Differentiable Manifolds
§12. The Tangent Bundle

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The Tangent Bundle as a Manifold

Objective

Let M be smooth manifold of dimension n. We would like to bundle together all the tangent spaces T_pM so as to get a smooth manifold, called the *tangent bundle*.

Definition

As a set, the *tangent bundle* of M is the disjoint union,

$$TM := \bigsqcup_{p \in M} T_pM = \{(p, v); p \in M,\ v \in T_pM\}.$$

Remarks

1. For $p \in M$ we identify the subset $\{p\} \times T_pM$ with the tangent space T_pM. This allows us to see T_pM as a subset of TM.
2. In particular, we write an element of TM either as (p, v) with $p \in M$ and $v \in T_pM$, or simply as v.

The Tangent Bundle as a Manifold

Remark

Let U be an open set in M. If $p \in U$, then $T_pU = T_pM$. Thus,

$$TU = \bigsqcup_{p \in U} T_pU = \bigsqcup_{p \in U} T_pM.$$

Definition

The *canonical map* $\pi : TM \to M$ is defined by

$$\pi\left((p,v)\right) = p, \qquad p \in M, \quad v \in T_pM.$$

Remarks

1. The map $\pi : TM \to M$ is onto.
2. If $p \in M$, then $\pi^{-1}(p) = T_pM$.

The Tangent Bundle as a Manifold

Example

Let U be an open in $\mathbb{R}^n$. If $p \in U$, then $T_pU = T_p\mathbb{R}^n = \mathbb{R}^n$. Recall that, if $(r^1, \ldots, r^n)$ are the standard coordinates on $\mathbb{R}^n$, then we identify

$$T_p\mathbb{R}^n \ni v = \sum v^i \frac{\partial}{\partial r^i}\bigg|_p \longleftrightarrow \langle v^1, \ldots, v^n \rangle \in \mathbb{R}^n.$$

Thus, the pair (p, v) is naturally identified with $(p, v^1, \ldots, v^n)$. Therefore, we have

$$TU = \bigsqcup_{p \in U} T_pU = \bigsqcup_{p \in U} \mathbb{R}^n = U \times \mathbb{R}^n.$$

The Tangent Bundle as a Manifold

Aim

We wish to equip TM with a smooth structure. Therefore, we need to do the following:

1. Define a topology on TM.
2. Construct a C^∞-atlas for TM.

The Tangent Bundle as a Manifold

Facts

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M.

- Here $\phi(U)$ is an open in $\mathbb{R}^n$, and so $T(\phi(U)) = \phi(U) \times \mathbb{R}^n$.
- For every $p \in U$, the differential $\phi_{*,p}$ is an isomorphism from $T_pM = T_pU$ onto $T_{\phi(p)}(\phi(U)) = \mathbb{R}^n$.
- Therefore, we define a map $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ by
$$\tilde{\phi}(p, v) = \big(\phi(p), \phi_{*,p} v\big), \qquad p \in U, \ v \in T_pU.$$
- This is a bijection with inverse $(x, v) \to (\phi^{-1}(x), \phi^{-1}_{*,\phi^{-1}(x)} v)$.
- This allows us to define a topology on TU by pulling back the topology of $\phi(U) \times \mathbb{R}^n$:
$$W \subset TU \text{ is open} \iff \tilde{\phi}(W) \text{ is open in } \phi(U) \times \mathbb{R}^n.$$
- With respect to this topology $\tilde{\phi}$ is a homeomorphism.

The Tangent Bundle as a Manifold

Remark

- If $p \in U$, then $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^1}\Big|_p \right\}$ is a basis of T_pM.
- The differential $\phi_{*,p} : T_pU \to T_{\phi(p)}V = \mathbb{R}^n$ maps $\frac{\partial}{\partial x^i}\Big|_p$ to $\frac{\partial}{\partial r^i}\Big|_{\phi(p)}$. Thus,

$$\sum v^i \frac{\partial}{\partial x^i}\bigg|_p \xrightarrow{\phi_*} \sum v^i \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} \longleftrightarrow \langle v^1, \ldots, v^n \rangle \in \mathbb{R}^n.$$

- If $\phi(p) = (x^1(p), \ldots, x^n(p))$ and $v = \sum v^i \frac{\partial}{\partial x^i}\Big|_p \in T_pM$, then

$$\tilde{\phi}(p, v) = \big(\phi(p), \phi_{*,p} v\big) = \big(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\big).$$

  In particular, this defines coordinates on TU.

The Tangent Bundle as a Manifold

Facts

- Let $(V, \psi) = (V, y^1, \ldots, y^n)$ be another chart of M such that $U \cap V \neq \emptyset$. Define $\tilde{\psi} : TV \to \psi(V) \times \mathbb{R}^n$ by
$$\tilde{\psi}(p, v) = \big(\psi(p), \psi_{*,p} v\big), \qquad p \in V, \ v \in T_pM.$$
- On $T(U \cap V) = (TU) \cap (TV)$ we have two topologies induced by the respective topologies of TU and TV.
- On $\tilde{\phi}(TU \cap TV) = \phi(U \cap V) \times \mathbb{R}^n$ we have
$$\tilde{\psi} \circ \tilde{\phi}^{-1}(r, v) = \big(\psi \circ \phi^{-1}(r), \psi_* \circ \phi_*^{-1} v\big) = \big(\psi \circ \phi^{-1}(r), (\psi \circ \phi^{-1})_{*,r} v\big).$$
- Here $(\psi \circ \phi^{-1})_{*,r}$ is the multiplication by the Jacobian matrix $J_{\psi \circ \phi^{-1}}(r) = \left[\partial(y^j \circ \phi^{-1})/\partial r^i(r)\right]$ whose entries depends smoothly on r.
- Therefore, $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ is smooth map. Its inverse map $\tilde{\phi} \circ \tilde{\psi}^{-1}$ is smooth as well, and so $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is a diffeomorphism.

Facts (Continued)

- $T(U \cap V)$ is open in TU and in TV.
- As $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ is a diffeomorphism, this is a homeomorphism.
- If $W \subset T(U \cap V)$, then

$$\begin{aligned} W \text{ open in } TU &\iff \tilde{\phi}(W) \text{ open in } \phi(U) \times \mathbb{R}^n, \\ &\iff \tilde{\psi} \circ \tilde{\phi}^{-1}\left[\tilde{\phi}(W)\right] \text{ open in } \psi(U \cap V) \times \mathbb{R}^n, \\ &\iff \tilde{\psi}(W) \text{ open in } \psi(V) \times \mathbb{R}^n, \\ &\iff W \text{ open in } TV. \end{aligned}$$

  Thus, TU and TV induce the same topology on $T(U \cap V)$.
- It follows that, if W is open in TU and X is open in TV, then $W \cap X = (W \cap T(U \cap V)) \cap (X \cap T(U \cap V))$ is open in $T(U \cap V)$ (by definition of the subspace topology).

The Tangent Bundle as a Manifold

Summary

(a) If (U, ϕ) is a chart for M, then $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ allows us to define a topology on TU by pulling back the topology of $\phi(U) \times \mathbb{R}^n$. This map then becomes a homeomorphism.

(b) If (V, ψ) is another chart for M, then the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ is a diffeomorphism.

(c) TU and TV induce the same topology on $U \cap V$. In particular, if W is open in TU and X is open in TV, then $W \cap X$ is open in $T(U \cap V)$.

In particular, (b) would allow us to get a C^∞ atlas for TM provided we can define a topology on TM by patching together the TU-topologies.

The Tangent Bundle as a Manifold

Reminder (Topological Bases; see Appendix A)

Let X be a topological space. A *basis for the topology of* X is a collection $\mathscr{B}$ of open sets such that, for every open $U \subset X$ and every $p \in U$, there is an open set $V \in \mathscr{B}$ such that $p \in V \subset U$.

Remark

If $\mathscr{B}$ is a basis for the topology of X, then every open set is the union of open sets in $\mathscr{B}$. We then say that $\mathscr{B}$ generates the topology of X.

Proposition (Proposition A.8)

Let X be a set and $\mathscr{B}$ a collection of subsets such that:

(i) $X = \cup_{V \in \mathscr{B}} V$.

(ii) *If $V_1, V_2 \in \mathscr{B}$ and $p \in V_1 \cap V_2$, then there is $W \in \mathscr{B}$ such that $p \in W \subset V_1 \cap V_2$.*

Then:

1. *$\mathscr{B}$ is a basis for a unique topology on X.*
2. *The open sets for this topology consists of unions of sets in $\mathscr{B}$.*

Remark

The condition (ii) holds automatically if $\mathscr{B}$ is closed under finite intersection.

The Tangent Bundle as a Manifold

Facts

Let $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of M (which defines its smooth structure). Define

$$\mathscr{B} = \bigcup_\alpha \{W;\ W \text{ is an open in } TU_\alpha\}.$$

Note that $TU_\alpha \in \mathscr{B}$.

- As $\cup U_\alpha = M$, we have
$$\bigcup_\alpha TU_\alpha = \bigcup_\alpha \bigsqcup_{p \in U_\alpha} T_pM = \bigsqcup_{p \in \cup U_\alpha} T_pM = \bigsqcup_{p \in M} T_pM = TM.$$
- If W_α is an open in TU_α and W_β is an open in TU_β, then $W_1 \cap W_2$ is open in $T(U_\alpha \cap U_\beta)$, and hence is contained in $\mathscr{B}$.

It follows that $\mathscr{B}$ satisfies the conditions (i) and (ii) of Proposition A.8, and so it's a basis for a unique topology on TM.

The Tangent Bundle as a Manifold

Definition

The topology of TM is the topology generated by $\mathscr{B}$. The open sets are unions of sets in $\mathscr{B}$.

Remark

Each TU_α is open in TM, since it is contained in $\mathscr{B}$.

Proposition (Proposition 12.4)

As a topological space TM is Hausdorff.

Remark

Each open TU_α is Hausdorff since it is homeomorphic to the open set $\phi_\alpha(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$. This can be used to show that TM is Hausdorff.

The Tangent Bundle as a Manifold

Proposition (Proposition 12.3)

The topology of TM is second countable.

Remarks

- It can be shown that the topology of M admits a countable basis $\{U_i\}_{i\in I}$ consisting of domains of charts (*cf.* Lemma 12.2 of Tu's book).
- Each TU_i is second countable since it is homeomorphic to an open in $\mathbb{R}^n \times \mathbb{R}^n$.
- If for each $i \in I$, we let $\{W_{i,j}\}_{j\in\mathbb{N}}$ be a countable basis for the topology of TU_i, then $\{W_{i,j}; i \in I,\ j \in \mathbb{N}\}$ is a countable basis for the topology of TM (see Tu's book).

The Tangent Bundle as a Manifold

Facts

- Each TU_α is an open in TM.
- We knows that $TM = \cup_\alpha TU_\alpha$.
- The local trivializations $\tilde{\phi}_\alpha : TU_\alpha \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n$ are homeomorphisms onto open sets in $\mathbb{R}^n \times \mathbb{R}^n$.
- All the transition maps $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ are smooth.

Proposition

The collection $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas for TM, and hence TM is a smooth manifold of dimension $2n$.

Remark

If $\{(V_\beta, \psi_\beta)\}$ is any C^∞-atlas for M, then we also get a C^∞ atlas $\{(TV_\beta, \tilde{\psi}_\beta)\}$ for TM. It is compatible with the atlas $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$, and so it defines the same smooth structure.

The Tangent Bundle as a Manifold

Facts

- The canonical map $\pi : TM \to M$ is such that $\pi(v) = p$ if $v \in T_pM$. It is onto.
- Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart for M. Then

$$\begin{aligned}\phi \circ \pi \circ \tilde{\phi}^{-1}&(r^1, \ldots, r^n, v^1, \ldots, v^n)\\ &= \phi \circ \pi \left[\phi^{-1}(r^1, \ldots, r^n), \sum v^i \partial/\partial x^i\right]\\ &= \phi \circ \phi^{-1}(r^1, \ldots, r^n) = (r^1, \ldots, r^n).\end{aligned}$$

- As (U, ϕ) and $(TU, \tilde{\phi})$ are charts this shows that π is C^∞.
- By the converse of the submersion theorem (exercise!) this also shows that π is a submersion.

Proposition

The canonical projection $\pi : TM \to M$ is a surjective submersion.

Vector Bundles

Definition

A *vector bundle of rank* r over a manifold M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ such that:

(i) For every $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a vector space of dimension r.

(ii) For each $p \in M$ there is an open neighborhood U of p in M and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ (called *trivialization of* E *over* U) such that

- $\pi \circ \phi^{-1}(q, \xi^1, \dots, \xi^r) = q$ for all $q \in U$ and $(\xi^1, \dots, \xi^r) \in \mathbb{R}^r$.
- For each $q \in U$, the restriction of ϕ to E_q is a vector space isomorphism from E_q onto $\{q\} \times \mathbb{R}^r$.

Vector Bundles

Remarks

- We sometimes write a vector bundle as $E \xrightarrow{\pi} M$.
- We may also think of a vector bundle as a triple (E, M, π). In this picture E is called the *total space*, M is called the *base space*, and π is called the *projection*.

Remark

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle and S a regular submanifold in M. Then $\pi^{-1}(S) \xrightarrow{\pi} S$ is a smooth vector bundle over S denoted $E_{|S}$ and called the *restriction of E to S*.

Example

- A *trivial vector bundle* is of the form $E = M \times \mathbb{R}^r$.
- In this case the projection $\pi : M \times \mathbb{R}^r \to M$ is just the projection onto the first factor.

Example

- The tangent bundle TM is a vector bundle of rank n.
- If $(U, x^1, \ldots, x^n)$ is a chart, then a trivialization of TM over U is the map $\psi : TU \to U \times \mathbb{R}^n$ given by

$$\psi\left(\sum v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = (p, v^1, \ldots, v^n), \qquad p \in U,\ v^i \in \mathbb{R}.$$

In particular, $(\phi \times \mathbb{1}_{\mathbb{R}^n}) \circ \psi = \tilde{\phi}$.

Remark

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle. Suppose that $(U, \psi) = (U, x^1, \ldots, x^n)$ is a chart for M and we have a local trivialization,

$$\phi : E_{|U} \longrightarrow U \times \mathbb{R}^r, \qquad \phi(\xi) = \left(\pi(\xi), c^1(\xi), \ldots, c^r(\xi)\right).$$

Then $(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi : E_{|U} \rightarrow \psi(U) \times \mathbb{R}^r$ is a diffeomorphism, and we have

$$\begin{aligned}(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi &= (\psi \times \mathbb{1}_{\mathbb{R}^r})(\pi, c^1, \ldots, c^r) \\ &= (x^1 \circ \pi, \ldots, x^n \circ \pi, c^1, \ldots, c^r).\end{aligned}$$

In particular, $(\pi^{-1}(U), (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi)$ is a chart for E. We call $x^1, \ldots, x^n$ the *base coordinates* and $c^1, \ldots, c^n$ the *fiber coordinates*

Definition (Bundle Maps)

Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be smooth vector bundles. A *bundle map* from E to F is given by a pair of smooth maps $(f, \tilde{f})$, $f : M \to N$, $\tilde{f} : E \to F$ such that:

(i) $\pi_F \circ \tilde{f} = f \circ \pi_E$, i.e., we have a commutative diagram,

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow{f} & N'. \end{array}$$

(ii) For every $p \in M$, the map $\tilde{f}$ restricts to a linear map $\tilde{f} : E_p \to F_{f(p)}$.

Vector Bundles

Example

Any smooth map $f : M \to N$ gives rise to a bundle map $(f, \tilde{f})$ from TM to TN with $\tilde{f} = f_*$. Namely,

$$\tilde{f}(v) = f_{*,p}(v) \qquad p \in M,\ v \in T_pM.$$

Remarks

- The smooth vector bundles define a category where the objects are smooth vector bundles and the morphisms are bundle maps.
- From this point of view, the tangent bundle construction defines a functor from the category of smooth manifolds to the category of smooth vector bundles.

Vector Bundles

Remark

- We may also consider the category of vector bundles over a fixed manifold M.
- In this case the morphisms are bundle maps $(f, \tilde{f})$ with $f = \mathbb{1}_M$.

Smooth Sections

Definition (Section of a Vector Bundle)

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle.

- A *section* of E is any map $s : M \rightarrow E$ such that $\pi \circ s = \mathbb{1}_M$, i.e., $s(p) \in E_p$ for all $p \in M$,
- A *smooth section* is a section which is smooth as a map from M to E.

Remarks

- The set of smooth sections of E is denoted $\Gamma(E)$ or $\Gamma(M, E)$.
- If U is an open subset of M, we denote by $\Gamma(U, E)$ the set of smooth sections of $E_{|U}$.
- Sections of $E_{|U}$ are called *local sections*, whereas sections defined on the entire manifold M are called *global sections*.

Smooth Sections

Definition (Vector Field)

- A *vector field* is a section of the tangent bundle TM.
- A *smooth vector field* is a smooth section of TM.

Remark

In other words, a vector field $X : M \to TM$ assigns to each $p \in M$ a tangent vector $X_p \in T_pM$.

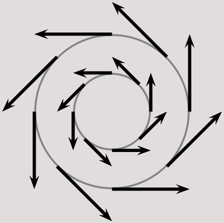
Smooth Sections

Example

On $\mathbb{R}^2$

$$X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \langle -y, x \rangle$$

is a smooth vector field on $\mathbb{R}^2$.



Smooth Sections

Proposition (Proposition 12.9)

Let E be a vector bundle over M. Then its set of smooth sections $\Gamma(E)$ is a module over the ring $C^\infty(M)$ with respect to the addition and scalar multiplication given by

$$(s_1 + s_2)(p) = s_1(p) + s_2(p), \quad s_i \in \Gamma(E), \quad p \in M,$$
$$(fs)(p) = f(p)s(p), \quad f \in C^\infty(M), \; s \in \Gamma(E), \; p \in M.$$

Remarks

- Here $s_1(p) + s_2(p)$ and $f(p)s(p)$ make sense as elements of the fiber E_p, since E_p is a vector space.
- If U is an open set, then $\Gamma(U, E)$ is a module over $C^\infty(U)$.

Smooth Frames

Definition (Frames of Vector Bundles)

Let E be a smooth vector bundle of rank r over M.

- A *frame* of E over an open $U \subset M$ is given by sections $s_1, \ldots, s_r$ such that $\{s_1(p), \ldots, s_r(p)\}$ is a basis of the fiber E_p for every $p \in U$.
- We say that the frame $\{s_1, \ldots, s_r\}$ is *smooth* when the sections $s_1, \ldots, s_r$ are smooth.

Remarks

- A frame of the tangent bundle is called a *tangent frame*, or simply a *frame*.
- For instance, $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is a smooth tangent frame over $\mathbb{R}^2$.

Smooth Frames

Example

Let $e_1, \ldots, e_r$ be the canonical basis of $\mathbb{R}^r$. For $i = 1, \ldots, r$, define $\tilde{e}_i : M \to M \times \mathbb{R}^r$ by

$$\tilde{e}_i(p) = (p, e_i), \qquad p \in M.$$

- Each map $\tilde{e}_i$ is a smooth section of the trivial bundle $M \times \mathbb{R}^r$.
- If $p \in M$, then $\{\tilde{e}_1(p), \ldots, \tilde{e}_r(p)\}$ is a basis of $\{p\} \times \mathbb{R}^r$.

Therefore, $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ is a smooth frame of $M \times \mathbb{R}^r$ over M.

Smooth Frames

Example (Frame of a trivialization)

Suppose E is a smooth vector bundle of rank r over M. Let $\phi : E_{|U} \rightarrow U \times \mathbb{R}^r$ be a trivialization over an open $U \subset M$.

- From the previous example $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ is a smooth frame of $U \times \mathbb{R}^r$ over U.
- As ϕ is smooth, $t_i = \phi^{-1} \circ \tilde{e}_i$ is a smooth map from U to $E_{|U}$.
- If $p \in U$, then $t_i(p) = \phi(\tilde{e}_i(p)) = \phi(p, e_i) \in E_p$, so t_i is a smooth section of E.
- The trivialization ϕ induces a linear isomorphism from E_p to $\{p\} \times \mathbb{R}^r$. It pullbacks the basis $\{\tilde{e}_i(p), \ldots, \tilde{e}_r(p)\}$ of $\{p\} \times \mathbb{R}^r$ to $\{t_1(p), \ldots, t_r(p)\}$, so the latter is a basis of E_p.
- Therefore, $\{t_1, \ldots, t_r\}$ is a smooth frame of E over U. It is called the *frame of the trivialization* (U, ϕ).

Smooth Frames

Facts

Let s be a section of E over U. If $p \in U$, then $s(p) \in E_p$ and $\{t_1(p), \ldots, t_r(p)\}$ is a basis of E_p. Thus, we may write

$$s(p) = \sum b^i(p) t_i(p), \qquad b^i(p) \in \mathbb{R}.$$

- If the coefficients $b_i(p)$ depends smoothly on p, then s is smooth.
- Conversely, suppose that s is a smooth section.
  - This implies that $\phi \circ s : U \to U \times \mathbb{R}^r$ is a smooth map.
  - If $p \in U$, then $\phi \circ s(p) = \phi\left[\sum b^i(p) t_i(p)\right] = \sum b^i(p)\phi[t_i(p)]$.
  - As $\phi[t_i(p)] = \phi[\phi^{-1}(\tilde{e}_i(p))] = \tilde{e}_i(p) = (p, e_i)$, we get

    $$\phi \circ s(p) = \sum b^i(p)(p, e_i) = (p, b^1(p), \ldots, b^r(p)).$$
  - As $\phi \circ s$ is a smooth map, the components $b^1(p), \ldots, b^r(p)$ must be smooth functions.

Smooth Frames

From the previous slide we obtain:

Lemma (Lemma 12.11)

Let $\phi : E_{|U} \to U \times \mathbb{R}^r$ be a trivialization of E over an open $U \subset M$ with frame $\{t_1, \ldots, t_n\}$. A section $s = \sum b^i t_i$ of E over U is smooth if and only if $b^1, \ldots, b^r$ are smooth functions.

More generally, we have:

Proposition (Proposition 12.12; see Tu's book)

Let $\{s_1, \ldots, s_r\}$ be a smooth frame of E over an open $U \subset M$. A section $s = \sum c^i s_i$ of E over U is smooth if and only if $c^1, \ldots, c^r$ are smooth functions.

Corollary

If $\{s_1, \ldots, s_r\}$ is a smooth frame of E over an open $U \subset M$, then this is a $C^\infty(U)$-basis of the $C^\infty(U)$-module $\Gamma(U, E)$.

Remark

Let $\{s_1, \ldots, s_r\}$ be a smooth frame of E over an open $U \subset M$. Define $\sigma : U \times \mathbb{R}^n \to E_{|U}$ by

$$\sigma(p, \xi^1, \ldots, \xi^r) = \sum \xi^i s_i(p), \qquad p \in U, \ \xi^i \in \mathbb{R}.$$

- The map σ is a smooth bijection that induces a linear isomorphism from $\{p\} \times \mathbb{R}^r$ onto E_p.
- It can be shown that the inverse map $\phi = \sigma^{-1} : E_{|U} \to U \times \mathbb{R}^r$ is smooth, and so this is a trivialization of E over U.
- The frame of (ϕ, U) is $\{s_1, \ldots, s_r\}$, since

$$\phi^{-1}(\tilde{e}_i(p)) = \sigma(p, e_i) = s_i(p).$$

It follows that we have a one-to-one correspondance between trivializations and smooth frames.

Smooth Frames

Example

Let $(U, x^1, \ldots, x^n)$ be a local chart for M.

- We know that $(U, x^1, \ldots, x^n)$ gives rise to the trivialization $\psi : TU \to U \times \mathbb{R}^n$ given by
$$\psi(v) = \left(p, v^1, \ldots, v^n\right) \quad \text{if } v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p \in T_pM, \ p \in U.$$
- In particular, as $\psi(\frac{\partial}{\partial x^i}\big|_p) = (p, e_i) = \tilde{e}_i(p)$, we have
$$t_i(p) = \psi^{-1}\big(\tilde{e}_i(p)\big) = \frac{\partial}{\partial x^i}\bigg|_p.$$
Thus, $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ is the frame of the trivialization (U, ψ). In particular, this is a smooth tangent frame over U.