Differentiable Manifolds
§11. The Rank of a Smooth Map

Sichuan University, Fall 2022

Constant Rank Theorem

Reminder

Let N be a manifold of dimension n and M a manifold of dimension m.

- The rank at $p \in N$ of a smooth map $f : N \to M$ is the rank of its differential $f_{*,p} : T_pN \to T_{f(p)}M$.
- The rank is always $\leq \min(m, n)$.

Constant Rank Theorem

Theorem (Constant Rank Theorem; Theorem B.4)

Let $f : U \to \mathbb{R}^m$ be a C^∞ map, where $U \subset \mathbb{R}^n$ is open. Assume that f has constant rank k near $p \in U$. Then there are:

- *A diffeomorphism F from a neighborhood of p onto a neighborhood of $0 \in \mathbb{R}^n$ with $F(p) = 0$,*
- *A diffeomorphism G from a neighborhood of $f(p)$ onto a neighborhood of $0 \in \mathbb{R}^m$ with $G(f(p)) = 0$,*

in such a way that

$$G \circ f \circ F^{-1}(r^1, \ldots, r^n) = (r^1, \ldots, r^k, 0, \ldots, 0).$$

Remark

If $k = m$, then

$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \ldots, r^n) = (r^1, \ldots, r^m).$$

Constant Rank Theorem

Theorem (Constant Rank Theorem for Manifolds; Theorem 11.1)

Suppose that M *is a manifold of dimension* m *and* N *is a manifold of dimension* n. *Let* $f : N \to M$ *be a smooth map that has constant rank* k *near a point* $p \in N$. *Then, there are a chart* (U, ϕ) *centered at* p *in* N *and a chart* (V, ψ) *centered at* $f(p)$ *in* M *such that, for all* $(r^1, \dots, r^n) \in \phi(U)$, *we have*

$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Constant Rank Theorem

Proof of Theorem 11.1.

- Let $(\overline{U}, \overline{\phi})$ and $(\overline{V}, \overline{\psi})$ be charts around p and $q = f(p)$.
- Apply the constant rank theorem to $\overline{\psi} \circ f \circ \overline{\phi}^{-1}$ to get
  - A diffeomorphism $F : W \to W'$ from a neighborhood W of $\overline{\phi}(p)$ onto a neighborhood of $0 \in \mathbb{R}^n$ with $F(\overline{\phi}(p)) = 0$,
  - A diffeomorphism $G : X \to X'$ from a neighborhood X of $\overline{\psi}(q)$ onto a neighborhood of $0 \in \mathbb{R}^m$ with $G(\overline{\psi}(q)) = 0$,

  in such a way that

$$
\begin{aligned}
(G \circ \overline{\psi}) \circ f \circ (F \circ \overline{\phi})^{-1}(r^1, \ldots, r^n) &= G \circ \left(\overline{\psi} \circ f \circ \overline{\phi}^{-1} \right) \circ F^{-1}(r^1, \ldots, r^n) \\
&= (r^1, \ldots, r^k, 0, \ldots, 0).
\end{aligned}
$$

- Set $U = \overline{\phi}^{-1}(W)$ and $V = \overline{\psi}^{-1}(X)$. Then $(U, F \circ \overline{\phi})$ and $(V, G \circ \overline{\psi})$ are charts centered p and q with the required property. $\square$

Constant Rank Theorem

Remark

Suppose that $(U, \phi) = (U, x^1, \dots, x^n)$ is a chart centered at p and $(V, \psi) = (V, y^1, \dots, y^m)$ is a chart centered at $f(p)$ such that

$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

- For any $q \in U$, we have $\phi(q) = \left(x^1(q), \dots, x^n(q)\right)$ and $\psi(f(q)) = \left(y^1 \circ f(q), \dots, y^m \circ f(q)\right)$.
- Thus,

$$\begin{aligned}
\left(y^1 \circ f(q), \dots, y^m \circ f(q)\right) = \psi\big(f(q)\big) &= \left(\psi \circ f \circ \phi^{-1}\right)\big(\phi(q)\big) \\
&= \left(\psi \circ f \circ \phi^{-1}\right)\left(x^1(q), \dots, x^n(q)\right) \\
&= \left(x^1(q), \dots, x^k(q), 0, \dots, 0\right).
\end{aligned}$$

- Therefore, relative to the local coordinates $(x^1, \dots, x^n)$ and $(y^1, \dots, y^m)$ the map f is such that

$$(x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^k, 0, \dots, 0).$$

Constant Rank Theorem

A consequence of the constant rank theorem is the following extension of the regular level set theorem (Theorem 9.9).

Theorem (Constant-Rank Level Set Theorem; Theorem 11.2)

Let $f : N \to M$ be a smooth map and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N, then $f^{-1}(c)$ is a regular submanifold of codimension k.

Remark

A neighborhood of a subset $A \subset N$ is an open set containing A.

Constant Rank Theorem

Example (Orthogonal group $O(n)$; Example 11.3)

The *orthogonal group* $O(n)$ is the subgroup of $GL(n, \mathbb{R})$ of matrices A such that $A^T A = I_n$ (identity matrix),

- This is the level set $f^{-1}(I_n)$, where $f : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $A \to A^T A$.
- It can be shown that f has constant rank (in fact it has rank $k = \frac{1}{2}n(n+1)$).
- Therefore, by the constant-rank level set theorem $O(n)$ is a regular submanifold of $GL(n, \mathbb{R})$ (of codimension $\frac{1}{2}n(n+1)$).

The Immersion and Submersion Theorems

Reminder

Suppose that M is a manifold of dimension m and N is a manifold of dimension n, and let $f : N \to M$ be a smooth map.

- f is an *immersion at p* if $f_{*,p} : T_pN \to T_{f(p)}M$ is injective.
- f is a *submersion at p* if $f_{*,p} : T_pN \to T_{f(p)}M$ is surjective.

Remark

Equivalently,

$$f \text{ is an immersion at } p \Longleftrightarrow \left(n \leq m \text{ and } \operatorname{rk} f_{*,p} = n\right),$$
$$f \text{ is a submersion at } p \Longleftrightarrow \left(n \geq m \text{ and } \operatorname{rk} f_{*,p} = m\right).$$

As we always have $\operatorname{rk} f_{*,p} \leq \min(m, n)$, we see that

$$f \text{ is an immersion/submersion at } p \Longleftrightarrow f_{*,p} \text{ has maximal rank.}$$

Facts

Set $k = \min(m, n)$, and denote by $\mathbb{R}^{m\times n}_{\max}$ the set of $m \times n$ matrices $A \in \mathbb{R}^{m\times n}$ of maximal rank.

- An $m \times n$-matrix has maximal rank if and only if it has a non-zero $k \times k$-minor.
- The minors are polynomials in the coefficients of matrices, and hence are continuous functions.
- Thus, if a matrix A has a non-zero $k \times k$-minor, then this minor is non-zero for any matrix that is sufficiently close to A, and so those matrices have maximal rank.
- It follows that $\mathbb{R}^{m\times n}_{\max}$ is a neighbourhood of each of its elements, and hence is an open set in $\mathbb{R}^{m\times n}$.

The Immersion and Submersion Theorems

Facts

Suppose that $f : N \to M$ is a smooth map. Let $(U, x^1, \ldots, x^n)$ be chart about p in M and $(V, y^1, \ldots, y^m)$ a chart about $f(p)$ in M. Set $U_{\max} = \{q \in U;\ f_{*,q} \text{ has maximal rank}\}$.

- If $q \in U$, then $f_{*,q} : T_q M \to T_{f(q)}$ is represented by the matrix $J(q) := \left[\partial f^i / \partial x^j(q)\right]$, with $f^i = y^i \circ f$, and hence $\operatorname{rk} f_{*,q} = \operatorname{rk} J(q)$. Thus,

$$U_{\max} = \{q \in U;\ J(q) \in \mathbb{R}^{m\times n}_{\max}\} = J^{-1}\left(\mathbb{R}^{m\times n}_{\max}\right).$$

- It can be shown that $q \to J(F)(q)$ is C^∞, and hence is continuous.
- As $\mathbb{R}^{m\times n}_{\max}$ is open, it follows that $U_{\max}$ is open as well.
- In particular, if f_* has maximal rank at p, then it has maximal rank near p.

As a consequence we obtain:

Proposition (Proposition 11.4)

If a smooth map $f : N \to M$ is a immersion (resp., a submersion) at a point $p \in N$, then it is an immersion (resp., submersion) near p. In particular, it has constant rank near p.

The Immersion and Submersion Theorems

Combining the previous proposition with the Constant Rank Theorem gives the following result.

Theorem (Theorem 11.5)

Let $f : N \to M$ be a smooth map.

1. **Immersion Theorem**. *If f is an immersion at p, then there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that near $\phi(p)$ we have*
$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \ldots, r^n) = (r^1, \ldots, r^n, 0, \ldots, 0).$$
2. **Submersion Theorem**. *If f is a submersion at p, then there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that near $\phi(p)$ we have*
$$\left(\psi \circ f \circ \phi^{-1}\right)(r^1, \ldots, r^m, r^{m+1}, \ldots, r^n) = (r^1, \ldots, r^m).$$

Remark

- The immersion theorem implies that if $f : N \to M$ is an immersion then, for every $p \in N$, there are a chart $(U, x^1, \ldots, x^n)$ centered at p in N and a chart $(V, y^1, \ldots, y^m)$ centered at $f(p)$ in M relative to which f is such that
$$(*) \qquad (x^1, \ldots, x^n) \longrightarrow (x^1, \ldots, x^n, 0, \ldots, 0).$$
- Conversely, If f satisfies $(*)$, then, setting $f^i = y^i \circ f$, we have
$$\left[\partial f^i / \partial x^j\right] = \begin{bmatrix} \partial x^i / \partial x^j \\ 0_{m-n} \end{bmatrix} = \begin{bmatrix} I_n \\ 0_{m-n} \end{bmatrix}.$$
In particular, $\left[\partial f^i / \partial x^j\right]$ has maximal rank, which implies that f is an immersion near p.

The Immersion and Submersion Theorems

Remark

- The submersion theorem implies that if $f : N \to M$ is a submersion then, for every $p \in N$, there are a chart $(U, x^1, \ldots, x^n)$ centered at p in N and a chart $(V, y^1, \ldots, y^m)$ centered at $f(p)$ in M relative to which f is such that
$$(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) \longrightarrow (x^1, \ldots, x^m).$$
- The projection $(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) \to (x^1, \ldots, x^m)$ is an open map (see Problem A.7). This implies that f maps any neighborhood of p onto a neighborhood of $f(p)$.
- As this is true for every $p \in N$, we see that f is an open map. Therefore, we obtain:

Corollary (Corollary 11.6)

Every submersion $f : N \to M$ is an open map.

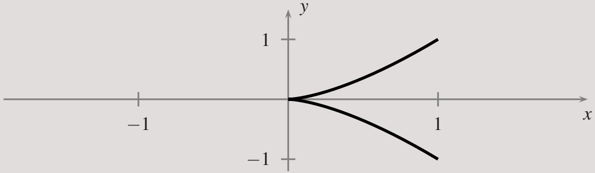
Images of Smooth Maps

Let us look at some examples of smooth maps $f : \mathbb{R} \to \mathbb{R}^2$.

Example (Example 11.7)

Let $f(t) = (t^2, t^3)$.

- This is a one-to-one map, since $t \to t^3$ is one-to-one.
- As $f'(0) = (0, 0)$ the differential $f_{*,0}$ is zero, and so f is not an immersion at 0.
- The image of f is the cuspidal cubic $y^2 = x^3$.



Images of Smooth Maps

Example (Example 11.8)

Let $f(t) = (t^2 - 1, t^3 - t)$.

- As $f'(t) = (2t, 3t^2 - 1) \neq (0, 0)$ the differential f_* is one-to-one everywhere, and hence f is an immersion.
- However, f is not one-one since $f(1) = f(-1) = (0, 0)$.
- The image of f is the nodal cubic $y^2 = x^2(x + 1)$ (see Tu's book).

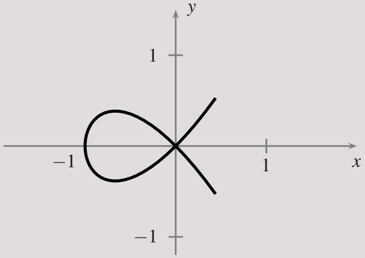
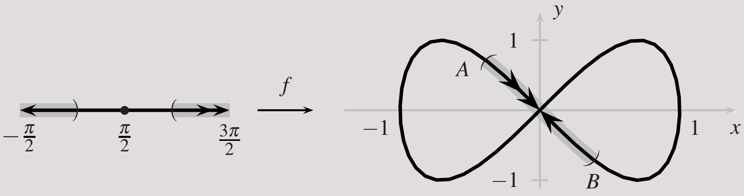


Image of Smooth Maps

Example (The Figure-eight; Example 11.12)

Set $I = (-\pi/2, 3\pi/2)$, and let $f : I \to \mathbb{R}^2$, $t \to (\cos t, \sin 2t)$.



- $f'(t) = (-\sin t, 2\cos 2t) \neq (0,0)$, and so f is an immersion.
- f is one-to-one, and so f is a bijection onto its image $f(I)$.
- The inverse map $f^{-1} : f(I) \to I$ is not continuous: if $t \to (3\pi/2)^-$, then $f(t) \to (0,0) = f(\pi/2)$, but
$$f^{-1}(f(t)) = t \to 3\pi/2 \notin I.$$
In particular, $f : I \to f(I)$ is not a homeomorphism.

Image of Smooth Maps

Summary

As the previous examples show:

- A one-to-one smooth map need not be an immersion.
- An immersion need not be one-to-one.
- A one-to-one immersion need not be a homeomorphism onto its image.

Definition

A smooth map $f : N \to M$ is called an *embedding* if f is an immersion and a homeomorphism onto its image $f(N)$ with respect to the subspace topology.

Remark

A one-to-one immersion $f : N \to M$ is an embedding if and only if it is an open map.

Image of Smooth Maps

The importance of embeddings stems from the following result.

Theorem (Theorem 11.13)

If $f : N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold in M.

Proof of Theorem 11.13.

- As f is an immersion, by the immersion theorem, for any $p \in N$, there are a chart $(U, x^1, \ldots, x^n)$ centered at p in N and a chart $(V, y^1, \ldots, y^m)$ centered at $f(p)$ relative to which f is such that $(x^1, \ldots, x^n) \longrightarrow (x^1, \ldots, x^n, 0, \ldots, 0)$. Thus,

$$f(U) = \{q \in V; y^{n+1}(q) = \cdots = y^m(q) = 0\}$$

- As $f : N \to f(N)$ is a homeomorphism, $f(U)$ is an open set in $f(N)$ with respect to the subspace topology. That is, there is an open V' in M such that $f(U) = V' \cap f(N)$.
- Thus,

  $$V \cap V' \cap f(N) = V \cap f(U) = f(U) = \{y^{n+1} = \cdots = y^m = 0\}.$$

  That is, $(V \cap V', y^1, \ldots, y^m)$ is an adapted chart relative to $f(N)$ near $f(p)$ in M.
- This shows that $f(N)$ is a regular submanifold.

$\square$

We have the following converse of the previous theorem.

Theorem (Theorem 11.14)

If N is a regular submanifold in M, then the inclusion $i : N \to M$ is an embedding.

Proof of Theorem 11.14.

Let N be a regular submanifold in M.

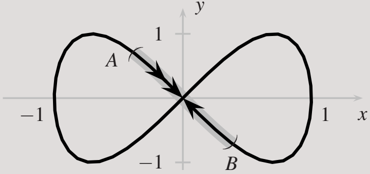
- As N has the subspace topology, the inclusion $i: N \to M$ is a homeomorphism onto its image.
- As N is a regular submanifold, near every $p \in N$, there is an adapted chart $(U, x^1, \ldots, x^m)$ near p in M such that $(U \cap N, x^1, \ldots, x^n)$ is a chart in N near p and $U \cap N = \{x^{n+1} = \cdots = x^m = 0\}$.
- Therefore, relative to the charts $(U \cap N, x^1, \ldots, x^n)$ and $(U, x^1, \ldots, x^m)$ the inclusion $i: N \to M$ is such that
$$(x^1, \ldots, x^n) \longrightarrow (x^1, \ldots, x^n, 0, \ldots, 0).$$
- By a previous remark, it follows that the map $i: N \to M$ is an immersion near p. $\square$

Remarks

1. The images of smooth embeddings are called *embedded submanifolds*.
2. The previous two results show that the regular submanifolds and embedded submanifolds are the same objects.
3. The images of one-to-one immersions are called *immersed submanifolds*.

Example

The figure-eight is an immersed submanifold in $\mathbb{R}^2$ (but this is not a regular submanifold).



Smooth Maps into a Submanifold

Question

Suppose that $f : N \to M$ is smooth map such that $f(N)$ is contained in a given subset $S \subset M$. If S is manifold, then is the induced map $f : N \to S$ smooth as well?

Theorem (Theorem 11.15)

Suppose that $f : N \to M$ is a smooth map whose image is contained in a regular submanifold S in M. Then the induced map $f : N \to S$ is smooth.

Remarks

1. The above result does not hold if S is only an immersed submanifold (see Tu's book).
2. The converse holds. As S is a regular submanifold, the inclusion $i : S \to M$ is smooth. Thus, if $f : N \to S$ is a smooth map, then $i \circ f : N \to M$ is a C^∞ map that induces f.

Smooth Maps into a Submanifold

Proof of Theorem 11.15.

Set $m = \dim M$ and $s = \dim S$, and let $p \in N$.

- As S is a regular submanifold and $f(p) \in S$, there is an adapted chart $(V, \psi) = (V, y^1, \ldots, y^m)$ near $f(p)$ in M. Then $(V \cap S, \psi_S) = (V \cap S, y^1, \ldots, y^s)$ is a chart near $f(p)$ in S.
- As f is a C^∞-map, the functions $y^i \circ f$ are C^∞ on $U := f^{-1}(V)$ (which is an open neighbourhood of p in N since f is continuous).
- On $f^{-1}(V)$ we have $\psi_S \circ f = (y^1 \circ f, \ldots, y^s \circ f)$, and so $\psi_S \circ f : f^{-1}(V) \to \mathbb{R}^s$ is a smooth map.
- As $(V \cap S, \psi_S)$ is chart for S, it follows from Proposition 6.15 that the induced map $f : f^{-1}(V) \to S$ is smooth, and hence is smooth near p. $\square$

Smooth Maps into a Submanifold

Example (Multiplication map of $\mathrm{SL}(n,\mathbb{R})$; Example 11.16)

$\mathrm{SL}(n,\mathbb{R})$ is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ of matrices of determinant 1.

- This is a regular submanifold in $\mathrm{GL}(n,\mathbb{R})$ (Example 9.11), and so the inclusion $\iota : \mathrm{SL}(n,\mathbb{R}) \hookrightarrow \mathrm{GL}(n,\mathbb{R})$ is a smooth map.
- By Example 6.21 we have a smooth multiplication map,
$$\mu : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R}).$$
- We thus get a smooth map,
$$\mu \circ (\iota \times \iota) : \mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R}).$$
- As it takes values in $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold in $\mathrm{GL}(n,\mathbb{R})$, we get a smooth multiplication map,
$$\mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \longrightarrow \mathrm{SL}(n,\mathbb{R}).$$

Smooth Maps into a Submanifold

Theorem 11.5 and its converse are especially useful when $M = \mathbb{R}^m$. In this case we have:

Corollary

Let S be a regular submanifold in $\mathbb{R}^m$ and $f : N \to \mathbb{R}^m$ a map such that $f(N) \subset S$. Set $f = (f^1, \dots, f^m)$. Then TFAE:

(i) *f is smooth as a map from N to S.*

(ii) *f is smooth as a map from N to $\mathbb{R}^m$.*

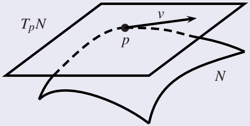
(iii) *The components $f^1, \dots, f^m$ are smooth functions on N.*

The Tangent Space to a Submanifold in $\mathbb{R}^m$

Facts

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function with no critical points on its zero set $N = f^{-1}(0)$.

- By the regular level set theorem N is a regular submanifold in $\mathbb{R}^{n+1}$ of dimension n.
- Then the inclusion $i : N \to \mathbb{R}^{n+1}$ is an embedding, and so, for every $p \in N$, the differential $i_* : T_pN \to T_p\mathbb{R}^{n+1}$ is injective.
- We thus can identify the tangent space T_pN with a subspace of $T_p\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$. More precisely, we regard it as a subspace of $\mathbb{R}^{n+1}$ through p.
- Thus, any $v \in T_pN$, is identified with a vector $\langle v^1, \ldots v^{n+1} \rangle$, which is then identified with the point $x = p + (v^1, \ldots, v^{n+1})$.



Facts

- Set $p = (p^1, \ldots, p^{n+1})$ and $x = (x^1, \ldots, x^{n+1})$. Let $c : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ be a smooth curve such that $c(0) = p$, $c'(0) = v$, and $c(t) \in N$, i.e., $f(c(t)) = 0$. Then
$$0 = \frac{d}{dt}\bigg|_0 f(c(t)) = \sum (c^i)'(0)\frac{\partial f}{\partial x^i}(c(0)) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$
- As $v^i = x^i - p^i$, we see that $(x^1, \ldots, x^{n+1})$ satisfies,
$$(*) \qquad \sum \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0.$$
- As p is a regular point, $\frac{\partial f}{\partial x^i}(p) \neq 0$ for some i, and so the solution set of $(*)$ has dimension n.
- As $\dim N = n$, the tangent space T_pN has dimension n, and so it is identified with the full solution set of $(*)$.

Therefore, we obtain the following result:

Proposition

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function with no critical points on its zero set $N = f^{-1}(0)$. If $p = (p^1, \ldots, p^{n+1})$ is a point in N, then the tangent space T_pN is defined by the equation,

$$(*) \qquad \sum \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0.$$

Remark

Equivalently, T_pN is identified with the hyperplane through p that is normal to the gradient vector $\langle \partial f/\partial x^1(p), \ldots, \partial f/\partial x^{n+1}(p) \rangle$.

Example (Tangent plane to a sphere)

The sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is the zero set of

$$f(x, y, z) = x^2 + y^2 + z^2 - 1.$$

- We have
$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z.$$
- Thus, at $p = (a, b, c) \in \mathbb{S}^2$ the tangent plane has equation,

$$\begin{aligned}
\frac{\partial f}{\partial x}(p)(x-a) + \frac{\partial f}{\partial y}(p)(y-b) + \frac{\partial f}{\partial z}(p)(z-c) &= 0, \\
\Longleftrightarrow a(x-a) + b(y-b) + c(z-c) &= 0, \\
\Longleftrightarrow ax + by + cz &= a^2 + b^2 + c^2, \\
\Longleftrightarrow ax + by + cz &= 1.
\end{aligned}$$