Noncommutative Geometry
Chapter 6:
Connes' Integration and Weyl's Laws. Birman-Schwinger Principle

Sichuan University, Fall 2022

Additional References

- McDonald, E.; Ponge, R.: *Cwikel estimates and negative eigenvalues of Schrödinger operators on noncommutative tori*. J. Math. Phys. **61** (2022), 043503.
- Ponge, R.: *Connes' integration and Weyl's laws*. Preprint, arXiv, July 2021. To appear in J. Noncomm. Geom..

Setup

Throughout this chapter $\mathcal{H}$ is a separable Hilbert space.

Weyl Operators

Definition

If $A \in \mathcal{L}(\mathcal{H})$, its real and imaginary parts are

$$\Re A := \frac{1}{2}(A + A^*), \qquad \Im A := \frac{1}{2i}(A - A^*).$$

Definition

If $A^* = A$, its positive and negative parts are

$$A^+ := \frac{1}{2}(|A| + A), \qquad A^- := \frac{1}{2}(|A| - A).$$

Remark

The operators A^+ and A^- are positive, and we have

$$A = A^+ - A^-, \qquad |A| = A^+ + A^-, \qquad A^+A^- = A^-A^+ = 0.$$

Remark

Assume A is compact (and selfadjoint).

- Let (ξ_j) be an orthonormal family such that $A\xi_j = \lambda_j(A)\xi_j$.
- Thus,
$$A = \sum_{j\geq 0} \lambda_j(A)|\xi_j\rangle\langle\xi_j|.$$
- We then have
$$A^+ = \sum_{\lambda_j(A)>0} \lambda_j(A)|\xi_j\rangle\langle\xi_j|, \qquad A^- = \sum_{\lambda_j(A)<0} (-\lambda_j(A))|\xi_j\rangle\langle\xi_j|.$$

Weyl Operators

Definition

If $A = A^*$ is compact, then $\pm\lambda_j^{\pm}(A)$, $j \geq 0$, are the positive eigenvalues of A such that

$$\lambda_0^{\pm}(A) \geq \lambda_1^{\pm}(A) \geq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

Remark

In other words,

$$\lambda_j^{\pm}(A) = \lambda_j(A^{\pm}) = \mu_j(A^{\pm}), \qquad j \geq 0.$$

Definition

We say that $A \in \mathcal{L}^{p,\infty}$ is a Weyl operator if one of the following conditions is satisfied:

(i) $A \geq 0$ and $\lim_{j\to\infty} j^{1/p}\lambda_j(A)$ exists.

(ii) $A = A^*$ and $\lim_{j\to\infty} j^{1/p}\lambda_j^{\pm}(A)$ both exist.

(iii) The real and imaginary parts both satisfy (ii).

Notation

The class of Weyl operators in $\mathcal{L}^{p,\infty}$ is denoted $\mathcal{W}^{p,\infty}$.

Weyl Operators

Definition

Let A be a Weyl operator in $\mathcal{L}^{p,\infty}$.

1. If $A \geq 0$, then we set
$$\Lambda(A) := \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j(A).$$
2. If $A^* = A$, then we set
$$\Lambda^{\pm}(A) := \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^{\pm}(A).$$
3. In general, we set
$$\Lambda^{\pm}(A) := \Lambda^{\pm}(\Re A) + i\Lambda^{\pm}(\Im A).$$

Proposition (Birman-Solomyak '70s)

1. *$\mathcal{W}^{p,\infty}$ is a closed subset of $\mathcal{L}^{p,\infty}$ on which the functions $\Lambda^{\pm}$ are continuous.*
2. *If $A \in \mathcal{W}^{p,\infty}$ and $B \in \mathcal{L}_0^{p,\infty}$, then $A + B \in \mathcal{W}^{p,\infty}$ and $\Lambda^{\pm}(A + B) = \Lambda^{\pm}(A)$.*

Remark

In particular, if $A_\ell \to A$ in $\mathcal{L}^{p,\infty}$ with $A_\ell = A_\ell^* \in \mathcal{W}^{p,\infty}$, then $A \in \mathcal{W}^{p,\infty}$, and

$$\Lambda^{\pm}(A) = \lim_{\ell \to \infty} \Lambda^{\pm}(A_\ell).$$

That is,

$$\lim_{j \to \infty} j^{\frac{1}{p}} \lambda_j^{\pm}(A) = \lim_{\ell \to \infty} \lim_{j \to \infty} j^{\frac{1}{p}} \lambda_j^{\pm}(A_\ell).$$

Weyl Operators

Definition

$|\mathcal{W}|^{p,\infty}$ consists of operators $A \in \mathcal{L}^{p,\infty}$ s.t. $|A| \in \mathcal{W}^{p,\infty}$. That is,

$$\lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j(|A|) = \lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A) \text{ exists.}$$

Remark

If $|A| \in \mathcal{W}^{p,\infty}$, then

$$\Lambda\left(|A|\right) = \lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j(|A|) = \lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A).$$

Proposition (Birman-Solomyak '70s)

1. $|\mathcal{W}|^{p,\infty}$ *is a closed subset of* $\mathcal{L}^{p,\infty}$ *on which* $A \to \Lambda(|A|)$ *is continuous.*
2. *If* $A \in |\mathcal{W}|^{p,\infty}$ *and* $B \in \mathcal{L}_0^{p,\infty}$, *then* $A + B \in |\mathcal{W}|^{p,\infty}$ *and* $\Lambda(|A+B|) = \Lambda(|A|)$.

Remark

In particular, if $A_\ell \to A$ in $\mathcal{L}^{p,\infty}$ with $A_\ell \in |\mathcal{W}|^{p,\infty}$, then

$$\lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A) = \lim_{\ell\to\infty} \lim_{j\to\infty} j^{\frac{1}{p}} \mu_j(A_\ell).$$

NC Integral and Weyl's Laws

Proposition

1. *If* $A \in \mathcal{W}^{1,\infty}$, *then* A *is strongly measurable, and we have*
$$⨍ A = \Lambda^+(A) - \Lambda^-(A).$$
2. *In particular, if* $A^* = A$, *then*
$$⨍ A = \lim_{j\to\infty} j\lambda_j^+(A) - \lim_{j\to\infty} j\lambda_j^-(A).$$

Proof.

- We have
$$A = (\Re A)^+ - (\Re A)^- + i\left((\Im A)^+ - (\Im A)^-\right).$$
- By linearity it is enough to prove the result for $(\Re A)^\pm$ and $(\Im A)^\pm$.
- This reduces the proof to the case $A \geq 0$.

□

Proof (Continued).

Assume $A \geq 0$, and set $c = \Lambda(A)$.

- As $A \in \mathcal{W}^{1,\infty}$ and $\Lambda(A) = c$, we have $\lim j\lambda_j(A) = c$, i.e.,
$$\lambda_j(A) = cj^{-1} + o(j^{-1}).$$
- Thus,
$$\sum_{j<N} \lambda_j(A) = c \log N + o(\log N).$$
- That is,
$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{j<N} \lambda_j(A) = c.$$
- This means that A is measurable and $⨍ A = c$. □

Proof (Continued).

It remains to show that A is strongly measurable.

- Let (ξ_j) be an orthonormal family such that $A\xi_j = \lambda_j(A)\xi_j$.
- Let $T_0 \in \mathcal{L}^{1,\infty}$ be such that $T_0\xi_j = (j+1)^{-1}\xi_j$.
- Set $B = A - cT_0$. We have

$$B\xi_j = \left(\lambda_j(A) - c(j+1)^{-1}\right)\xi_j$$

- It then can be shown that $\mu_j(B) = \mathrm{o}(j^{-1})$, i.e., $B \in \mathcal{L}_0^{1,\infty}$.
- Here $T_0 \in \mathcal{M}_s$ and $\mathcal{L}_0^{1,\infty} \subset \mathcal{M}_s$.
- Thus, $A = cT_0 + B \in \mathcal{M}_s$.

□

NC Integral and Weyl's Laws

Corollary

If $A \in |\mathcal{W}|^{1,\infty}$, *then* $|A|$ *is strongly measurable, and*

$$⨍ |A| = \lim_{j\to\infty} j\mu_j(A).$$

Example: Riemannian Manifolds

Setup

- $(M^n, g) =$ compact Riemannien manifold.
- $\Delta_g : C^\infty(M) \to C^\infty(M)$ is the Laplacian.
- $\nu_g(x) := \sqrt{\det(g(x))}d^n x$ is the Riemannian measure.
- $\mathrm{Vol}_g(M) := \int_M d\nu_g(x)$ is the volume of (M, g).

Facts

1. Δ_g is essentially selfadjoint on $L^2_g(M)$.
2. Its spectrum consists of isolated non-negative eigenvalues with finite multiplicity,
$$0 = \lambda_0(\Delta_g) \leq \lambda_1(\Delta_g) \leq \lambda_2(\Delta_g) \leq \cdots ,$$
where each eigenvalue is repeated according to multiplicity.

Reminder

- Let $(e_j) \subset C^\infty(M)$ be an orthonormal eigenbasis of Δ_g,

$$\Delta_g e_j = \lambda_j(\Delta_g) e_j, \qquad j \geq 0.$$

- For $q > 0$, the operator Δ_g^{-q} is defined by

$$\Delta_g^{-q} e_j = \begin{cases} \lambda_j(\Delta_g)^{-q} e_j & \text{if } \lambda_j(\Delta_g) > 0, \\ 0 & \text{if } \lambda_j(\Delta_g) = 0. \end{cases}$$

Fact

Δ_g^{-q} is a positive compact operator with

$$\lambda_j\left(\Delta_g^{-q}\right) = \lambda_{j+k}(\Delta_g)^{-q}, \quad k := \dim\ker \Delta_g.$$

Example: Riemannian Manifolds

Proposition (Weyl's Law)

As $j \to \infty$, we have

$$\lambda_j(\Delta_g) \sim \left(\frac{j}{c(n)\,\mathrm{Vol}_g(M)}\right)^{\frac{2}{n}}, \qquad c(n) := (2\pi)^{-n}|\mathbb{B}^n|.$$

Consequences

- As $\lambda_j(\Delta_g^{-q}) = \lambda_{j+k}(\Delta_g)^{-q}$, we get

$$\lambda_j\left(\Delta_g^{-q}\right) \sim \left(\frac{c(n)\,\mathrm{Vol}_g(M)}{j}\right)^{\frac{2q}{n}} = \mathrm{O}(j^{-\frac{2q}{n}}).$$

- In particular,

$$\Delta_g^{-q} \in \mathcal{L}_{\frac{n}{2q},\infty}.$$

- That is, Δ_g^{-q} is an infinitesimal of order $2qn^{-1}$.

Example: Riemannian Manifolds

For $q = n/2$, we get:

$$\lambda_j\left(\Delta_g^{-\frac{n}{2}}\right) \sim \frac{c(n)\,\mathrm{Vol}_g(M)}{j}.$$

Therefore, we obtain:

Proposition

The operator $\Delta_g^{-n/2}$ is strongly measurable, and we have

$$\fint \Delta_g^{-\frac{n}{2}} = c(n)\,\mathrm{Vol}_g(M).$$

Remark

In other words the NC integral recaptures the Riemannian volume.

Semiclassical Weyl's Law

Setup

- $(M^n, g) =$ compact Riemannien manifold.
- $\Delta_g : C^\infty(M) \to C^\infty(M)$ is the Laplacian.
- $\nu_g(x) := \sqrt{\det(g(x))} d^n x$ is the Riemannian measure.
- $V(x) =$ real-valued C^∞-potential, i.e., $V \in C^\infty(M, \mathbb{R})$.

Facts

1. The Schrödinger operator $\Delta_g + V$ is bounded from below and has discrete spectrum.
2. In particular, it has finitely many negative eigenvalues.

Definition

$N^-(\Delta_g + V)$ is the number of negative eigenvalues counted with multiplicity.

Semiclassical Weyl's Law

Theorem (Semiclassical Weyl's Law)

Under the semiclassical limit $h \to 0^+$*, we have*

$$\lim_{h \to 0^+} h^n N^- \left(h^2 \Delta_g + V\right) = c(n) \int_M |V_-(x)|^{\frac{n}{2}} d\nu_g(x),$$

where $V_-(x) = \max(0, -V(x))$ *is the negative part of* V*.*

Remark (Birman-Solomyak, Cwikel, Rozenblum)

The above asymptotics continue to hold provided one of the following conditions hold:

- $n \geq 3$ and $V \in L_g^{n/2}(M)$.
- $n = 2$ and $V \in L_g^p(M)$, $p > 1$.
- $n = 1$ and $V \in L_g^1(M)$.

Remark

The previous results also hold for Schrödinger operators on $\mathbb{R}^n$ and bounded domains $\Omega \subset \mathbb{R}^n$ with Dirichlet or Neumann conditions.

Counting Functions of Compact Operators

Definition

Let $A = A^*$ be compact. For $\lambda > 0$, we set

$$N^{\pm}(A; \lambda) := \#\{j;\ \lambda_j^{\pm}(A) > \lambda\}.$$

Remark

In other words, $N^{\pm}(A; \lambda)$ is the number of positive/negative eigenvalues (counted with multiplicity) that are $> \lambda$.

Definition

If $\lambda > 0$, then $\mathcal{F}^{\pm}(A;\lambda)$ is the class of subspaces $F \subset \mathcal{H}$ such that

$$\pm \langle A\xi|\xi\rangle > \lambda \langle \xi|\xi\rangle \quad \forall \xi \in F \setminus 0.$$

Lemma (Glazman's Lemma)

For all $\lambda > 0$, we have

$$N^{\pm}(A;\lambda) = \max\left\{\dim F; F \in \mathcal{F}^{\pm}(A;\lambda)\right\}.$$

Facts

1. We always have

$$\liminf_{\lambda\to 0^+} \lambda^p N^{\pm}(A;\lambda) = \liminf_{j\to\infty} j\lambda_j^{\pm}(A)^p,$$
$$\limsup_{\lambda\to 0^+} \lambda^p N^{\pm}(A;\lambda) = \limsup_{j\to\infty} j\lambda_j^{\pm}(A)^p.$$

2. In particular, we have

$$\lim_{\lambda\to 0^+} \lambda^p N^{\pm}(A;\lambda) = \left(\lim_{j\to\infty} j^{\frac{1}{p}}\lambda_j^{\pm}(A)\right)^p,$$

provided any of the above limits exist.

Birman-Schwinger Principle

Setup

- $H =$ selfadjoint (unbounded) operator with non-negative spectrum.
- $V =$ selfadjoint operator such that $(1+H)^{-1/2}V(1+H)^{-1/2}$ is compact.

Facts

- The operator $H_V := H + V$ is selfadjoint.
- The negative part of its spectrum is discrete, i.e., it consists of isolated eigenvalues with finite multiplicity.

Definition

$N^{-}(H_V)$ is the number of negative eigenvalues counted with multiplicity.

Birman-Schwinger Principle

Remarks

- The quadratic form of H is
$$Q_H(\xi,\xi) = \langle H\xi|\xi\rangle = \langle H^{1/2}\xi|H^{1/2}\xi\rangle.$$
- Its domain is $\mathrm{dom}(Q_H) = \mathrm{dom}(1+H)^{1/2}$.
- H_V is the (selfadjoint) operator whose quadratic form is
$$\begin{aligned} Q_{H_V}(\xi,\xi) &= Q_H(\xi,\xi) + Q_V(\xi,\xi) \\ &= Q_H(\xi,\xi) + \langle V\xi|\xi\rangle\,. \end{aligned}$$
- It has same domain as Q_H.

Definition

1. $\mathcal{F}^-(H_V)$ consists of all subspaces $E \subset \mathrm{dom}(Q_H)$ such that
$$Q_{H_V}(\xi,\xi) < 0 \quad \forall \xi \in E \setminus 0.$$
2. $\mathcal{F}_0^+(H_V)$ consists of all subspaces $E \subset \mathrm{dom}(Q_H)$ such that
$$Q_{H_V}(\xi,\xi) \geq 0 \quad \forall \xi \in E.$$

Lemma (Glazman's Lemma)

We have
$$\begin{aligned} N^-(H_V) &= \max\left\{\dim E; E \in \mathcal{F}^-(H_V)\right\}, \\ &= \min\left\{\operatorname{codim} E; E \in \mathcal{F}_0^+(H_V)\right\}. \end{aligned}$$

Birman-Schwinger Principle

Theorem (Birman-Schwinger Principle)

Assume $0 \notin \mathrm{Sp}(H)$, *and set* $H_V = H^{-1/2} V H^{-1/2}$. *Then,*

$$N^-(H_V) = N^-(K_V; 1).$$

Proof.

- By Glazman's lemma,

$$N^-(H_V) = \max\left\{\dim E; E \in \mathcal{F}^-(H_V)\right\},$$
$$N^-(K_V; 1) = \max\left\{\dim F; F \in \mathcal{F}^-(K_V; 1)\right\}.$$

- Here $\mathcal{F}^-(H_V)$ consists of subspaces $E \subset \mathrm{dom}(Q_H)$ such that

$$Q_{H_V}(\xi, \xi) < 0 \quad \text{on } E \setminus 0.$$

- $\mathcal{F}^-(K_V; 1)$ consists of subspaces $F \subset \mathcal{H}$ such that

$$-\langle K_V \eta | \eta \rangle > \langle \eta | \eta \rangle \quad \text{on } F \setminus 0.$$

□

Proof (Continued).

- Let $\xi \in \mathrm{dom}(Q_H)$ and set $\eta = H^{1/2}\xi$ (i.e., $\xi = H^{-1/2}\eta$).
- We have

$$\begin{aligned} Q_{H_V}(\xi,\xi) &= Q_H(\xi,\xi) + Q_V(\xi,\xi) \\ &= \langle H^{1/2}\xi | H^{1/2}\xi \rangle + \langle V\xi|\xi \rangle \,. \end{aligned}$$

- Here $\langle H^{1/2}\xi|H^{1/2}\xi\rangle = \langle \eta|\eta\rangle$, and

$$\langle V\xi|\xi\rangle = \langle VH^{-1/2}\eta|H^{-1/2}\eta\rangle = \langle H^{-1/2}VH^{-1/2}\eta|\eta\rangle = \langle K_V\eta|\eta\rangle \,.$$

- Thus, $Q_{H_V}(\xi,\xi) = \langle\eta|\eta\rangle + \langle K_V\eta|\eta\rangle$, and hence

$$\begin{aligned} Q_{H_V}(\xi,\xi) < 0 &\iff \langle\eta|\eta\rangle + \langle K_V\eta|\eta\rangle < 0 \\ &\iff -\langle K_V\eta|\eta\rangle > \langle\eta|\eta\rangle \,. \end{aligned}$$

□

Birman-Schwinger Principle

Proof (Continued).

- As $H^{1/2}$ is a bijection from $\mathrm{dom}(Q_H) = \mathrm{dom}(H^{1/2})$ onto $\mathcal{H}$, we get
$$E \in \mathcal{F}^-(H_V) \Longleftrightarrow H^{1/2}(E) \in \mathcal{F}^-(K_V;1).$$
- Thus, $H^{1/2}$ yields a one-to-one correspondence between $\mathcal{F}^-(H_V)$ and $\mathcal{F}^-(K_V;1)$.
- As this preserves the dimension, we get
$$\begin{aligned} N^-(H_V) &= \max\left\{\dim E; E \in \mathcal{F}^-(H_V)\right\} \\ &= \max\left\{\dim H^{1/2}(E); E \in \mathcal{F}^-(H_V)\right\} \\ &= \max\left\{\dim F; F \in \mathcal{F}^-(K_V;1)\right\} \\ &= N^-(K_V;1). \end{aligned}$$

The proof is complete. □

Birman-Schwinger Principle

Definition

If $\lambda \leq 0$, then $N^-(H_V; \lambda)$ is the number of eigenvalues counted with multiplicity that are $< \lambda$.

Remark

$N^-(H_V; \lambda) = N^-(H_V - \lambda)$.

Applying the previous result to $H - \lambda$ we get:

Theorem (Birman-Schwinger Principle)

Let $\lambda < 0$, *and set* $K_V(\lambda) = (H - \lambda)^{-1/2} V (H - \lambda)^{-1/2}$. *Then,*

$$N^-(H_V; \lambda) = N^-(K_V(\lambda); 1).$$

Remark

- The above proof is due to Birman-Solomyak.
- It is much simpler than various other proofs in the literature.

Birman-Schwinger Principle

Theorem (Borderline Birman-Schwinger Principle)

Assume 0 *is an isolated eigenvalue of* H *with finite multiplicity. Set* $K_V = H^{-1/2} V H^{-1/2}$*. Then:*

$$N^-(K_V; 1) \leq N^-(H_V) \leq N^-(K_V; 1) + \dim \ker H.$$

Proof.

Set $\mathcal{H}_1 = (\ker H)^\perp$. Let Π_1 be the orthoprojection onto $\mathcal{H}_1$. Set

$$H_1 = H_{|\mathcal{H}_1}, \quad V_1 = \Pi_1 V \Pi_1, \quad H_{V_1} = H_1 + V_1, \quad K_{V_1} = H_1^{-\frac{1}{2}} V_1 H_1^{-\frac{1}{2}}.$$

- With respect to the splitting $\mathcal{H} = \ker H \oplus \mathcal{H}_1$, we have

$$H_V = \begin{pmatrix} * & * \\ * & H_{V_1} \end{pmatrix}, \qquad K_V = H^{-1/2} V H^{-1/2} = \begin{pmatrix} 0 & 0 \\ 0 & K_{V_1} \end{pmatrix}.$$

- In particular, $Q_{H_V} = Q_{H_{V_1}}$ on $\mathcal{H}_1$, and K_V and K_{V_1} have the same no non-zero eigenvalues with same multiplicity. □

Birman-Schwinger Principle

Proof (Continued).

- As $0 \notin \mathrm{Sp}(H_1)$, by the Birman-Schwinger principle,
$$N^-(H_{V_1}) = N^-(K_{V_1}; 1).$$
- As K_V and K_{V_1} have the same non-zero eigenvalues,
$$N^-(K_{V_1}; 1) = N^-(K_V; 1).$$
- By Glazman's Lemma,
$$N^-(H_V) = \max\left\{\dim E; E \in \mathcal{F}^-(H_V)\right\},$$
where $\mathcal{F}^-(H_V)$ consists of subspaces $E \subset \mathrm{dom}(Q_H)$ such that $Q_{H_V} < 0$ on $E \setminus 0$.
- As $Q_{H_V} = Q_{H_{V_1}}$ on $\mathcal{H}_1$, we have $\mathcal{F}^-(H_{V_1}) \subset \mathcal{F}^-(H_V)$.
- Thus,
$$N^-(H_V) \geq \max\left\{\dim E; E \in \mathcal{F}^-(H_{V_1})\right\} = N^-(H_{V_1}).$$

□

Proof (Continued).

- By Glazman's Lemma we also have

$$N^-(H_V) = \min\left\{\operatorname{codim}_{\mathcal{H}} F; F \in \mathcal{F}_0^+(H_V)\right\},$$

where $\mathcal{F}_0^+(H_V)$ consists of subspaces $F \subset \mathcal{H}$ such that $Q_{H_V} \geq 0$ on F.

- Once again $Q_{H_V} = Q_{H_{V_1}}$ on $\mathcal{H}_1$, and so $\mathcal{F}_0^+(H_{V_1}) \subset \mathcal{F}_0^+(H_V)$.
- Moreover, if $F \subset \mathcal{H}_1$, then

$$\operatorname{codim}_{\mathcal{H}} F = \operatorname{codim}_{\mathcal{H}_1} F + \dim\ker H.$$

- Therefore, we have

$$\begin{aligned} N^-(H_V) &\leq \min\left\{\operatorname{codim}_{\mathcal{H}} F; F \in \mathcal{F}_0^+(H_{V_1})\right\} \\ &\leq \min\left\{\operatorname{codim}_{\mathcal{H}_1} F; F \in \mathcal{F}_0^+(H_{V_1})\right\} + \dim\ker H \\ &\leq N^-(H_{V_1}) + \dim\ker H. \end{aligned}$$

□

Proof (Continued).

- To sum up, we have

$$N^-(H_{V_1}) = N^-(K_{V_1};1) = N^-(K_V;1),$$
$$N^-(H_{V_1}) \leq N^-(H_V) \leq N^-(H_{V_1}) + \dim\ker H.$$

- Thus,

$$N^-(K_V;1) \leq N^-(H_V) \leq N^-(K_V;1) + \dim\ker H.$$

The proof is complete. □

Birman-Schwinger Principle

Corollary (Semiclassical BSP)

Assume that

- 0 *is an isolated eigenvalue of* H *with finite multiplicity.*
- $H^{-1/2}VH^{-1/2}$ *is a Weyl operator in* $\mathcal{L}^{p,\infty}$, $p > 0$.

Then, we have

$$\lim_{h\to 0^+} h^{2p} N^-\left(h^2 H + V\right) = \left(\lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^-\left(H^{-\frac{1}{2}}VH^{-\frac{1}{2}}\right)\right)^p$$
$$= \fint \left(H^{-\frac{1}{2}}VH^{-\frac{1}{2}}\right)_-^p .$$

Birman-Schwinger Principle

Proof.

Set $K_V = H^{-1/2}VH^{-1/2}$; this is a Weyl operator in $\mathcal{L}^{p,\infty}$.

- Reminder:

$$\lim_{\lambda\to 0^+} \lambda^p N^-(K_V;\lambda) = \left(\lim_{j\to\infty} j^{\frac{1}{p}}\lambda_j^-(K_V)\right)^p,$$

- The limit in the r.h.s. exists, since K_V is a Weyl operator.
- We have

$$N^-(h^2H+V) = N^-(H+h^{-2}V) = N^-(H_{h^{-2}V}),$$
$$N^-(K_{h^{-2}V};1) = N^{-1}(h^{-2}K_V;1) = N^-(K_V;h^2).$$

- By the Borderline Birman-Schwinger Principle, we have

$$N^-(K_{h^{-2}V};1) \leq N^-(H_{h^{-2}V}) \leq N^-(K_{h^{-2}V};1) + \dim\ker H.$$

- Thus,

$$N^-(K_V;h^2) \leq N^-(h^2H+V) \leq N^-(K_V;h^2) + \dim\ker H.$$

□

Proof.

- The previous inequalities implies that

$$N^-(K_V;h^2) \leq N^-(h^2H+V) \leq N^-(K_V;h^2) + \dim\ker H.$$

- Thus,

$$\begin{aligned}\lim_{h\to 0^+} h^{2p} N^-\left(h^2H+V\right) &= \lim_{h\to 0^+} h^{2p} N^-(K_V;h^2)\\ &= \lim_{\lambda\to 0^+} \lambda^p N^-(K_V;\lambda)\\ &= \left(\lim_{j\to\infty} j^{\frac{1}{p}} \lambda_j^-\left(K_V\right)\right)^p.\end{aligned}$$

□

Proof.

- As $\lambda_j^-(K_V)^p = \lambda_j((K_V)_-^p)$, we get

$$\lim_{j\to\infty} j\lambda_j\left((K_V)_-^p\right) = \lim_{j\to\infty} j\lambda_j^-(K_V)^p = \left(\lim_{j\to\infty} j^{\frac{1}{p}}\lambda_j^-(K_V)\right)^p.$$

- Therefore, $(K_V)_-^p$ is strongly measurable, and we have

$$⨍ (K_V)_-^p = \left(\lim_{j\to\infty} j^{\frac{1}{p}}\lambda_j^-(K_V)\right)^p.$$

□

Birman-Schwinger Principle

Remarks

- The previous result provides a semiclassical interpretation of the NC integral.
- This draws a link between NC geometry and semiclassical analysis.

Example: Riemannian Manifolds

Setup

- (M^n, g) = compact Riemannien manifold.
- $\Delta_g : C^\infty(M) \to C^\infty(M)$ is the Laplacian.
- $\nu_g(x) := \sqrt{\det(g(x))}d^n x$ is the Riemannian measure.

Remark

In what follows $c(n) = (2\pi)^{-n}|\mathbb{B}^n|$.

Example: Riemannian Manifolds

Theorem (Birman-Solomyak '70s)

Let $q > 0$ *and* $f \in C^\infty(M)$.

1. $\Delta_g^{-q/4} f \Delta_g^{-q/4}$ *and* $|\Delta_g^{-q/4} f \Delta_g^{-q/4}|$ *are Weyl operators in* $\mathcal{L}^{q^{-1}n,\infty}$ *(and hence are infinitesimals of order* qn^{-1}*).*
2. *We have*
$$\lim_{j\to\infty} j^{\frac{q}{n}} \mu_j \left(\Delta_g^{-\frac{q}{4}} f \Delta_g^{-\frac{q}{4}}\right) = \left(c(n) \int_M |f(x)|^{\frac{n}{q}} d\nu_g(x)\right)^{\frac{q}{n}}.$$
3. *If* $f(x)$ *is real-valued, then*
$$\lim_{j\to\infty} j^{\frac{q}{n}} \lambda_j^{\pm} \left(\Delta_g^{-\frac{q}{4}} f \Delta_g^{-\frac{q}{4}}\right) = \left(c(n) \int_M f_{\pm}(x)^{\frac{n}{q}} d\nu_g(x)\right)^{\frac{q}{n}},$$
where $f_{\pm}(x) = \max(0, \pm f(x))$ *are the positive/negative parts of* f.

Corollary

If $q > 0$ *and* $V \in C^\infty(M, \mathbb{R})$, *then*

$$\lim_{h \to 0^+} h^n N^- \left(h^{2q} \Delta_g^q + V \right) = c(n) \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x)$$
$$= ⨍ \left(\Delta_g^{-\frac{q}{2}} V \Delta_g^{-\frac{q}{2}} \right)_-^{\frac{n}{2q}}.$$

Proof.

- Apply the semiclassical version of the Birman-Schwinger principle to $H = \Delta_g^q$ to get

$$\lim_{h\to 0^+} h^n N^- \left(h^{2q}\Delta_g^q + V\right) = \lim_{h\to 0^+} (h^q)^{2\cdot\frac{n}{2q}} N^- \left((h^q)^2\Delta_g + V\right)$$
$$= \left(\lim_{j\to\infty} j^{\frac{2q}{n}} \lambda_j^- \left(\Delta_g^{-\frac{q}{2}} V \Delta_g^{-\frac{q}{2}}\right)\right)^{\frac{n}{2q}}$$
$$= c(n) \int_M V_-(x)^{\frac{n}{2q}} d\nu_g(x)$$

- The limits are also equal to $⨍(\Delta_g^{-q/2} V \Delta_g^{-q/2})_-^{n/2q}$. □

Theorem (Connes' Integration Formula)

For every $f \in C^\infty(M)$, *the operator* $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ *is strongly measurable, and we have*

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n) \int_M f(x) d\nu_g(x).$$

Example: Riemannian Manifolds

Proof.

- By linearity it is enough to prove the result for $f \geq 0$.
- If $f \geq 0$, then by Birman-Solomyak's result for $q = n$ we have

$$\begin{aligned} \lim_{j\to\infty} j\lambda_j\left(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right) &= \lim_{j\to\infty} j\lambda_j^+\left(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right) \\ &= c(n)\int_M f(x)d\nu_g(x). \end{aligned}$$

- It follows that $\Delta_g^{-n/4} f \Delta_g^{-n/4}$ is strongly measurable, and

$$⨍ \Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}} = c(n)\int_M f(x)d\nu_g(x).$$

□

Example: Riemannian Manifolds

Remarks

- Connes (CMP '88) established the formula and showed measurability.
- Strong measurability was later proved by Lord-Potapov-Sukochev (Adv. Math. '13).
- It was only recently realized that the result can be deduced from Birman-Solomyak's results.

Example: Riemannian Manifolds

Birman-Solomyak's results further allow us to get an integration formula for absolute values.

Theorem

For every $f \in C^\infty(M)$*, the operator* $|\Delta_g^{-n/4} f \Delta_g^{-n/4}|$ *is strongly measurable, and we have*

$$\fint \left|\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right| = c(n) \int_M |f(x)| d\nu_g(x).$$

Example: Riemannian Manifolds

Proof.

- By Birman-Solomyak's result for singular values for $q = n$ we get
$$\lim_{j\to\infty} j\mu_j\left(\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right) = c(n)\int_M |f(x)|d\nu_g(x).$$
- It follows that $|\Delta_g^{-n/4} f \Delta_g^{-n/4}|$ is strongly measurable, and
$$⨍ \left|\Delta_g^{-\frac{n}{4}} f \Delta_g^{-\frac{n}{4}}\right| = c(n)\int_M |f(x)|d\nu_g(x).$$

□