Noncommutative Geometry
Chapter 7:
Cwikel Estimates. Semiclassical Analysis and Connes' Integration Formula

Sichuan University, Fall 2022

Singular Values and Schatten Classes

Additional References

- Ponge, R.: *Connes' integration and Weyl's laws*. Preprint, arXiv, July 2021. To appear in J. Noncomm. Geom..
- Ponge, R.: *Weyl's laws and Connes' integration formulas for matrix-valued LlogL-Orlicz potentials*. Math. Phys. Anal. Geom. **25**, 10 (2022), 33 pages.

Setup

Throughout this chapter $\mathcal{H}$ is a separable Hilbert space.

Reminder: SC Weyl's Laws and Integration Formula

Setup

- $(M^n, g) =$ compact Riemannien manifold.
- $\Delta_g : C^\infty(M) \to C^\infty(M)$ is the Laplacian.
- $\nu_g(x) := \sqrt{\det(g(x))} d^n x$ is the Riemannian measure.

Remark

In what follows $c(n) = (2\pi)^{-n}|\mathbb{B}^n|$.

Proposition (Birman-Solomyak '70s)

If $q > 0$ *and* $f \in C^\infty(M, \mathbb{R})$, *then*

$$\lim_{j\to\infty} j^{\frac{2q}{n}} \lambda_j^{\pm}\left(\Delta_g^{-\frac{q}{2}} f \Delta_g^{-\frac{q}{2}}\right) = \left(c(n) \int_M f_\pm(x)^{\frac{n}{2q}} d\nu_g(x)\right)^{\frac{2q}{n}},$$

where $f_\pm(x) = \max(0, \pm f(x))$ *are the positive/negative parts of* f.

Reminder: SC Weyl's Laws and Integration Formula

Corollary (Semiclassical Weyl's Law)

If $q > 0$ *and* $V \in C^\infty(M, \mathbb{R})$, *then*

$$\lim_{h \to 0^+} h^n N^- \left(h^{2q} \Delta_g^q + V \right) = c(n) \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x).$$

Proof's Idea.

By the Birman-Schwinger principle:

$$\lim_{h \to 0^+} h^n N^- \left(h^{2q} \Delta_g^q + V \right) = \left(\lim_{j \to \infty} j^{\frac{2q}{n}} \lambda_j^- \left(\Delta_g^{-\frac{q}{2}} V \Delta_g^{-\frac{q}{2}} \right) \right)^{\frac{n}{2q}}$$

$$= c(n) \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x).$$

□

Reminder: SC Weyl's Laws and Integration Formula

Corollary (Connes' Integration Formula)

For every $f \in C^\infty(M)$, the operator $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ is strongly measurable, and we have

$$\fint \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M f(x) d\nu_g(x).$$

Proof.

- By linearity it is enough to prove the result for $f \geq 0$.
- If $f \geq 0$, then Birman-Solomyak's result for $q = n$ delivers

$$\lim_{j \to \infty} j\lambda_j \left(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} \right) = c(n) \int_M f(x) d\nu_g(x).$$

- This gives the result.

□

Extension to L^p-Potentials

Questions

1. Do the SC Weyl's laws still hold for L^p-potentials?
2. Does Connes' integration formula continues to hold for L^1-functions?

Remark

To solve these questions it is enough to extend Birman-Solomyak's asymptotics to L^p-functions.

Extension to L^p-Potentials

Theorem (Birman-Solomyak, Cwikel, Rozenblum '70s)

We have the semiclassical Weyl's law,

$$\lim_{h\to 0^+} h^n N^-\left(h^2\Delta_g + V\right) = c(n)\int_M V_-(x)^{\frac{n}{2}} d\nu_g(x),$$

provided that:

- $n \geq 3$ *and* $V \in L_g^{n/2}(M)$, *or*
- $n = 2$ *and* $V \in L_g^r(M)$, $r > 1$, *or*
- $n = 1$ *and* $V \in L_g^1(M)$.

Remark

For $n = 2$ the above SC Weyl's law need not hold for $V \in L_g^1(M)$.

Extension to L^p-Potentials

Theorem (Kalton-Lord-Potapov-Sukochev '13)

1. *If* $f \in L^r_g(M)$, $r > 1$, *then Connes' integration formula holds,*

$$\fint \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M f(x) d\nu_g(x).$$

2. *The formula need not hold for* $f \in L^1_g(M)$.

Reminder: Birman-Solomyak Perturbation Theory

Proposition (Birman-Solomyak)

Assume that:

- $A_\ell \to A$ *in* $\mathcal{L}^{p,\infty}$ *with* $A_\ell^* = A_\ell$.
- $\lim_{j\to\infty} j^{1/p}\lambda_j^{\pm}(A_\ell)$ *exist for all* ℓ.

Then:

$$\lim_{j\to\infty} j^{\frac{1}{p}}\lambda_j^{\pm}(A) = \lim_{\ell\to\infty}\lim_{j\to\infty} j^{\frac{1}{p}}\lambda_j^{\pm}(A_\ell).$$

Reminder: Birman-Schwinger Principle

Setup

- $H =$ selfadjoint (unbounded) operator with non-negative spectrum.
- $V =$ selfadjoint operator such that $(1+H)^{-1/2}V(1+H)^{-1/2}$ is compact.
- $N^-(H+V) =$ number of negative eigenvalues of $H+V$.

Proposition (Birman-Schwinger Principle)

If 0 is an isolated eigenvalue of H, and $H^{-1/2}VH^{-1/2} \in \mathcal{L}^{p,\infty}$, then

$$\lim_{h\to 0^+} h^{2p} N^-\left(h^2 H + V\right) = \left(\lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^-\left(H^{-\frac{1}{2}} V H^{-\frac{1}{2}}\right)\right)^p,$$

provided any of these limits exists.

Combining the Birman-Schwinger principle with Birman-Solomyak's theory gives:

Proposition

Assume that:

- $H^{-1/2}V_\ell H^{-1/2} \to H^{-1/2}VH^{-1/2}$ *in* $\mathcal{L}^{p,\infty}$.
- $\lim_{h\to 0^+} h^{2p}N^-(h^2H + V_\ell)$ *exists for all* ℓ.

Then:

$$\lim_{h\to 0^+} h^{2p}N^-(h^2H + V) = \lim_{\ell\to\infty} \lim_{h\to 0^+} h^{2p}N^-(h^2H + V_\ell).$$

In particular, we have:

Proposition (Birman-Solomyak, Simon)

Assume that:

- $V_\ell \to V$ *in* $L_g^{n/2}(M)$ *with* $V_\ell \in C^\infty(M, \mathbb{R})$.
- $\Delta_g^{-1/2} V_\ell \Delta_g^{-1/2} \to \Delta_g^{-1/2} V \Delta_g^{-1/2}$ *in* $\mathcal{L}^{n/2,\infty}$.

Then:

$$\lim_{h \to 0^+} h^n N^-(h^2 \Delta_g + V) = c(n) \int_M |V_-(x)|^{\frac{n}{2}} d\nu_g(x).$$

Proof.

By the previous result and the SC Weyl's laws for C^∞-potentials, we have:

$$\begin{aligned}\lim_{h\to 0^+} h^n N^-(h^2\Delta_g + V) &= \lim_{\ell\to\infty}\lim_{h\to 0^+} h^n N^-(h^2\Delta_g + V_\ell) \\ &= c(n)\lim_{\ell\to\infty}\int_M |(V_\ell)_-(x)|^{\frac{n}{2}} d\nu_g(x) \\ &= c(n)\int_M |V_-(x)|^{\frac{n}{2}} d\nu_g(x).\end{aligned}$$

□

This leads to the following:

Problem

Find a Banach space $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ of functions on M such that:

(i) $\mathcal{V}$ embeds continuously in $L_g^{n/2}(M)$.

(ii) $C^\infty(M)$ is a dense subspace of $\mathcal{V}$.

(iii) The map $f \to \Delta_g^{-1/2} f \Delta_g^{-1/2}$ is continuous from $\mathcal{V}$ to $\mathcal{L}^{n/2,\infty}$, i.e., we have an estimate,

$$\left\| \Delta_g^{-\frac{1}{2}} f \Delta_g^{-\frac{1}{2}} \right\|_{\frac{n}{2},\infty} \leq C_n \|f\|_{\mathcal{V}}.$$

Remark (Birman-Solomyak, Simon)

This is merely an operator-theoretic question.

Semiclassical Perturbation Theory

Proposition (Birman-Solomyak, Simon)

If the conditions (i)–(iii) are satisfied, then, for every real-valued potential $V \in \mathcal{V}$, we have

$$\lim_{h \to 0^+} h^n N^-(h^2 \Delta_g + V) = c(n) \int_M |V_-(x)|^{\frac{n}{2}} d\nu_g(x).$$

Proof.

- By (ii) there is $V_\ell \to V$ in $\mathcal{V}$ with $V_\ell \in C^\infty(M, \mathbb{R})$.
- By (i) $V_\ell \to V$ in $L_g^{n/2}(M)$.
- By (iii) $\Delta_g^{-1/2} V_\ell \Delta_g^{-1/2} \to \Delta_g^{-1/2} V \Delta_g^{-1/2}$.
- By the previous result, we then have

$$\lim_{h \to 0^+} h^n N^-(h^2 \Delta_g + V) = c(n) \int_M |V_-(x)|^{\frac{n}{2}} d\nu_g(x).$$

□

Cwikel Estimates

Theorem (Cwikel, Birman-Solomyak)

We have the following estimates:

1. *If* $0 < q < n/2$, *then*
$$\left\|\Delta_g^{-\frac{q}{2}} f \Delta_g^{-\frac{q}{2}}\right\|_{\frac{n}{2q},\infty} \leq C_{nq} \|f\|_{L^{\frac{n}{2q}}}.$$
2. *If* $q = n/2$ *and* $r > 1$, *then*
$$\left\|\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right\|_{1,\infty} \leq C_{nr} \|f\|_{L^r}.$$
3. *If* $q > n/2$, *then*
$$\left\|\Delta_g^{-\frac{q}{2}} f \Delta_g^{-\frac{q}{2}}\right\|_{\frac{n}{2q},\infty} \leq C_{nq} \|f\|_{L^1}.$$

Cwikel Estimates

Remarks

1. The estimates for $q < n/2$ were proved by Cwikel (and conjectured by Barry Simon).
2. The estimates for $q > n/2$ were established by Birman-Solomyak.
3. The estimates for $q = n/2$ are deduced from the estimates in the previous cases by an interpolation argument (Birman-Solomyak).

Cwikel Estimates

For $q = 1$ we get:

Theorem (Cwikel, Birman-Solomyak)

We have the following estimates:

1. *If* $n \geq 3$, *then*
$$\left\|\Delta_g^{-\frac{1}{2}} f \Delta_g^{-\frac{1}{2}}\right\|_{\frac{n}{2},\infty} \leq C_n \|f\|_{L^{\frac{n}{2}}}.$$
2. *If* $n = 2$ *and* $r > 1$, *then*
$$\left\|\Delta_g^{-\frac{1}{2}} f \Delta_g^{-\frac{1}{2}}\right\|_{1,\infty} \leq C_{nr} \|f\|_{L^r}.$$
3. *If* $n = 1$, *then*
$$\left\|\Delta_g^{-\frac{1}{2}} f \Delta_g^{-\frac{1}{2}}\right\|_{\frac{1}{2},\infty} \leq C_{nq} \|f\|_{L^1}.$$

Cwikel Estimates

Corollary (Birman-Solomyak, Cwikel, Rozenblum)

We have the semiclassical Weyl's law,

$$\lim_{h\to 0^+} h^n N^- \left(h^2\Delta_g + V\right) = c(n)\int_M V_-(x)^{\frac{n}{2}} d\nu_g(x),$$

provided that:

- $n \geq 3$ *and* $V \in L_g^{n/2}(M)$, *or*
- $n = 2$ *and* $V \in L_g^r(M)$, $r > 1$, *or*
- $n = 1$ *and* $V \in L_g^1(M)$.

More generally, we have:

Corollary (Birman-Solomyak, Rozenblum)

Let $q > 0$*. We have the semiclassical Weyl's law,*

$$\lim_{h \to 0^+} h^n N^- \left(h^{2q} \Delta_g^q + V \right) = c(n) \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x),$$

provided that:

- $q < n/2$ *and* $V \in L_g^{n/2q}(M)$*, or*
- $q = n/2$ *and* $V \in L_g^r(M)$*,* $r > 1$*, or*
- $q > n/2$ *and* $V \in L_g^1(M)$*.*

Remark

The previous results hold *verbatim* for:

- Schrödinger operators $\Delta + V$ on $\mathbb{R}^n$ with $n \geq 3$.
- Schrödinger operators $\Delta + V$ on bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with Dirichlet/Neumann boundary conditions.

Connes' Integration Formula

For $q = n/2$ we have:

Theorem (Cwikel, Birman-Solomyak)

If $r > 1$, then

$$\left\|\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right\|_{1,\infty} \leq C_{nr} \|f\|_{L^r}.$$

Corollary

If $f_\ell \to f$ in $L^r_g(M)$, $r > 1$, then

$$\Delta_g^{-n/4} f_\ell \Delta_g^{-n/4} \longrightarrow \Delta_g^{-n/4} f \Delta_g^{-n/4} \quad \text{in } \mathcal{L}^{1,\infty}.$$

Corollary (Kalton-Lord-Potapov-Sukochev)

For every $f \in L^r_g(M)$, $r > 1$, the operator $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ is strongly measurable, and we have

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M f(x) d\nu_g(x).$$

Proof.

Let $f \in L_g^r(M)$, $r > 1$.

- We need to show that, for every normalized positive trace φ on $\mathcal{L}^{1,\infty}$, we have
$$\varphi\left(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right) = c(n) \int_M f(x) d\nu_g(x).$$
- Every positive trace on $\mathcal{L}^{1,\infty}$ is continuous.
- Let $f_\ell \to f$ in $L_g^r(M)$ with $f_\ell \in C^\infty(M)$.
- By the previous result $\Delta_g^{-n/4} f_\ell \Delta_g^{-n/4} \to \Delta_g^{-n/4} f \Delta_g^{-n/4}$ in $\mathcal{L}^{1,\infty}$.
- Thus,
$$\varphi\left(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right) = \lim_{\ell \to \infty} \varphi\left(\Delta_g^{-\frac{n}{4}} f_\ell \Delta_g^{-\frac{n}{4}}\right).$$

□

Proof.

- As $f_\ell \in C^\infty(M)$, by Connes' original integration formula,

$$\varphi\left(\Delta_g^{-\frac{n}{4}} f_\ell \Delta_g^{-\frac{n}{4}}\right) = \fint \Delta_g^{-\frac{n}{4}} f_\ell \Delta_g^{-\frac{n}{4}} = c(n) \int_M f_\ell(x) d\nu_g(x).$$

- Thus,

$$\begin{aligned}\varphi\left(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right) &= \lim_{\ell\to\infty} \varphi\left(\Delta_g^{-\frac{n}{4}} f_\ell \Delta_g^{-\frac{n}{4}}\right)\\ &= c(n) \lim_{\ell\to\infty} \int_M f_\ell(x) d\nu_g(x)\\ &= c(n) \int_M f(x) d\nu_g(x).\end{aligned}$$

This completes the proof. □

Connes' Integration Formula for L^r-Functions

Remarks

1. The original proof of Kalton-Lord-Potapov-Sukochev did not use Cwikel estimates.
2. Recently, versions of Cwikel estimates were also obtained for NC Euclidean spaces (Levitina-Sukochev-Zanin), NC tori (McDonald-RP), and nilpotent graded groups (McDonald-Sukochev-Zanin).

Critical Cwikel Estimates

Problem

Find the largest Banach space $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ of functions on M such that:

(i) $\mathcal{V}$ embeds continuously in $L^1_g(M)$.

(ii) Each space $L^r_g(M)$, $r > 1$, embeds into $\mathcal{V}$.

(iii) $C^\infty(M)$ is a dense subspace of $\mathcal{V}$.

(iv) The map $f \to \Delta_g^{-n/4} f \Delta_g^{-n/4}$ is continuous from $\mathcal{V}$ to $\mathcal{L}^{1,\infty}$, i.e., we have an estimate,

$$\left\| \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} \right\|_{1,\infty} \leq C_n \|f\|_{\mathcal{V}}.$$

Critical Cwikel Estimates

Proposition

If the conditions (i)–(iv) are satisfied, then Connes' integration formula holds for all $f \in \mathcal{V}$, i.e.,

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M f(x) d\nu_g(x) \qquad \forall f \in \mathcal{V}.$$

Definition (Zygmund)

$L\log L(M)$ consists of measurable functions $f : M \to \mathbb{C}$ such that

$$\int_M |f(x)| \log(1 + |f(x)|) d\nu_g(x) < \infty.$$

Proposition

$L\log L(M)$ *is a Banach space with respect to the norm,*

$$\|f\|_{L\log L} := \inf\left\{\lambda > 0; \int_M |\lambda^{-1} f(x)| \log(1+\lambda^{-1}|f(x)|) d\nu_g(x) < 1\right\}.$$

Remark

We have (strict) continuous inclusions,

$$L^r_g(M) \subsetneq L\log L(M) \subsetneq L^1_g(M), \qquad r > 1.$$

Critical Cwikel Estimates

Proposition

If $f \in L\log L(M)$, then the operator $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ is bounded.

Remark

The above result is a consequence of Moser-Trüdinger's inequality.

In fact, we have:

Theorem (Solomyak '95, Rozenblum '22, Sukochev-Zanin '22)

If $f \in L\log L(M)$, then $\Delta_g^{-n/4} f \Delta_g^{-n/4} \in \mathcal{L}^{1,\infty}$, and we have

$$\left\| \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} \right\|_{1,\infty} \leq C_n \|f\|_{L\log L}.$$

Critical Cwikel Estimates

Remarks

- Solomyak obtained the estimate in even dimension.
- This was recently extended to tori of any dimension by Sukochev-Zanin. This allows us to get the estimate for closed manifolds of any dimension.
- Rozenblum independently obtained the estimate for an even larger class of potentials of the form $V = f\mu$, where μ is a Borel measure supported on a submanifold $\Sigma \subset M$ and $f \in L\log L(\Sigma)$.

Remark (Solomyak, Sukochev-Zanin)

It can be shown that $L\log L(M)$ is largest Orlicz spaces on which the critical Cwikel estimate holds.

Critical Cwikel Estimates

As a consequence of the critical Cwikel estimate from the previous slide, we get the following extension of Connes' integration formula:

Theorem (Rozenblum '22, Sukochev-Zanin '22, RP '22)

For every $f \in L\log L(M)$, *the operator* $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ *is strongly measurable, and we have*

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M V(x) d\nu_g(x).$$

Critical Cwikel Estimates

We also get a critical Semiclassical Weyl's law:

Theorem (Solomyak '95, Rozenblum '22, RP '22)

Let $V \in L\log L(M)$ be real-valued. We have

$$\lim_{h \to 0^+} h^n N^- \left(h^n \Delta_g^{\frac{n}{2}} + V\right) = c(n) \int_M V_-(x) d\nu_g(x).$$

In particular, for $n = 2$ we get:

Theorem (Solomyak '95)

Let $V \in L\log L(M)$ be real-valued. We have

$$\lim_{h \to 0^+} h^2 N^- \left(h^2 \Delta_g + V\right) = \frac{1}{4\pi} \int_M V_-(x) d\nu_g(x).$$

Remark

Thanks to Rozenblum's results the above results further hold for potentials $V = f\mu$, where μ is a Borel measure supported on a submanifold $\Sigma \subset M$ and $f \in L\log L(\Sigma)$.

Remark

- The previous results hold *verbatim* for Schrödinger operators on bounded domains $\Omega \subset \mathbb{R}^n$ with Dirichlet/Neumann boundary conditions.
- The results also hold on $\mathbb{R}^n$ provided we restrict ourselves to the subspace,

$$\mathcal{V} = \left\{ f \in L\log L(\mathbb{R}^n);\ \int |f(x)| \log(1+|x|)dx < \infty \right\}.$$