
- Thus,
$$k_P(x,x) = \kappa^* \left[ k_{P_\kappa}(x,x) \right].$$
- By the first part, $k_{P_\kappa}(x,x) \in C^\infty(V)$, and so $k_P(x,x)$ is $C^\infty$ on $U$.
- As this is true on any chart domain, this gives the result. □

**Setup**

$M^n$ is a compact manifold.

**Remark**

- If $P \in \Psi^m(M)$, $\Re m < -n$, then $k_P(x, y)$ is $C^0$.
- If $\rho$ is any nowhere-vanishing density, then we may write
$$k_P(x, y) = K_\rho(x, y)\rho(y), \qquad K_\rho \in C(M \times M).$$
- If $u \in C^\infty(M)$, then
$$Pu(y) = \int K_\rho(x, y)\rho(y) = \int K_\rho(x, y)d\rho(y).$$
- Moreover, we have
$$\int k_P(x, x) = \int K_\rho(x, x)\rho(x) = \int K_\rho(x, x)d\rho(x).$$

**Reminder**

Suppose that $\mu$ is a smooth measure on $M$. Let $K(x,y) \in C^0(M \times M)$, and define $T_K : L^2_\mu(M) \rightarrow L^2_\mu(M)$ by

$$T_K u(x) = \int K(x,y)u(y)d\mu(y), \quad u \in L^2_\rho(M)$$

Then the following hold:

1. If $T_K \geq 0$, then $T_K$ is trace-class.
2. If $T_K$ is trace-class, then
$$\mathrm{Tr}\left[T_K\right] = \int K(x,y)d\mu(y).$$

# Trace Formula for ΨDOs

**Lemma**

*Given any $m > 0$ and smooth measure $\mu$ on $M$, we always can find an operator $A \in \Psi^m(M)$ such that*

- *$A$ is elliptic.*
- *It is selfadjoint with respect to $\mu$ and has positive spectrum.*

**Proposition**

*If $P \in \Psi^m(M)$ with $\Re m < -n$, then $P$ is trace-class, and we have*

$$\mathrm{Tr}[P] = \int_M k_P(x,x).$$

**Proof.**

Let $\rho$ be a nowhere-vanishing $C^\infty$ density on $M$.

- As $\Re m < -n$, we know that $P$ has a continuous kernel.
- Thus, there is $K_\rho \in C^0(M \times M)$ such that

$$Pu(x) = \int_M K_\rho(x,y)u(y)d\rho(y), \qquad u \in C^\infty(M).$$

- If $P \geq 0$ with respect to $\rho$, then $P$ is trace-class.

□

**Proof.**

- In general, set $a = |\Re m| > n$.
- Pick $A \in \Psi^{a}(M)$ which is elliptic and selfadjoint w.r.t. $\mu$ and has positive spectrum.
- As $A$ is elliptic its inverse $A^{-1}$ is in $\Psi^{-a}(M)$.
- As $A^{-1} \geq 0$, the first part ensures that $A^{-1} \in \mathcal{L}^1$.
- Note that $P = A^{-1}(AP)$.
- Here $AP \in \Psi^{m-a}$, as $\Re(m-a) = 0$, we see that $AP$ is bounded.
- As $\mathcal{L}^1$ is an ideal, $P = A^{-1}(AP)$ is in $\mathcal{L}^1$.
- Finally, as $P$ is trace-class and has a continuous kernel, we have

$$\mathrm{Tr}[P] = \int_M K_\rho(x,x)d\rho(x) = \int_M k_P(x,x).$$

The proof is complete. □

# Complex Powers of Elliptic ΨDOs

## Setup

- $M^n$ = closed manifold with smooth measure $\mu$.
- $P \in \Psi^m(M)$, $m > 0$, elliptic, selfadjoint and $\geq 0$ with principal symbol $p_m(x, \xi)$.

## Example

$P = Q^*Q$ with $Q \in \Psi^{m/2}(M)$ elliptic.

# Complex Powers of Elliptic ΨDOs

## Reminder

- $P$ with domain $W_2^m(M)$ is selfadjoint.
- Its spectrum can be arranged as a non-decreasing sequence,

$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots ,$$

  where each eigenvalue is repeated according to multiplicity.
- Each eigenspace $E_\lambda(P) := \ker(P - \lambda)$, $\lambda \in \mathrm{Sp}(P)$, is a finite dimensional subspace of $C^\infty(M)$.
- $P$ admits an orthonormal eigenbasis $(e_j)_{j\geq 0} \subset C^\infty(M)$ with

$$Pe_j = \lambda_j(P)e_j, \qquad j = 0, 1, 2, \ldots .$$

**Definition**

$\Pi_\lambda(P)$, $\lambda \in \operatorname{Sp}(P)$, is the orthogonal projection onto $E_\lambda(P)$.

**Lemma**

$\Pi_\lambda(P)$ *is a smoothing operator.*

**Proof.**

- $E_\lambda(P)$ is a finite dimensional subspace of $C^\infty(M)$.
- Let $\{\xi_1, \ldots \xi_N\} \subset C^\infty(M)$ be an orthonormal basis. We have

$$\Pi_\lambda(P)u(x) = \sum \langle \xi_j | u \rangle \, \xi_j(x) = \sum \xi_j(x) \Big( \int_M \overline{\xi_j(y)} u(y) d\mu(y) \Big)$$

- Thus,

$$\Pi_\lambda(P)u(x) = \int_M K(x,y) d\mu(y), \quad K(x,y) := \sum \xi_j(x) \overline{\xi_j(y)}.$$

- As $K(x,y) \in C^\infty(M \times M)$, it follows that $\Pi_\lambda(P) \in \Psi^{-\infty}(M)$. □

**Corollary**

*Set* $\tilde{P} := P + \Pi_0(P)$. *Then:*

1. $\tilde{P}$ *is in* $\Psi^m(M)$ *and has same principal symbol as* $P$. *In particular, it is elliptic.*
2. $P$ *is selfadjoint and has positive spectrum.*

**Proof.**

- The first part follows from the fact that $\Pi_0(P)$ is a smoothing operator.
- The operator $\tilde{P}$ is formally selfadjoint since $P$ and $\Pi_0(P)$ both are, and hence $\tilde{P}$ is selfadjoint.
- $\tilde{P} = P$ on $\ker \Pi_0(P) = (\ker P)^{\perp}$.
- $\tilde{P} = \Pi_0(P) = 1$ on $\operatorname{ran}(\Pi_0(P)) = \ker P$.
- Thus,

$$\operatorname{Sp}(\tilde{P}) = [\operatorname{Sp}(P) \setminus \{0\}] \cup \{1\} \subset (0, \infty).$$

□

**Remark**

- For $z \in \mathbb{C}$, the complex power $P^z$ is defined by using the Borel functional calculus for $P$ for $f(t) = \mathbb{1}_{(0,\infty)} t^z$.
- Equivalently, $P^z$ is the operator on $L^2_\mu(M)$ such that

$$P^z e_j = \begin{cases} \lambda_j(P)^z e_j & \text{if } \lambda_j(P) > 0, \\ 0 & \text{if } \lambda_j(P) = 0. \end{cases}$$

**Facts**

- For $\Re z \leq 0$ this a bounded operator.
- For $\Re z > 0$ this is a selfadjoint unbounded operator.
- We have

$$P^{z_1} P^{z_2} = P^{z_1 + z_2}, \qquad P^z|_{z=0} = 1 - \Pi_0(P).$$

# Complex Powers of Elliptic ΨDOs

**Theorem (Seeley)**

*$P^z$ is an operator in $\Psi^{mz}(M)$ whose principal symbol is $p_m(x,\xi)^z$.*

**Corollary**

*Let $P \in \Psi^m(M)$, $m \in \mathbb{C}$, be elliptic. Then:*

1. *$|P| := \sqrt{P^*P}$ is an operator in $\Psi^{\Re m}(M)$ whose principal symbol is $|p_m(x,\xi)|$.*
2. *$|P|^z$, $z \in \mathbb{C}$, is an operator in $\Psi^{z\Re m}(M)$ whose principal symbol is $|p_m(x,\xi)|^z$.*

# Complex Powers of Elliptic ΨDOs

**Setup**

$g =$ Riemannian metric on $M$.

**Proposition**

*The power $\Delta_g^z$, $z \in \mathbb{C}$, is an operator in $\Psi^{2z}(M)$ whose principal symbol is $|\xi|_g^z$.*

**Proof.**

- $\Delta_g$ is a 2nd order elliptic operator whose principal symbol is $|\xi|_g$.
- It is selfadjoint and $\geq 0$ with respect to the inner product defined by the Riemannian density.

□

The previous result holds *verbatim* for $1+\Delta_g$. In particular, we have:

**Proposition**

*Set* $\Lambda_g := \sqrt{1+\Delta_g}$. *Then:*

1. $\Lambda_g$ *is an operator in* $\Psi^1(M)$ *whose principal symbol is* $|\xi|_g$.
2. $\Lambda_g$ *is selfadjoint and its spectrum is contained in* $[1,\infty)$.
3. $\Lambda_g^z$, $z\in\mathbb{C}$, *is an operator in* $\Psi^z(M)$ *whose principal symbol is* $|\xi|_g^z$.

**Remark**

The fact that $\Lambda_g$ is invertible implies the group property,

$$\Lambda_g^{z_1}\Lambda_g^{z_2} = \Lambda_g^{z_1+z_2}, \qquad \Lambda_g|_{z=0} = 1.$$

In particular,

$$\left(\Lambda_g^z\right)^{-1} = \Lambda_g^{-z}, \qquad z\in\mathbb{C}.$$

**Proposition**

*Let $s \in \mathbb{R}$. Then $\Lambda_g^s : W_2^{t+s}(M) \to W_2^t(M)$ is an isomorphism for every $t \in \mathbb{R}$.*

**Proof.**

- As $\Lambda_g^s$ is a ΨDO of order $s$ it gives rise to a continuous operator,
$$\Lambda^s : W_2^{t+s}(M) \longrightarrow W_2^t(M).$$
- Its inverse is $\Lambda_g^{-s} : W_2^t(M) \to W_2^{t+s}(M)$.

□

**Corollary**

*If* $s \geq 0$, *then*

$$W_2^s(M) = \left\{u \in L^2(M);\ \Lambda_g^s u \in L^2(M)\right\}.$$

*Moreover,* $u \rightarrow \|\Lambda_g^s u\|$ *is a continuous norm on* $W_2^s(M)$.

**Proof.**

As $\Lambda_g^{-s} : L^2(M) \rightarrow W_2^s(M)$ is an isomorphism, we have

$$\begin{aligned} W_2^s(M) &= \operatorname{ran}\left\{\Lambda_g^{-s} : L^2(M) \longrightarrow W_2^s(M)\right\} \\ &= \left\{u \in L^2(M);\ \Lambda_g^s u \in L^2(M)\right\}. \end{aligned}$$

□

### Setup

$M^n$ = smooth manifold of dimension $n$.

### Definition

$\Psi^m(M, \mathbb{C}^r)$, $m \in \mathbb{C}$, consists of continuous operators $P : C_c^\infty(M, \mathbb{C}^r) \to C^\infty(M, \mathbb{C}^r)$ of the form,

$$Pu = \Big( \sum_j P_{ij} u_j \Big), \quad u = (u_j), \quad P_{ij} \in \Psi^m(M).$$

### Remark

In other words $\Psi^m(M, \mathbb{C}^r) \simeq M_r(\Psi^m(M)) = \Psi^m(M) \otimes M_r(\mathbb{C})$.

## Remarks

1. All the properties of scalar ΨDOs extend *verbatim* to ΨDOs with matrix coefficients.
2. We may also define ΨDOs mapping $C_c^\infty(M, \mathbb{C}^r)$ to $C^\infty(M, \mathbb{C}^{r'})$ with $r' \neq r$.

# ΨDOs on Vector Bundles

## Setup

$\mathcal{E}$ = smooth vector bundle over $M$ of rank $r$.

## Facts

Let $\tau : \mathcal{E}_{|U} \rightarrow U \times \mathbb{C}^r$ be a trivialization.

- We have a pushforward map $\tau_* : C^\infty(U, \mathcal{E}) \rightarrow C^\infty(U, \mathbb{C}^r)$ given by
$$\tau \circ u(x) = (x, \tau_* u(x)), \qquad u \in C^\infty(U, \mathcal{E}).$$
- We also have a pullback map $\tau^* : C^\infty(U, \mathbb{C}^r \rightarrow C^\infty(U, \mathcal{E}))$ given by
$$\tau^* v(x) = \tau^{-1}(x, v(x)), \qquad v \in C^\infty(U, \mathbb{C}^r).$$
- If $P : C_c^\infty(U, \mathcal{E}) \rightarrow C^\infty(U, \mathcal{E})$ is a linear operator, then the pushforward $\tau_* P : C_c^\infty(U, \mathbb{C}^r) \rightarrow C^\infty(U, \mathbb{C}^r)$ is defined by
$$(\tau^* P)v = \tau_* \left[ P(\tau^* v) \right], \qquad v \in C^\infty(U, \mathbb{C}^r).$$

## Facts

Let $\tau_1 : \mathcal{E}_{|U} \rightarrow U \times \mathbb{C}^r$ be another trivialization, and set

$$\tau_1 \circ \tau^{-1}(x,\xi) = (x, A(x)\xi), \qquad (x,\xi) \in U \times \mathbb{C}^r,$$

with $A(x) \in C^\infty(U, M_r(\mathbb{C}))$.

- If $u \in C^\infty(U, \mathcal{E})$, then
$$(\tau_1)_* u(x) = A(x)^{-1}(\tau_* u)(x).$$
- If $v \in C^\infty(U, \mathbb{C}^r)$, then
$$\tau_1^* v = \tau^*(Av).$$
- Thus, for any operator $P : C_c^\infty(U, \mathcal{E}) \rightarrow C^\infty(U, \mathcal{E})$, we have
$$(\tau_1)_* P = A^{-1}(\tau_* P)A.$$
- Note that $A(x)$ is an invertible element of $C^\infty(U, M_r(\mathbb{C}))$, and hence this is an invertible element of $\Psi^0(U, \mathbb{C}^r)$.

**Definition**

$\Psi^m(M,\mathcal{E})$, $m \in \mathbb{C}$, consists of continuous linear operators $P : C_c^\infty(M,\mathcal{E}) \rightarrow C^\infty(M,\mathcal{E})$ such that

$$\tau_* P \in \Psi^m(M,\mathbb{C}^r) \quad \text{for every trivialization } \tau : \mathcal{E}_{|U} \rightarrow U \times \mathbb{C}^r.$$

**Remark**

Let $\tau_1 : \mathcal{E}_{|U} \rightarrow U \times \mathbb{C}^r$ be another trivialization with $\tau_1 \circ \tau^{-1}(x,\xi) = (x, A(x)\xi)$.

- We have
$$(\tau_1)_* P = A^{-1}(\tau_* P)A.$$
- As $A(x)$ is an invertible element of $\Psi^0(U,\mathbb{C}^r)$, we get
$$(\tau_1)_* P \in \Psi^m(U,\mathbb{C}^r) \Longleftrightarrow \tau_* P \in \Psi^m(U,\mathbb{C}^r)$$
- This shows that the above definition is consistent.

# ΨDOs on Vector Bundles

**Remarks**

1. All the properties of scalar ΨDOs extend *mutantis mutandis* to ΨDOs acting on sections of vector bundles.
2. We may also define ΨDOs mapping sections of $\mathcal{E}$ to sections of another bundle $\mathcal{F}$.

**Notation**

$\pi : TM^* \to M =$ canonical submersion.

**Proposition**

*If $P \in \Psi^m(M, \mathcal{E})$, then its principal symbol makes sense as a section,*

$$\sigma_m(P)(x,\xi) \in C^\infty\left(T^*M \setminus 0, \pi^* \operatorname{End}(\mathcal{E})\right),$$

*such that*

$$\sigma_m(P)(x,\lambda\xi) = \lambda^m \sigma_m(P)(x,\xi) \qquad \forall \lambda > 0.$$

**Remark**

In particular $\sigma_m(x,\xi) \in \operatorname{End}(\mathcal{E}_x)$ for all $(x,\xi) \in T^*M \setminus 0$.

# ΨDOs on Vector Bundles

**Definition**

$S_m(T^*M, \mathcal{E})$ consists of $\sigma(x,\xi) \in C^\infty(T^*M \setminus 0, \pi^* \operatorname{End}(\mathcal{E}))$ s.t.

$$\sigma(x, \lambda\xi) = \lambda^m \sigma(x,\xi) \qquad \forall \lambda > 0.$$

**Proposition**

*The principal symbol map gives rise to an exact sequence,*

$$0 \longrightarrow \Psi^{m-1}(M, \mathcal{E}) \hookrightarrow \Psi^m(M\mathcal{E}) \xrightarrow{\sigma_m} S_m(T^*M, \mathcal{E}) \longrightarrow 0.$$

## Setup

- $\rho =$ is a smooth positive density on $M$.
- $(\cdot|\cdot)_{\mathcal{E}} =$ smooth Hermitian metric on $\mathcal{E}$.
- $L^2(M,\mathcal{E})$ is the completion of $C_c^\infty(M,\mathcal{E})$ w.r.t.

$$\langle u|v\rangle = \int_M (u(x)|v(x))_{\mathcal{E}}\, \rho(x), \qquad u, v \in C_c^\infty(M,\mathcal{E}).$$

## Remark

- If $M$ is compact, as a topological vector space $L^2(M,\mathcal{E})$ is independent of the choice of $\rho$ and $(\cdot|\cdot)_{\mathcal{E}}$.
- All the corresponding norms are equivalent to each other.

**Definition**

Let $P : C_c^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ be a linear operator. A formal adjoint is any operator $P^* : C_c^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ such that

$$\langle Pu|v\rangle = \langle u|P^*v\rangle \qquad \forall u, v \in C_c^\infty(M,\mathcal{E}).$$

**Proposition**

*Let $P \in \Psi^m(M,\mathcal{E})$. Then:*

1. *$P$ admits a formal adjoint $P^* \in \Psi^{\overline{m}}(M,\mathcal{E})$.*
2. *$\sigma_{\overline{m}}(P)(x,\xi) = \sigma_m(P)(x,\xi)^*$.*

**Remark**

If $P$ is formally selfadjoint, then $\sigma_m(P)(x,\xi)$ is a selfadjoint element of $\mathrm{End}(\mathcal{E}_x)$ for all $(x,\xi) \in T^*M \setminus 0$.

**Assumption**

$M$ is a closed manifold.

**Proposition**

*Let $P \in \Psi^m(M, \mathcal{E})$, $m > 0$, be elliptic and formally selfadjoint.*

1. $P$ *is essentially selfadjoint.*
2. $\mathrm{Sp}(P)$ *is an unbounded discrete set of real eigenvalues with finite multiplicity.*
3. *Each eigenspace* $\ker(P - \lambda)$ *is a finite dimensional subspace of* $C^\infty(M, \mathcal{E})$.

# ΨDOs on Vector Bundles

**Proposition**

*Let $P \in \Psi^m(M, \mathcal{E})$, $m > 0$, be elliptic and selfadjoint and such that $\sigma_m(P)(x, \xi) > 0$ in $\text{End}(\mathcal{E}_x)$.*

1. *$P$ is bounded from below.*
2. *Its spectrum can be arranged as a sequence,*

$$0 \leq \lambda_0(P) \leq \lambda_1(P) \leq \lambda_2(P) \leq \cdots,$$

*where each eigenvalue is repeated according to multiplicity.*

**Remark**

- The notation $\sigma_m(P)(x, \xi) > 0$ means that $\sigma_m(P)(x, \xi)$ is a positive invertible element of $\text{End}(\mathcal{E}_x)$.
- That is, $\sigma_m(P)(x, \xi)$ is selfadjoint and has positive spectrum.
- In particular, this ensures that $P$ is elliptic.

**Theorem (Weyl's Law)**

*Let* $P \in \Psi^m(M, \mathcal{E})$, $m > 0$, *be elliptic and selfadjoint with* $\sigma_m(P)(x,\xi) > 0$. *As* $j \to \infty$ *we have*

$$\lambda_j(P) \sim j^{\frac{m}{n}} \left( \frac{1}{n} \int_{S^*M} \mathrm{Tr}_{\mathcal{E}} \left[ \sigma_m(P)(x,\xi)^{-\frac{n}{m}} \right] dx đ\xi \right)^{-\frac{m}{n}}.$$

# ΨDOs on Vector Bundles

## Proposition

*Let* $P \in \Psi^m(M, \mathcal{E})$, $m > 0$, *be elliptic and selfadjoint with* $\sigma_m(P)(x,\xi) > 0$.

1. $P^z \in \Psi^{mz}(M, \mathcal{E})$ *for all* $z \in \mathbb{C}$.
2. $\sigma_{mz}(P^z)(x,\xi) = \sigma_m(P)(x,\xi)^z$.

## Remark

Here $\sigma_m(P)(x,\xi)^z$ is defined by holomorphic functional calculus in $\mathrm{End}(\mathcal{E}_x)$.

## Corollary

*Let* $Q \in \Psi^m(M, \mathcal{E})$, $\Re m > 0$, *be elliptic.*

1. $|Q| := \sqrt{Q^*Q} \in \Psi^{\Re m}(M, \mathcal{E})$.
2. $|Q|^z \in \Psi^{z\Re m}(M, \mathcal{E})$ *for all* $z \in \mathbb{C}$.
3. $\sigma_{z\Re m}(|Q|^z)(x,\xi) = |\sigma_m(P)(x,\xi)|^z$.

# ΨDOs on Vector Bundles

## Facts

- If $P : C_c^\infty(M, \mathbb{C}^r) \to C^\infty(M, \mathbb{C}^r)$ is continuous, then it has Schwartz kernel,
$$k_P(x, y) \in C^\infty(M, \mathbb{C}^r) \hat{\otimes} \mathcal{D}'(M, \mathbb{C}^r).$$
- Here $\mathcal{D}'(M, \mathbb{C}^r) = C_c^\infty(M, \mathbb{C}^r)'$.
- If $k_P(x, y)$ is $C^\infty$, then $k_P(x, y)$ is in

$$C^\infty(M, \mathbb{C}^r) \hat{\otimes} C^\infty(M, |\Lambda|(M) \otimes \mathbb{C}^r) = C^\infty(M \times M, \pi_2^* |\Lambda|(M) \otimes M_r(\mathbb{C})).$$

where $\pi_2 : M \times M \to M$ is the projection onto the 2nd factor.

# ΨDOs on Vector Bundles

## Definition (Exterior Tensor Product)

If $\mathcal{E}$ and $\mathcal{F}$ are $C^\infty$-vector bundles over $M$, then $\mathcal{E} \boxtimes \mathcal{F}$ is the $C^\infty$-vector over $M \times M$ whose fiber at $(x, y) \in M \times M$ is $\mathcal{E}_x \otimes \mathcal{E}_y$. That is,

$$\mathcal{E} \boxtimes \mathcal{F} = \bigsqcup_{(x,y)\in M\times M} \mathcal{E}_x \otimes \mathcal{E}_y.$$

## Remark

- If $\mathcal{F} = \mathcal{E}^*$, then
$$(\mathcal{E} \boxtimes \mathcal{E}^*)_{x,y} = \mathcal{E}_x \otimes \mathcal{E}_y^* = \mathrm{Hom}(\mathcal{E}_y, \mathcal{E}_x).$$
- In particular,
$$(\mathcal{E} \boxtimes \mathcal{E}^*)_{x,x} = \mathrm{Hom}(\mathcal{E}_x, \mathcal{E}_x) = \mathrm{End}(\mathcal{E}_x).$$

## Facts

- If $P : C_c^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ is continuous, then it has Schwartz kernel,
$$k_P(x,y) \in C^\infty(M, \mathcal{E}) \hat{\otimes} \mathcal{D}'(M, \mathcal{E}).$$
- Here $\mathcal{D}'(M, \mathcal{E}) = C_c^\infty(M, \mathcal{E})'$.
- If $k_P(x,y)$ is $C^\infty$, then $k_P(x,y)$ is in

$$C^\infty(M, \mathcal{E}) \hat{\otimes} C^\infty(M, |\Lambda|(M) \otimes \mathcal{E}^*) = C^\infty\left(M \times M, \mathcal{E} \boxtimes (\mathcal{E}^* \otimes |\Lambda|(M))\right).$$

**Setup**

$\Gamma = \{(x,x);\ x \in M\}$ diagonal of $M \times M$.

**Proposition**

*Let $P \in \Psi^m(M,\mathcal{E})$, $m \in \mathbb{C}$, have Schwartz kernel $k_P(x,y)$. Then $k_P(x,y)$ is $C^\infty$ on $(M \times M) \setminus \Gamma$.*

**Proposition**

*Let* $P \in \Psi^m(M, \mathcal{E})$, $\Re m < -n$ *have Schwartz kernel* $k_P(x, y)$. *Then:*

1. $k_P(x, y)$ *is* $C^0$ *on* $M \times M$, *i.e.,*

$$k_P(x, y) \in C^0\left(M \times M, \mathcal{E} \boxtimes (\mathcal{E}^* \otimes |\Lambda|(M))\right).$$

2. *We have*

$$k_P(x, x) \in C^\infty\left(M, \mathcal{E} \otimes \mathcal{E}^* \otimes |\Lambda|(M)\right) = C^\infty\left(M, \operatorname{End}(\mathcal{E}) \otimes |\Lambda|(M)\right).$$

*That is,* $k_P(x, x)$ *is an* $\operatorname{End}(\mathcal{E})$*-valued smooth density on* $M$.

**Remark**

We have a natural smooth bundle map $\operatorname{tr}_{\mathcal{E}} : \mathcal{E} \to M \times \mathbb{C}$, where $M \times \mathbb{C}$ is the trivial line bundle. Namely,

$$\operatorname{tr}_{\mathcal{E}}[A] := \operatorname{tr}_{\mathcal{E}_x}[A], \qquad x \in M,\ A \in \operatorname{End}(\mathcal{E}_x).$$

**Corollary**

*Let $P \in \Psi^m(M, \mathcal{E})$, $\Re m < -n$ have Schwartz kernel $k_P(x, y)$. Then:*

$$\operatorname{tr}_{\mathcal{E}}[k_P(x, x)] \in C^\infty(M, |\Lambda|(M)).$$

*That is, $\operatorname{tr}_{\mathcal{E}}[k_P(x, x)]$ is a smooth density on $M$.*

**Proposition**

*Assume* $M$ *is compact, and let* $P \in \Psi^m(M, \mathcal{E})$, $\Re m < -n$ *have Schwartz kernel* $k_P(x,y)$. *Then:*

1. $P$ *is trace-class.*
2. *We have*
$$\mathrm{Tr}[P] = \int_M \mathrm{tr}_{\mathcal{E}}[k_P(x,x)].$$