

Noncommutative Geometry
Chapter 5:
Connes' Quantized Calculus

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Additional References

- Lord, S.; Sukochev, F.; Zanin, D.: *Singular traces: theory and applications*. De Gruyter, 2012.
- Ponge, R.: *Connes' integration and Weyl's laws*. Preprint, arXiv, July 2021. To appear in J. Noncomm. Geom..

Setup

Throughout this chapter $\mathcal{H}$ is a separable Hilbert space.

Quantized Calculus (Connes)

Classical	Quantum (Connes)
Complex variable	Operator on Hilbert space $\mathcal{H}$
Real variable	Selfadjoint operator on $\mathcal{H}$
Infinitesimal variable	Compact operator on $\mathcal{H}$
Infinitesimal of order α	Compact operator s.t. $\mu_j(T) = O(j^{-\alpha})$
Integral $\int f(x)dx$	NC integral $\oint T$

Here the $\mu_j(A)$ are the singular values of A .

Infinitesimal Operators

Intuitive Definition

An infinitesimal is an object that it is smaller than any positive number.

Remark

For an operator $T \in \mathcal{L}(\mathcal{H})$ the condition

$$\|T\| < \epsilon \quad \text{for all } \epsilon > 0$$

gives the solution $T = 0$!

Definition (Infinitesimal Operator)

An operator $T \in \mathcal{L}(\mathcal{H})$ is **infinitesimal** if, for all $\epsilon > 0$, there is a subspace $E \subset \mathcal{H}$, $\dim E < \infty$, such that

$$\|T|_{E^\perp}\| < \epsilon.$$

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. Then TFAE

- ① T is a compact operator.
- ② T is the norm-limit of finite rank operators.
- ③ $\mu_j(T) \rightarrow 0$ as $j \rightarrow \infty$.
- ④ For all $\epsilon > 0$, there is $E \subset \mathcal{H}$, $\dim E < \infty$, s.t. $\|T|_{E^\perp}\| < \epsilon$.

Consequence

An operator T is an infinitesimal if and only if it is compact.

Infinitesimal Operators

Definition

A (compact) operator T is an infinitesimal of order α , $\alpha > 0$, if

$$\mu_j(T) = O(j^{-\alpha}) \quad \text{as } j \rightarrow \infty.$$

Remark

In other words, T is an infinitesimal of order α if and only if it belongs to the weak Schatten class $\mathcal{L}_{p,\infty}$ with $p = \alpha^{-1}$.

From the properties of weak Schatten classes we get:

Proposition

For $j = 1, 2$, let T_j be infinitesimal of order α_j . Then

- ① $T_1 + T_2$ is infinitesimal of order $\min(\alpha_1, \alpha_2)$.
- ② $T_1 T_2$ is infinitesimal of order $\alpha_1 + \alpha_2$.

Ansatz for a NC Integral (Connes)

The NC integral should have the following properties:

- ① It is defined on infinitesimals of order 1, i.e., on the weak trace class $\mathcal{L}^{1,\infty}$.
- ② It should take non-negative values on positive operators.
- ③ It vanishes on infinitesimals of order > 1 .
- ④ It vanishes on the commutator space,

$$\text{Com}(\mathcal{L}^{1,\infty}) = \text{Span} \{ [A, T]; A \in \mathcal{L}(\mathcal{H}), T \in \mathcal{L}^{1,\infty} \}.$$

That is, it should be a positive trace on $\mathcal{L}^{1,\infty}$.

Eigenvalue Sequences

Setup

T = compact operator on $\mathcal{H}$.

Definition

Let $\lambda \in \operatorname{Sp}(T) \setminus 0$.

- 1 The root space relatively to λ is

$$E_\lambda(T) = \bigcup_{\ell \geq 1} \ker(T - \lambda)^\ell.$$

- 2 $\dim E_\lambda(T)$ is called the algebraic multiplicity of λ .

Facts (see Gohberg-Krein)

- 1 The algebraic multiplicity is always finite (if $\lambda \neq 0$).
- 2 On $E_\lambda(T)$ the operator T takes the form $T = \lambda + N_\lambda$, where N_λ is nilpotent.
- 3 If T is normal, then $E_\lambda(T) = \ker(T - \lambda)$.

Eigenvalue Sequences

Definition

An eigenvalue sequence $\lambda(T) = \{\lambda_j(T)\}_{j \geq 0}$ is any sequence s.t.:

- 1 $\lambda_j(T)$ is an eigenvalue of T and is repeated according to algebraic multiplicity.
- 2 $|\lambda_0(T)| \geq |\lambda_1(T)| \geq \dots$.

Remarks

- 1 An eigenvalue sequence need not be unique.
- 2 If $T \geq 0$, then $\lambda_j(T) = \mu_j(T)$.
- 3 We shall denote by $\lambda(T)$ any eigenvalue sequence for T .

Eigenvalue Sequences

Proposition (Weyl)

For all $N \geq 1$, we have

$$\sum_{j < N} |\lambda_j(T)| \leq \sum_{j < N} \mu_j(T).$$

Remark

In general we don't have $|\lambda_j(T)| \leq \mu_j(T)$.

Theorem (Lidskii)

If $T \in \mathcal{L}^1$, then

$$\mathrm{Tr}(T) = \sum_{j \geq 0} \lambda_j(T).$$

Eigenvalue Sequences

Definition

A (compact) operator Q is called quasi-nilpotent if $\text{Sp}(Q) = \{0\}$.

Fact

Q is quasi-nilpotent if and only if $\lim_{n \rightarrow \infty} \|Q^n\|^{\frac{1}{n}} = 0$.

Proof.

By Gel'fand-Mazur theorem,

$$\lim_{n \rightarrow \infty} \|Q^n\|^{\frac{1}{n}} = \sup\{|\lambda|; \lambda \in \text{Sp}(Q)\}.$$



Theorem (Ringrose)

Any compact operator T can be put in the form,

$$T = A + Q,$$

where A and Q are compact operators such that

- A is normal and $\lambda(A) = \lambda(T)$.*
- Q is quasi-nilpotent.*

Lemma (see Reed-Simon)

Every $A \in \mathcal{L}(\mathcal{H})$ is linear combination of 4 unitaries.

Proof.

- If $A = A^*$ and $\|A\| = 1$, then

$$A = \frac{1}{2}(U + U^*) \quad \text{where } U = A + i\sqrt{1 - A^2}.$$

- In general $A = c_1 A_1 + ic_2 A_2$ with $c_i \geq 0$ and A_i as above. $\square$

Lemma

Let $\varphi : \mathcal{L}^{1,\infty} \rightarrow \mathbb{C}$ be a linear functional. TFAE:

(i) φ is a trace, i.e.,

$$\varphi(AT) = \varphi(TA) \quad \forall T \in \mathcal{L}^{1,\infty} \quad \forall A \in \mathcal{L}(\mathcal{H}).$$

(ii) φ is unitarily invariant, i.e.,

$$\varphi(U^*TU) = \varphi(T) \quad \forall T \in \mathcal{L}^{1,\infty} \quad \forall U \in \mathcal{L}(\mathcal{H}) \text{ unitary.}$$

Proof.

- If U is unitary, then

$$U^*TU - T = (U^*T)U - U(U^*T) = [U^*T, U]$$

- Here $U^*T \in \mathcal{L}^{1,\infty}$. Thus, if φ is a trace, then

$$\varphi(U^*TU) - \varphi(T) = \varphi([U^*T, U]) = 0.$$

- If φ is unitarily invariant, then

$$\varphi(UT) = \varphi(U^*(UT)U) = \varphi(TU).$$

- Thanks to the previous lemma the unitaries span $\mathcal{L}(\mathcal{H})$.
- Thus, by linearity $\varphi(TA) = \varphi(AT)$ for all $A \in \mathcal{L}(\mathcal{H})$, i.e., φ is a trace.



Proposition

Any positive trace on $\mathcal{L}^{1,\infty}$ is continuous.

Remark

This is a folk result.

- It can be shown that any positive linear form on a C^* -algebra is continuous (see, e.g., Murphy's book).
- The same arguments show that any positive linear form on $\mathcal{L}^{1,\infty}$ is continuous.

Proposition (Connes-McDonald-Sukochev-Zanin '19)

Every continuous trace on $\mathcal{L}^{1,\infty}$ is the linear combination of 4 positive traces.

Spectral Characterization of Commutators

Lemma (see Lord-Sukochev-Zanin's book)

If $S, T \in \mathcal{L}^{1,\infty}$, then

$$\sum_{j < N} \lambda_j(S + T) = \sum_{j < N} \lambda_j(S) + \sum_{j < N} \lambda_j(T) + O(1).$$

Corollary

If $(\lambda_j(T))_{j \geq 0}$ and $(\lambda'_j(T))_{j \geq 0}$ are two eigenvalue sequences, then

$$\sum_{j < N} \lambda'_j(T) = \sum_{j < N} \lambda_j(T) + O(1).$$

Proof.

Apply the lemma to $S = 0$ with $\lambda_j(S + T) = \lambda'_j(T)$. □

Corollary

If $T \in \text{Com}(\mathcal{L}^{1,\infty})$, then

$$\sum_{j < N} \lambda_j(T) = O(1).$$

Spectral Characterization of Commutators

Proof.

- As the unitaries span $\mathcal{L}(\mathcal{H})$, the space $\text{Com}(\mathcal{L}^{1,\infty})$ is span by operators of the form,

$$[T, U] = TU - UT = U^*(UT)U - UT.$$

with U unitary and $T \in \mathcal{L}^{1,\infty}$.

- Substituting U^*T for T shows that $\text{Com}(\mathcal{L}^{1,\infty})$ is span by operators of the form $U^*TU - T$.
- As $U^*TU = U^{-1}TU$ has same spectrum as T , we may take $\lambda_j(U^*TU) = \lambda_j(T)$ to get

$$\sum_{j < N} \lambda_j(U^*TU - T) = \sum_{j < N} \lambda_j(U^*TU) - \sum_{j < N} \lambda_j(T) + O(1) = O(1).$$

□

Spectral Characterization of Commutators

Theorem (Dykema-Figiel-Weiss-Wodzicki)

If $S, T \in \mathcal{L}^{1,\infty}$, then

$$S - T \in \text{Com}(\mathcal{L}^{1,\infty}) \iff \sum_{j < N} \lambda_j(S) = \sum_{j < N} \lambda_j(T) + O(1).$$

In particular,

$$T \in \text{Com}(\mathcal{L}^{1,\infty}) \iff \sum_{j < N} \lambda_j(T) = O(1).$$

Remarks

- This is a special case of a more general result for operator ideals.
- The proof for $\mathcal{L}^{1,\infty}$ is much simpler (see LSZ book).

Spectral Characterization of Commutators

Corollary

$$\mathcal{L}^1 \subset \text{Com}(\mathcal{L}^{1,\infty}).$$

Proof.

- If $T \in \mathcal{L}^1$, then by Weyl's inequality,

$$\sum_{j < N} |\lambda_j(T)| \leq \sum_{j < N} \mu_j(T) \leq \sum_{j \geq 0} \mu_j(T) < \infty.$$

- Thus, $\sum \lambda_j(T) = O(1)$, and hence $T \in \text{Com}(\mathcal{L}^{1,\infty})$. □

Corollary

Every trace on $\mathcal{L}^{1,\infty}$ vanishes on trace-class operators, including infinitesimal operators of order > 1 .

Dixmier Traces

Notation

- $\ell^\infty = C^*$ -algebra of **bounded** complex-valued sequences.
- $\mathfrak{c}_0$ = closed ideal of sequences converging to **0**.

Definition

For $T \in \mathcal{L}^{1,\infty}$ set

$$\Lambda_N(T) = \frac{1}{\log N} \sum_{j < N} \lambda_j(T), \quad N \geq 1.$$

Lemma

- 1 The sequence $(\Lambda_N(T))_{N \geq 1}$ is bounded.
- 2 If $(\lambda'_j(T))_{j \geq 1}$ is another eigenvalue sequence for T , then

$$\frac{1}{\log N} \sum_{j < N} \lambda'_j(T) - \frac{1}{\log N} \sum_{j < N} \lambda_j(T) \in \mathfrak{c}_0.$$

Proof.

- By Weyl's inequality $\sum_{j < N} |\lambda_j(T)| \leq \sum_{j < N} \mu_j(T)$.
- As $\mu_j(T) = O(j^{-1})$, we have $\sum_{j < N} \mu_j(T) = O(\log N)$.
- Thus,

$$|\Lambda_N(T)| \leq \frac{1}{\log N} \sum_{j < N} |\lambda_j(T)| \leq \frac{1}{\log N} \sum_{j < N} \mu_j(T) = O(1).$$

That is, $(\Lambda_N(T))_{N \geq 1}$ is a bounded sequence.

- If $(\lambda'_j(T))_{j \geq 1}$ is another eigenvalue sequence for T , then

$$\sum_{j < N} \lambda'_j(T) = \sum_{j < N} \lambda_j(T) + O(1).$$

- Thus,

$$\frac{1}{\log N} \sum_{j < N} \lambda'_j(T) - \frac{1}{\log N} \sum_{j < N} \lambda_j(T) = O((\log N)^{-1}) = o(1).$$

In particular, the above sequence is in $\mathfrak{c}_0$.



Consequence

The class of $(\Lambda_N(T))$ in $\ell^\infty/\mathfrak{c}_0$ does not depend on the choice of the eigenvalue sequence.

Definition

The map $\tau : \mathcal{L}^{1,\infty} \rightarrow \ell^\infty/\mathfrak{c}_0$ given by

$$\tau(T) = \text{class of } \left\{ \frac{1}{\log N} \sum_{k < N} \lambda_k(T) \right\}_{N \geq 1} \text{ in } \ell^\infty/\mathfrak{c}_0.$$

Lemma

τ is a positive linear map that vanishes on $\text{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.

Proof.

- If $S, T \in \mathcal{L}^{1,\infty}$, then

$$\sum_{j < N} \lambda_j(S + T) = \sum_{j < N} \lambda_j(S) + \sum_{j < N} \lambda_j(T) + O(1).$$

- Thus,

$$\Lambda_N(S + T) - \Lambda_N(S) - \Lambda_N(T) = O((\log N)^{-1}) = o(1).$$

That is, $\Lambda(S + T) - \Lambda(S) - \Lambda(T) \in \mathfrak{c}_0$, and hence $\tau(S + T) = \tau(S) + \tau(T)$.

- If $T \in \text{Com}(\mathcal{L}^{1,\infty})$, then $\sum_{j < N} \lambda_j(T) = O(1)$, and so

$$\Lambda_N(T) = O((\log N)^{-1}) = o(1).$$

That is, $\Lambda(T) \in \mathfrak{c}_0$, and hence $\tau(T) = 0$.



Proof (Continued).

- If $T \in \mathcal{L}_0^{1,\infty}$, then $\mu_j(T) = o(j^{-1})$, and so we have

$$\sum_{j < N} \mu_j(T) = o(\log N).$$

- By Weyl's inequality,

$$\sum_{j < N} |\lambda_j(T)| \leq \sum_{j < N} \mu_j(T).$$

- Thus,

$$|\Lambda_N(T)| \leq \frac{1}{\log N} \sum_{j < N} \mu_j(T) = o(1).$$

That is, $\Lambda(T) \in \mathfrak{c}_0$, and so $\tau(T) = 0$.



Dixmier Traces

Definition

A state on a unital C^* -algebra $\mathcal{A}$ is a positive linear form $\omega : \mathcal{A} \rightarrow \mathbb{C}$ such that $\omega(1) = 1$.

Remark

Every state is continuous.

Definition

An extended limit is any positive linear map $\lim_{\omega} : \ell^{\infty} \rightarrow \mathbb{C}$ s.t.:

- (i) $\lim_{\omega} 1 = 1$.
- (ii) $\lim_{\omega} a_j = 1$ if $(a_j) \in c_0$.

Remark

- If $a_j \rightarrow L$, then $(a_j) - L \in c_0$.
- Thus, for every extended limit $\lim_{\omega}$, we have

$$\lim_{\omega} a_j = \lim_{\omega} L = L \lim_{\omega} 1 = L.$$

Remark

- Any state ω on $\ell^\infty/\mathfrak{c}_0$ defines an extended limit by

$$\lim_\omega a_j = \omega([a]), \quad a = (a_j) \in \ell^\infty$$

where $[a]$ is the class of a in $\ell^\infty/\mathfrak{c}_0$.

- Conversely, any extended limit descends to a state on $\ell^\infty/\mathfrak{c}_0$.
- We thus have a one-to-one correspondence,

$$\{\text{extended limits}\} \longleftrightarrow \{\text{states on } \ell^\infty/\mathfrak{c}_0\}.$$

Remark

If $(a_j) \in \ell^\infty$ is real-valued, for every extended limit $\lim_\omega$ we have

$$\liminf a_j \leq \lim_\omega a_j \leq \limsup a_j.$$

Lemma

Given any $(a_j) \in \ell^\infty$, TFAE:

- (i) $a_j \rightarrow L$.
- (ii) $\lim_\omega a_j = L$ for every extended limit $\lim_\omega$.

Definition

If $\lim_\omega$ is an extended limit, then $\mathrm{Tr}_\omega : \mathcal{L}^{1,\infty} \rightarrow \mathbb{C}$ is given by

$$\mathrm{Tr}_\omega(T) := \lim_\omega \left\{ \frac{1}{\log N} \sum_{j < N} \lambda_j(T) \right\}, \quad T \in \mathcal{L}^{1,\infty}.$$

Proposition (Dixmier)

- ① Tr_ω is a positive linear trace on $\mathcal{L}^{1,\infty}$.
- ② It is annihilated by $\mathcal{L}_0^{1,\infty}$, and hence it vanishes on infinitesimals of order > 1 .

Proof.

- If ω is the state on $\ell^\infty/\mathfrak{c}_0$ defined by $\lim_\omega$, then

$$\mathrm{Tr}_\omega(T) = \lim_\omega \Lambda_N(T) = \omega([\Lambda(T)]) = \omega \circ \tau(T).$$

- It then follows from properties of τ and states that Tr_ω that:
 - It is a positive linear form on $\mathcal{L}^{1,\infty}$.
 - It vanishes on $\mathrm{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.
 - In particular, this is a trace.



Definition

Tr_ω is called the Dixmier trace associated with the extended limit $\lim_\omega$.

Measurable Operators

Definition (Connes)

- 1 An operator $T \in \mathcal{L}^{1,\infty}$ is called measurable if the value of $\mathrm{Tr}_\omega(T)$ does not depend on the extended limit.
- 2 We denote by $\mathcal{M}$ the class of measurable operators.
- 3 If $T \in \mathcal{M}$, we define its NC integral by

$$\int T := \mathrm{Tr}_\omega(A),$$

where Tr_ω is any Dixmier trace.

Proposition (Connes, Lord-Sukochev-Zanin)

Given $T \in \mathcal{L}^{1,\infty}$, TFAE:

- 1 T is measurable and $\int T = L$.
- 2 We have

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) = L.$$

Measurable Operators

Proof.

We have

$$\begin{aligned} T \text{ meas. \& } \int T = L &\iff \operatorname{Tr}_\omega(T) = L \quad \forall \lim_\omega, \\ &\iff \lim_\omega \left\{ \frac{1}{\log N} \sum_{j < N} \lambda_j(T) \right\} = L \quad \forall \lim_\omega, \\ &\iff \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) = L. \end{aligned}$$

□

Consequence

If T is measurable, then

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) = \int T.$$

Proposition

- ① $\mathcal{M}$ is a closed subspace of $\mathcal{L}^{1,\infty}$ that contains $\text{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.
- ② $f : \mathcal{M} \rightarrow \mathbb{C}$ is a positive linear functional that vanishes on $\text{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.
- ③ In particular, this is a positive trace that annihilates infinitesimals of order > 1 .

Remarks

- The C^* -algebra ℓ^∞/c_0 is not separable.
- The existence of states follows from Hahn-Banach theorem.
- In the non-separable case the proof relies on the Axiom of Choice.

Question (Connes, Fudan U. '17)

- Show the existence of a limit for measurable operators without using extended limits.
- Produce a purely spectral theoretic construction of the NC integral.

Reminder

If $(\lambda_j(T))$ and $(\lambda'_j(T))$ are two eigenvalue sequences for $T \in \mathcal{L}^{1,\infty}$, then

$$\frac{1}{\log N} \sum_{j < N} \lambda'_j(T) = \frac{1}{\log N} \sum_{j < N} \lambda_j(T) + o(1).$$

Lemma

Let $T \in \mathcal{L}^{1,\infty}$. TFAE:

- (i) $\lim_{N \rightarrow \infty} (\log N)^{-1} \sum_{j < N} \lambda_j(T)$ exists for some eigenvalue sequence.
- (ii) $\lim_{N \rightarrow \infty} (\log N)^{-1} \sum_{j < N} \lambda_j(T)$ exists for every eigenvalue sequence.

Definition (Lord-Sukochev-Zanin)

- ① An operator $T \in \mathcal{L}^{1,\infty}$ is called Tauberian if

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) \text{ exists.}$$

- ② The class of Tauberian operators is denoted $\mathcal{T}$.

Definition

For $T \in \mathcal{T}$ set

$$\mathcal{f}' T := \lim_{N \rightarrow \infty} (\log N)^{-1} \sum_{j < N} \lambda_j(T).$$

Proposition

- ① $\mathcal{T}$ is a closed subspace of $\mathcal{L}^{1,\infty}$ that contains $\text{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.
- ② $f' : \mathcal{T} \rightarrow \mathbb{C}$ is a positive linear functional that vanishes on $\text{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$.
- ③ In particular, this is a positive trace that annihilates infinitesimals of order > 1 .

Tauberian Approach

Proof.

- Reminder (LSZ Lemma): if $S, T \in \mathcal{L}^{1,\infty}$, then

$$\sum_{j < N} \lambda_j(S + T) = \sum_{j < N} \lambda_j(S) + \sum_{j < N} \lambda_j(T) + O(1).$$

- Thus,

$$\frac{1}{\log N} \sum_{j < N} \lambda_j(S + T) = \frac{1}{\log N} \sum_{j < N} \lambda_j(S) + \frac{1}{\log N} \sum_{j < N} \lambda_j(T) + o(1).$$

- Therefore, if $S, T \in \mathcal{T}$, then

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(S + T) &= \\ \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(S) + \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) &= f' S + f' T. \end{aligned}$$

- $S + T \in \mathcal{T}$ and $f'(S + T) = f' S + f' T$.



Proof (Continued).

- Reminder: if $T \in \text{Com}(\mathcal{L}^{1,\infty})$, then $\sum_{j < N} \lambda_j(T) = O(1)$, and hence

$$\frac{1}{\log N} \sum_{j < N} \lambda_j(T) = o(1).$$

- Thus,

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) = 0.$$

- That is,

$$T \in \mathcal{T} \quad \text{and} \quad \int' T = 0.$$



Proof (Continued).

- Reminder: if $T \in \mathcal{L}_0^{1,\infty}$, then $\mu_j(T) = o(j^{-1})$, and hence

$$\sum_{j < N} \mu_j(T) = o(\log N).$$

- By Weyl's inequality,

$$\left| \sum_{j < N} \lambda_j(T) \right| \leq \sum_{j < N} |\lambda_j(T)| \leq \sum_{j < N} \mu_j(T)$$

- Thus,

$$\frac{1}{\log N} \left| \sum_{j < N} \lambda_j(T) \right| \leq \frac{1}{\log N} \sum_{j < N} \mu_j(T) = o(1).$$

- It follows that $(\log N)^{-1} \sum \lambda_j(T) \rightarrow 0$.
- As before, this implies that

$$T \in \mathcal{T} \quad \text{and} \quad \int' T = 0.$$

Tauberian Approach

The two approaches agree.

Proposition

$\mathcal{T} = \mathcal{M}$ and $f' = f$.

Proof.

- We know that

$$T \in \mathcal{M} \iff \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) \text{ exists.}$$

- Thus, $\mathcal{T} = \mathcal{M}$.
- Moreover, if $T \in \mathcal{M} = \mathcal{T}$, then

$$f' T = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T) = f T.$$



Tauberian Approach

Corollary (Spectral Invariance)

Let $S, T \in \mathcal{L}^{1,\infty}$ have the same non-zero eigenvalues with same multiplicity. Then:

- 1 S is measurable if and only if T is measurable.
- 2 In this case $f S = f T$.

Proof.

- The assumptions imply that $\lambda_j(S) = \lambda_j(T)$.
- Thus,

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(S) = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T),$$

provided any of the above limit exists.

- Therefore, $S \in \mathcal{M}$ iff $T \in \mathcal{M}$, and in this case $f S = f T$. $\square$

Strong Measurability

Definition

A trace $\varphi : \mathcal{L}^{1,\infty} \rightarrow \mathbb{C}$ is called normalized if

$$(T \geq 0 \text{ and } \lambda_j(T) = (j+1)^{-1}) \implies \varphi(T) = 1.$$

Remark

Every Dixmier trace Tr_ω is a normalized trace.

Proof.

- If $\lambda_j(T) = (j+1)^{-1}$, then

$$\frac{1}{\log N} \sum_{j < N} \lambda_j(T) = \frac{1}{\log N} \sum_{j < N} \frac{1}{j+1} \rightarrow 1.$$

- Thus T is measurable and $\oint T = 1$.
- In particular, $\text{Tr}_\omega(T) = 1$.



Strong Measurability

Remark

There are many normalized positive traces on $\mathcal{L}^{1,\infty}$ that are not Dixmier traces.

Definition

An operator $T \in \mathcal{L}^{1,\infty}$ is called strongly measurable (or positively measurable) if $\varphi(T)$ takes the same value as φ ranges over all normalized positive traces.

Remark

If T is strongly measurable, then: it is measurable, and, for every normalized positive trace $\varphi : \mathcal{L}^{1,\infty} \rightarrow \mathbb{C}$, we have

$$\varphi(T) = \int T = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{j < N} \lambda_j(T).$$

Reminder

- 1 Every positive linear form on $\mathcal{L}^{1,\infty}$ is continuous.
- 2 Every continuous trace on $\mathcal{L}^{1,\infty}$ is linear combinations of 4 positive traces (Connes *et al.*).

Remark

It can be shown that every positive trace is normalized up to scalar multiple.

Consequence

The space of continuous traces on $\mathcal{L}^{1,\infty}$ is spanned by normalized positive traces.

Notation

$T_0 =$ any positive operator in $\mathcal{L}^{1,\infty}$ such that $\lambda_j(T) = (j+1)^{-1}$.

Lemma

Given any $T \in \mathcal{L}^{1,\infty}$, TFAE:

- (i) T is strongly measurable and $\int T = L$.
- (ii) $\varphi(T) = \varphi(T_0)L$ for every continuous trace on $\mathcal{L}^{1,\infty}$.

Strong Measurability

Notation

$\mathcal{M}_s$ = class of strongly measurable operators.

Proposition

- $\mathcal{M}_s$ is a closed subspace of $\mathcal{L}^{1,\infty}$.
- It contains $\text{Com}(\mathcal{L}^{1,\infty})$ and $\mathcal{L}_0^{1,\infty}$. In particular, it contains all infinitesimals of order > 1 .
- It does not depend on the inner product of $\mathcal{L}(\mathcal{H})$.

Remark

- In fact, $\mathcal{M}_s$ contains the closure $\overline{\text{Com}(\mathcal{L}^{1,\infty})}$.
- This closure contains $\text{Com}(\mathcal{L}^{1,\infty}) \cup \mathcal{L}_0^{1,\infty}$.

Proposition

Let $T \in \mathcal{L}^{1,\infty}$ be such that

$$\sum_{j < N} \lambda_j(T) = L \cdot \log N + O(1).$$

Then T is strongly measurable and $\oint T = L$.

Proof.

- The assumptions imply that T is measurable and $\int T = L$.
- We have

$$\sum_{j < N} \lambda_j(T_0) = \sum_{j < N} (j+1)^{-1} = \log N + O(1).$$

- Thus,

$$\sum_{j < N} \lambda_j(T) = L \cdot \log N + O(1) = \sum_{j < N} \lambda_j(T_0) + O(1).$$

- We know that by a theorem of Dykema *et al.* this implies that $T - LT_0 \in \text{Com}(\mathcal{L}^{1,\infty})$.
- Here $T_0 \in \mathcal{M}_s$ and $\text{Com}(\mathcal{L}^{1,\infty})$, and so $T \in \mathcal{M}_s$. □