

Commutative Algebra

Chapter 6: Chain Conditions

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Reminder

- An order relation on a set Σ is a reflexive and transitive relation $\leq$ on Σ such that

$$x \leq y \text{ and } y \leq x \implies x = y.$$

- A set equipped with an order relation is called *partially ordered*.

Example

Let Σ be the set of submodules of a module M over a ring A . Then $(\Sigma, \subseteq)$ and $(\Sigma, \supseteq)$ are both partially ordered sets.

Proposition (Proposition 6.1)

Let $(\Sigma, \leq)$ be a partially ordered set. TFAE:

- (i) Every increasing sequence $x_1 \leq x_2 \leq \cdots \leq x_j \leq \cdots$ is stationary (i.e., there is n such that $x_j = x_n$ for all $j \geq n$).
- (ii) Every non-empty subset of Σ has a maximal element.

Chain Conditions

Proof.

- Let $S \subset \Sigma$ have no maximal element, i.e., for every $x \in S$ there is $y \in S$ such that $x \leq y$ and $y \neq x$.
- Starting with any $x_0 \in S$ we can construct by induction a sequence $(x_n)_{n \geq 0} \subset S$ such that $x_n \leq x_{n+1}$ and $x_n \neq x_{n+1}$, and hence (i) fails.
- By contraposition (i) implies (ii).
- Suppose that (ii) holds, let $x_1 \leq x_2 \leq \dots$ be an increasing sequence.
- There is $n \geq 1$ such that x_n is maximal in the set $\{x_j; j \geq 0\}$.
- If $j \geq n$, then $x_n \leq x_j$, and so $x_j = x_n$, since x_n is maximal.
- Thus, the sequence is stationary, and hence (ii) implies (i).

The proof is complete. □

Definition

Let M be a module over a ring A and denote by Σ be the set of its submodules.

- For the partially ordered set $(\Sigma, \subseteq)$ the condition (i) is called the *ascending chain condition* (a.c.c.) and the condition (ii) is called the *maximal condition*.
- For the partially ordered set $(\Sigma, \supseteq)$ the condition (i) is called the *descending chain condition* (d.c.c.) and the condition (ii) is called the *minimal condition*.

Definition

Let M be a module over A .

- 1 We say that M is *Noetherian* (after Emmy Noether) if M satisfies a.c.c. or the maximal condition.
- 2 We say that M is *Artinian* (after Emil Artin) if M satisfies d.c.c. or the minimal condition.

Remark

Let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M = 0$. Then:

- M is an $A/\mathfrak{a}$ -module.
- M is Noetherian (resp., Artinian) over A if and only if it is Noetherian (resp., Artinian) over $A/\mathfrak{a}$.

Example

Any finite group is both Noetherian and Artinian (as a module over $\mathbb{Z}$).

Proof of the Noetherian Property.

Denote by $|G|$ the order of G (i.e., its number of elements).

- Let $G_1 \subseteq G_2 \subseteq \dots$ be an ascending chain of subgroups.
- Then $|G_1| \leq |G_2| \leq \dots \leq |G|$ is a bounded non-decreasing sequence of integers, and hence $\exists n$ s.t. $|G_j| = |G_n|$ for $j \geq n$.
- As $G_j \supseteq G_n$, it follows that $G_j = G_n$ for $j \geq n$.
- Thus, the chain is stationary, and hence G is a Noetherian $\mathbb{Z}$ -module. □

Proof of the Artinian Property.

- Let $G_1 \supseteq G_2 \supseteq \dots$ be a descending chain of subgroups.
- Then $|G_1| \geq |G_2| \geq \dots \geq 0$ is a non-increasing sequence of non-negative integers, and so $\exists n$ s.t. $|G_j| = |G_n|$ for $j \geq n$.
- As $G_j \subseteq G_n$, it follows that $G_j = G_n$ for $j \geq n$.
- Thus, the chain is stationary, and hence G is an Artinian $\mathbb{Z}$ -module.



Chain Conditions

Example (see also Proposition 6.10)

Let V be a vector space over a field k . Then:

- 1 If $\dim V < \infty$, then V is both Noetherian and Artinian.
- 2 If $\dim V = \infty$, then V is neither Noetherian nor Artinian.

Proof ($\dim V = \infty$ Case).

- If $\dim V = \infty$, then V contains a linearly independent sequence $(v_j)_{j \geq 1}$.
- Set $V_j = \text{Span}\{x_k; k \leq j\}$. Then $V_1 \subsetneq V_2 \subsetneq \dots$ is a non-stationary ascending chain, and hence V is not Noetherian.
- Set $W_j = \text{Span}\{x_k; k \geq j\}$. Then $V = W_1 \supsetneq W_2 \supsetneq \dots$ is a non-stationary descending chain, and hence V is not Artinian.



Chain Conditions

Proof ($\dim V < \infty$ Case).

- Suppose that $\dim V < \infty$.
- Let $V_1 \subseteq V_2 \subseteq \cdots$ be an ascending chain of vector subspaces.
- Here $0 \leq \dim V_1 \leq \dim V_2 \leq \cdots \leq \dim V$, and hence $(\dim V_j)_{j \geq 0}$ is a bounded non-decreasing sequence of integers.
- Thus, $\exists n$ s.t. $\dim V_j = \dim V_n$ for $j \geq n$.
- As $V_j \supseteq V_n$, it follows that $V_j = V_n$ for $j \geq n$, i.e., we have a stationary chain, and hence V is Noetherian.
- Let $V_1 \supseteq V_2 \supseteq \cdots$ be a descending chain of vector subspaces.
- Here $\dim V \geq \dim V_1 \geq \dim V_2 \geq \cdots \geq 0$, and hence $(\dim V_j)_{j \geq 0}$ is a bounded non-increasing sequence of integers.
- Thus, $\exists n$ s.t. $\dim V_j = \dim V_n$ for $j \geq n$.
- As $V_j \subseteq V_n$, it follows that $V_j = V_n$ for $j \geq n$, i.e., we have a stationary chain, and hence V is Artinian.



Example

The ring $\mathbb{Z}$ (as a module over itself) is Noetherian, but is not Artinian.

Proof that $\mathbb{Z}$ is not Artinian.

- Every ideal of $\mathbb{Z}$ is principal, i.e., of the form (m) with $m \in \mathbb{Z}$.
- We have $(m_1) \subseteq (m_2)$ if and only if m_2 divides m_1 .
- Let $m \in \mathbb{Z} \setminus 0$. Then $(m) \supsetneq (m^2) \supsetneq (m^3) \supsetneq \cdots$ is a non-stationary descending chain of ideals.
- Thus, $\mathbb{Z}$ is not Artinian. □

Proof that $\mathbb{Z}$ is Noetherian.

- Let $(m_1) \subseteq (m_2) \subseteq \dots$ be an ascending chain of ideals (with $m_j \geq 0$).
- Suppose the chain is non-stationary, i.e., $(m_j) \subsetneq (m_{j+1})$ for all $j \geq 1$.
- Thus m_{j+1} divides m_j and $m_{j+1} \neq m_j$, so $m_j \geq 2m_{j+1}$, i.e., $m_{j+1} \leq 2^{-1}m_j$.
- By induction $m_j \leq 2^{-j+1}m_1 < 1$ for j large (impossible).
- Thus, the chain is stationary, and hence $\mathbb{Z}$ is Noetherian. □

Chain Conditions

Example

If k is a field, then, in the same way as with $\mathbb{Z}$, it can be shown that the polynomial ring $k[x]$ (seen as a module over itself) is Noetherian, but not Artinian.

Example

If k is a field, then the polynomial ring $A = k[x_1, x_2, \dots]$ with an infinite number of variables is neither Noetherian nor Artinian.

Proof.

- If $M_j = k[x_1, \dots, x_j]$, $j \geq 1$, then $M_1 \subsetneq M_2 \subsetneq \dots$ is a non-stationary ascending chain.
- Thus, A is not Noetherian.
- If $(x_1) \supsetneq (x_1^2) \supsetneq \dots$ is a non-stationary descending chain.
- Thus, A is not Artinian.



Chain Conditions

Proposition (Proposition 6.2)

Let M be a module over A . TFAE:

- (i) M is Noetherian.
- (ii) Every submodule of M is finitely generated.

Proof of (i) $\Rightarrow$ (ii).

- Assume M is a Noetherian, and let $N \subseteq M$ be a submodule.
- Let Σ be the set of all finitely generated submodules of N ,
- Here $\Sigma \neq \emptyset$, since $0 \in \Sigma$, it admits a maximal element N_0 .
- Suppose that $N_0 \subsetneq N$. Let $x \in N \setminus N_0$. Then $N'_0 = N_0 + Ax$ is finitely generated and $N'_0 \supsetneq N_0$, and hence N_0 is not maximal (contradiction).
- Thus, $N = N_0$, and hence N is finitely generated. □

Proof of (ii) $\Rightarrow$ (i).

- Suppose that every submodule is finitely generated.
- Let $M_1 \subseteq M_2 \subseteq \cdots$ be an ascending sequence of submodules.
- $N = \bigcup_{j=1}^{\infty} M_j$ is a submodule, and hence is finitely generated by $x_1, \dots, x_r$.
- For each i there is n_i such that $x_i \in M_{n_i}$. Set $n = \max(n_1, \dots, n_r)$. $\top$
- If $j \geq n$, then $x_i \in M_{n_i} \subseteq M_n \subseteq M_j$ for all i , and hence $N \subseteq M_n \subseteq M_j$.
- As $M_n \subseteq M_j \subseteq N$, we deduce that $M_n = M_j = N$ for all $j \geq n$.
- Thus, the chain is stationary, and hence M is Noetherian.

The proof is complete. □

Proposition (Proposition 6.3)

Let $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ be an exact sequence of A -modules.

- (i) M is Noetherian if and only if M' and M'' are both Noetherian.
- (ii) M is Artinian if and only if M' and M'' are both Artinian.

Remarks

- As α is injective, if $N \subseteq M'$, then $\alpha^{-1}(\alpha(N)) = N$. Thus,

$$N_1 = N_2 \iff \alpha(N_1) = \alpha(N_2)$$

- As β is surjective, if $N \subseteq M''$, then $\beta(\beta^{-1}(N)) = N$. Thus,

$$N_1 = N_2 \iff \beta^{-1}(N_1) = \beta^{-1}(N_2)$$

Chain Conditions

Proof of Proposition 6.3(i).

- Assume M is Noetherian.
- Let $M'_1 \subseteq M'_2 \subseteq \cdots$ be an ascending chain of submod. of M' .
- $\alpha(M'_1) \subseteq \alpha(M'_2) \subseteq \cdots$ is an ascending chain of submod. of M .
- As M is Noetherian, there is n such that if $j \geq n$, then $\alpha(M'_j) = \alpha(M'_n)$ for $j \geq n$, and hence $M'_j = M'_n$.
- Thus, we have a stationary chain, and hence M' is Noetherian.
- Let $M''_1 \subseteq M''_2 \subseteq \cdots$ be an ascending chain of submod. of M'' .
- $\beta^{-1}(M''_1) \subseteq \beta^{-1}(M''_2) \subseteq \cdots$ is an ascending chain of submodules of M .
- As M is Noetherian, there is n such that if $j \geq n$, then $\beta^{-1}(M''_j) = \beta^{-1}(M''_n)$ for $j \geq n$, and hence $M''_j = M''_n$.
- Thus, we have a stationary chain, and hence M'' is Noetherian.



Chain Conditions

Proof of Proposition 6.3(i); Continued.

- Assume M' and M'' are Noetherian, and let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of submod. of M .
- $\alpha^{-1}(M_1) \subseteq \alpha^{-1}(M_2) \subseteq \dots$ is an ascending chain of submod. of M' , and hence $\alpha^{-1}(M_j) = \alpha^{-1}(M_{n'})$ for $j \geq n'$.
- $\beta(M_1) \subseteq \beta(M_2) \subseteq \dots$ is an ascending chain of submodules of M'' , and hence $\beta(M_j) = \beta(M_{n''})$ for $j \geq n''$.
- Set $n = \max(n', n'')$, and let $x \in M_j$, $j \geq n$.
- As $\beta(x) \in \beta(M_j) = \beta(M_n)$, $\exists y \in M_n$ s.t. $\beta(x) = \beta(y)$.
- Thus, $x - y \in \ker \beta = \text{ran } \alpha$, and hence $\exists z \in M'$ s.t. $\alpha(z) = x - y \in M_j$, and hence $z \in \alpha^{-1}(M_j) = \alpha^{-1}(M_n)$.
- Thus, $x = y + \alpha(z) \in M_n + M_n = M_n$, and hence $M_j \subseteq M_n$.
- Thus, $M_j = M_n$ for $j \geq n$, i.e., the chain is stationary, and hence M is Noetherian.

The proof is complete.



Chain Conditions

Corollary (Corollary 6.4)

If $M_1, \dots, M_n$ are Noetherian (resp., Artinian) modules over A , then so is their direct sum $\bigoplus_{i=1}^n M_i$.

Proof.

- Assume that the modules M_i are Noetherian. We proceed by induction on n .
- For $n = 1$ the result is immediate.
- Suppose that $\bigoplus_{i=1}^{n-1} M_i$ is Noetherian. We have an exact sequence,

$$0 \longrightarrow M_n \longrightarrow \bigoplus_{i=1}^n M_i \longrightarrow \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0.$$

- As M_n and $\bigoplus_{i=1}^{n-1} M_i$ are Noetherian, $\bigoplus_{i=1}^n M_i$ is Noetherian by Proposition 6.3.
- This proves the Noetherian case. The Artinian case is proved similarly.



Definition

A ring A is called *Noetherian* (resp., *Artinian*) when it is Noetherian (resp., Artinian) as a module over itself, i.e., it satisfies a.c.c. (resp., d.c.c.) on *ideals*.

Examples

- 1 Any field k is both Noetherian and Artinian (the only ideals are 0 and k).
- 2 The ring $\mathbb{Z}$ is Noetherian, but not Artinian (see slide 11).
- 3 Any principal ideal domain is Noetherian (this follows from Proposition 6.2).

Example

Let $A = k[x_1, x_2, \dots]$ be the polynomial ring with infinite variables over a field k .

- A is not Noetherian (see slide 13).
- It is an integral domain, and so its fraction field $k = \text{Frac}(A)$ is Noetherian.
- This shows that a subring of a Noetherian ring need not be Noetherian.

Example

Let X be a compact Hausdorff space. Then the ring $C(X)$ of continuous functions on X is not Noetherian.

Chain Conditions

Proposition (Proposition 6.5)

Let A be a Noetherian (resp., Artinian) ring and M a finitely generated A -module. Then M is Noetherian (resp., Artinian).

Proof (Noetherian Case).

- Assume A is Noetherian, and let $x_1, \dots, x_n$ generate M .
- Define $\phi : A^n \rightarrow M$, $\phi(a_1, \dots, a_n) = a_1x_1 + \dots + a_nx_n$.
- We then have an exact sequence,

$$0 \longrightarrow \ker \phi \longrightarrow A^n \longrightarrow M \longrightarrow 0.$$

- By Corollary 6.4 A^n is Noetherian.
- Thus, M is Noetherian by Proposition 6.3.

The proof is complete. □

Reminder

A ring A is Noetherian (resp., Artinian) if and only if any ascending (resp., descending) chain of ideals is stationary.

Chain Conditions

Proposition (Proposition 6.6)

Let A be a Noetherian (resp., Artinian) ring and $\mathfrak{a}$ an ideal of A . Then $A/\mathfrak{a}$ is a Noetherian (resp., Artinian) ring.

Proof of Proposition 6.6.

- We have an exact sequence of A -modules,

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0.$$

- If A is Noetherian, then $A/\mathfrak{a}$ is a Noetherian A -module by Proposition 6.3.
- If $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$ is an ascending chain of ideals of $A/\mathfrak{a}$, then this is an ascending chain of A -submodules of $A/\mathfrak{a}$.
- As $A/\mathfrak{a}$ is a Noetherian A -module, this chain is stationary, and hence $A/\mathfrak{a}$ is Noetherian as an $A/\mathfrak{a}$ -module.
- Likewise, if A is Artinian, then the ring $A/\mathfrak{a}$ is Artinian as well.

The proof is complete. □

Chain Conditions

Definition

Let M be a module over A .

- A *chain* of submodules of M is a strictly descending finite sequence of the form,

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n = 0.$$

- We call n the *length* of the chain.
- A *composition series* is a maximal chain, i.e., we cannot insert any module between M_i and M_{i+1} .

Remarks

- The composition series condition is equivalent to requiring each module M_i/M_{i+1} to be *simple*, i.e., it has no submodules but 0 and itself.
- If $M = A$ and the M_i are ideals, then this is equivalent to requiring M_i/M_{i+1} to be a field.

Chain Conditions

Proposition (Proposition 6.7)

Suppose that M has a composition series of length n . Then:

- (i) Every composition series has length n .*
- (ii) Every chain in M can be extended to a composition series.*

Definition

- If M has a composition series, then we denote by $\ell(M)$ the length of any composition series.
- Otherwise we set $\ell(M) = \infty$.
- We call $\ell(M)$ the *length* of M .

Remark

We say that M is of *finite length* if it admits a composition series.

Proposition (Proposition 6.8)

Let M be a module over A . TFAE:

- (i) M is of finite length.
- (ii) M is both Noetherian and Artinian.

Proof of (i) $\Rightarrow$ (ii).

- If M is of finite length, then by Proposition 6.7 all chains are bounded.
- Thus, every ascending or descending chain is stationary, and hence M is both Noetherian and Artinian.



Chain Conditions

Proof of (ii) $\Rightarrow$ (i).

- Suppose that M is Noetherian and Artinian.
- If $N \subseteq M$ is a non-zero sub-module, then the set of proper submodules of N is non-empty.
- As M is Noetherian, this set admits a maximal element.
- That is, there is submodule $N' \subsetneq N$ such that N/N' is simple.
- Thus, starting with $M_0 = M$ by induction we can construct a chain $M_0 \supseteq M_1 \supseteq \cdots$ such that either $M_j = 0$ or M_j/M_{j+1} is simple.
- As M is Artinian, this chain must be stationary, and so there is n such that $M_n = 0$ for all $j \geq n$.
- Taking the smallest such n then yields a composition series $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$, and hence M is of finite length.

The proof is complete. □

Remark (Jordan-Hölder Theorem; see Carlson + Gaillard)

Let M be an A -module of finite length n . If $(M_i)_{0 \leq i \leq n}$ and $(M'_i)_{0 \leq i \leq n}$ are two composition series for M . Then the quotients M_{i-1}/M_i and M'_{i-1}/M'_i are isomorphic for $i = 1, \dots, n$,

Proposition (Proposition 6.9)

The length is an additive function on finite-length A -modules. That is, if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is any exact sequence of finite-length A -modules, then

$$\ell(M) = \ell(M') + \ell(M'').$$

Proof of Proposition 6.9 (Sketch).

- Let $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ be an exact sequence.
- Let $M'_0 \supsetneq \cdots \supsetneq M'_{n'}$ be a comp. series for M' .
- Let $M''_0 \supsetneq \cdots \supsetneq M''_{n''}$ be a comp. series for M'' .
- Note that

$$\beta^{-1}(M''_{n''}) = \beta^{-1}(0) = \ker \beta = \operatorname{ran} \alpha = \alpha(M') = \alpha(M'_0).$$

- Then

$$M = \beta^{-1}(M''_0) \supsetneq \cdots \supsetneq \beta^{-1}(M''_{n''}) = \alpha(M'_0) \supsetneq \cdots \supsetneq \alpha(M'_{n'}) = 0$$

is a composition series for M .

- Thus, M of finite length $\ell(M) = n' + n'' = \ell(M') + \ell(M'')$.

This gives the result. □

Proposition (Proposition 6.10)

Let V be a vector space over a field k . TFAE:

- (i) V has finite dimension.
- (ii) V has finite length.
- (iii) V is Noetherian.
- (iv) V is Artinian.

Furthermore, if these conditions hold, then $\ell(V) = \dim V$.

Proof of Proposition 6.10.

- Assume $\dim V = n$ and let $\{v_1, \dots, v_n\}$ be a basis for V .
- Set $V_j = \text{Span}\{v_{j+1}, \dots, v_n\}$, $0 \leq j \leq n-1$, and $V_n = \{0\}$.
- Then $V = V_0 \supsetneq V_1 \supsetneq \dots \supsetneq V_n = 0$ is a composition series for V , and hence V is of finite length $n = \dim V$. Thus (i) $\Rightarrow$ (ii).
- It is immediate that (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).
- If $\dim V = \infty$, then V is not Noetherian or Artinian (see Slide 9). By contraposition (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i).

The proof is complete. □

Corollary (Corollary 6.11)

Let A be a ring such that there are maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ so that $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$. Then A is Noetherian if and only if it is Artinian.