Noncommutative Geometry
Chapter 4:
Singular Values and Schatten Classes

Sichuan University, Fall 2022

Additional References

- Gohberg, I.C.; Krein, M.G.: *Introduction to the theory of linear nonselfadjoint operators*. American Mathematical Society, 1969.
- Simon, B.: *Trace ideals and their applications*. American Mathematical Society, 2005.

Setup

Throughout this chapter $\mathcal{H}$ is a separable Hilbert space.

Singular Values

Definition (Singular Values)

Let $T \in \mathcal{L}(\mathcal{H})$. Given any $n \geq 0$, the $(n+1)$-th singular value (a.k.a. characteristic value) of T is

$$\mu_n(T) := \inf\{\|T_{|E^\perp}\|;\ \dim E = n\}.$$

Remark

It follows from the above definition that

$$\mu_n(\lambda T) = |\lambda|\mu_n(T) \qquad \forall \lambda \in \mathbb{C}.$$

Singular Values

Notation

$\mathcal{R}_n$ = the space of operators $T \in \mathcal{L}(\mathcal{H})$ of rank $\leq n$.

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. Then:

$$\mu_n(T) = \text{dist}(T, \mathcal{R}_n),$$
$$\mu_m(T) \leq \mu_n(T) \qquad \forall m \geq n.$$

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. Then:

$$\mu_n(T) = \mu_n(T^*) = \mu_n(|T|),$$
$$\mu_n(ATB) \leq \|A\| \mu_n(T) \|B\| \qquad \forall A, B \in \mathcal{L}(\mathcal{H}),$$
$$\mu_n(U^* TU) = \mu_n(T) \qquad \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary}.$$

Remark

Let $\mathcal{H}'$ be another (separable) Hilbert space.

- If $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and let $B \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, then
$$\mu_n(ATB) \leq \|A\|_{\mathcal{L}(\mathcal{H}',\mathcal{H})}\mu_n(T)\|B\|_{\mathcal{L}(\mathcal{H},\mathcal{H}')}.$$
- If $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is unitary, then
$$\mu_n(U^*TU) = \mu_n(T).$$
- Thus, the singular values are invariant under unitary isomorphisms.

Singular Values

Proposition

Let $S, T \in \mathcal{L}(\mathcal{H})$ *and let* $m, n \in \mathbb{N}_0$. *Then:*

$$\mu_{m+n}(S+T) \leq \mu_m(S) + \mu_n(T),$$
$$|\mu_m(S) - \mu_n(T)| \leq \|S - T\|,$$
$$\mu_{m+n}(ST) \leq \mu_m(S)\mu_n(T).$$

Singular Values

Proposition

Let $T \in \mathcal{L}(\mathcal{H})$. *TFAE:*

1. T *is a compact operator.*
2. T *is the norm-limit of finite-rank operators.*
3. $\mu_n(T) \to 0$ *as* $n \to 0$.
4. *For any* $\epsilon > 0$ *there exists a finite-dimensional subspace* E *of* $\mathcal{H}$ *such that* $\|T_{|E^\perp}\| < \epsilon$.

Singular Values

Theorem (Min-Max Principle)

If T is compact, then

$$\mu_n(T) = (n+1)\text{-th eigenvalue of } |T| \text{ counted with multiplicity.}$$

Remark

This implies that eigenvalues of *positive* compact operators are continuous, since the singular values are continuous.

Singular Values

Corollary

Let $T = U|T|$ be the polar decomposition of T and $(\xi_n)_{n\geq 0}$ an orthonormal family such that $|T|\xi_n = \mu_n(T)\xi_n$ for all n. Then

$$T = \sum_{n\geq 0} \mu_n(T) |U\xi_n\rangle\langle\xi_n|,$$

where the series converges in norm.

Corollary

Assume that $f(t)$, $t \geq 0$, is non-negative, non-decreasing and converges to 0 as $t \to 0^+$. Then

$$\mu_n(f(|T|)) = f(\mu_n(T)) \qquad \forall n \geq 0,$$
$$f(|T|) = \sum_{n \geq 0} f(\mu_n(T)) |\xi_n\rangle\langle\xi_n|,$$

where the series converges in norm.

Example

Let $p > 0$. Applying the above result to $f(t) = t^p$ gives

$$\mu_n(|T|^p) = \mu_n(T)^p \qquad \forall n \geq 0.$$

The Trace-Class $\mathcal{L}^1$

Definition

- For $T \in \mathcal{L}(\mathcal{H})$ we set
$$\|T\|_1 := \sum_{n\geq 0} \mu_n(T).$$
- The trace-class $\mathcal{L}^1$ is the set of operators T such that $\|T\|_1 < \infty$.

Remarks

- We always have $\|T\| = \mu_0(T) \leq \|T\|$.
- As $\|T\|_1 = \sum \mu_n(T) < \infty \Rightarrow \lim \mu_n(T) = 0$, any trace-class operator is compact.

The Trace-Class $\mathcal{L}^1$

Proposition

The properties of singular values imply the following:

$$\|T\|_1 = \|T^*\|_1 = \||T|\|_1,$$
$$\|\lambda T\|_1 = |\lambda|\|T\|_1, \qquad \lambda \in \mathbb{C},$$
$$\|S + T\|_1 \leq \|S\|_1 + \|T\|_1,$$
$$\|ATB\|_1 \leq \|A\|\|T\|_1\|B\|, \qquad A, B \in \mathcal{L}(\mathcal{H}).$$

Remark

The top equalities imply that

$$T \in \mathcal{L}^1 \Longleftrightarrow T^* \in \mathcal{L}^1 \Longleftrightarrow |T| \in \mathcal{L}^1.$$

The Trace-Class $\mathcal{L}^1$

Proposition

1. $\mathcal{L}^1$ *is a two-sided ideal of* $\mathcal{L}(\mathcal{H})$.
2. $\|\cdot\|_1$ *is a norm which respect to which* $\mathcal{L}^1$ *is a Banach space.*

Proposition

Assume $T \in \mathcal{L}^1$. *Let* $T = U|T|$ *be its polar decomposition and* $(\xi_n)_{n\geq 0}$ *an orthonormal family such that* $|T|\xi_n = \mu_n(T)\xi_n$ *for all* n. *Then*

$$T = \sum_{n\geq 0} \mu_n(T)|U\xi_n\rangle\langle\xi_n|,$$

where the series converges in $\mathcal{L}^1$.

Corollary

The space of finite-rank operators is dense in $\mathcal{L}^1$.

The Trace-Class $\mathcal{L}^1$

Lemma

Let $T \in \mathcal{L}(\mathcal{H})$ be positive. Then, for every orthonormal basis $(\xi_n)_{n\geq 0}$ of $\mathcal{H}$, we have

$$\|T\|_1 = \sum_{n\geq 0} \langle \xi_n | T\xi_n \rangle .$$

Lemma

Let $T \in \mathcal{L}^1$.

1. *For every orthonormal basis $(\xi_n)_{n\geq 0}$ of $\mathcal{H}$, the series*

   $$\sum_{n\geq 0} \langle \xi_n | T\xi_n \rangle$$

   converges absolutely.
2. *The value of its sum does not depend on the basis $(\xi_n)_{n\geq 0}$.*

The Trace-Class $\mathcal{L}^1$

Definition

Let $T \in \mathcal{L}(\mathcal{H})$. Then the trace of T is

$$\mathrm{Tr}(T) := \sum_{n \geq 0} \langle \xi_n | T \xi_n \rangle ,$$

where $(\xi_n)_{n \geq 0}$ is any orthonormal basis.

Example

We have

$$\mathrm{Tr}(|T|) = \sum \langle \xi_n | |T| \xi_n \rangle = \left\| |T| \right\|_1 = \|T\|_1 = \sum \mu_n(T).$$

The Trace-Class $\mathcal{L}^1$

Proposition

1. $T \to \mathrm{Tr}(T)$ *is a continuous linear form on* $\mathcal{L}^1$ *such that*

$$|\mathrm{Tr}(T)| \leq \mathrm{Tr}(|T|)| = \|T\|_1,$$
$$\mathrm{Tr}(T^*) = \overline{\mathrm{Tr}(T)}.$$

2. *This is a trace on the ideal* $\mathcal{L}^1$, *i.e.,*

$$\mathrm{Tr}(AT) = \mathrm{Tr}(TA) \qquad \forall T \in \mathcal{L}^1 \ \forall A \in \mathcal{L}(\mathcal{H}).$$

Example

Let $\xi, \eta \in \mathcal{H}$. Assume that $(\xi_n)_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}$ such that $\xi_0 = \|\xi\|^{-1}\xi$.

- For all $n \geq 0$, we have

$$\langle \xi_n | (|\xi\rangle\langle\eta|)\xi_n\rangle = \langle \xi_n|\xi\rangle \langle \eta|\xi_n\rangle = \delta_{n,0}\|\xi\| \langle \eta|\xi_0\rangle = \delta_{n,0} \langle \eta|\xi\rangle .$$

- Thus,

$$\operatorname{Tr}(|\xi\rangle\langle\eta|) = \sum_{n\geq 0} \langle \xi_n|(|\xi\rangle\langle\eta|)\xi_n\rangle = \langle \eta|\xi\rangle .$$

Example

Assume $T \in \mathcal{L}^1$ is normal, and let $(\xi_n)_{n\geq 0}$ be an orthonormal eigenbasis of $\mathcal{H}$, i.e., $T\lambda_n = \lambda_n \xi_n$ for all $n \geq 0$.

- We have

$$\mathrm{Tr}(T) = \sum_{n\geq 0} \langle \xi_n | T\xi_n \rangle = \sum_{n\geq 0} \lambda_n \langle \xi_n | \xi_n \rangle = \sum_{n\geq 0} \lambda_n.$$

- That is, the trace of T is the sum of its eigenvalues.

The Trace-Class $\mathcal{L}^1$

Notation

If $S \in \mathcal{L}^1$ and $T \in \mathcal{L}(\mathcal{H})$, then $ST \in \mathcal{L}^1$, and so we may set

$$(S, T) := \mathrm{Tr}(ST).$$

Remark

We have

$$|(S, T)| = |\mathrm{Tr}(ST)| \leq \|ST\|_1 \leq \|S\|_1 \|T\|.$$

The Trace-Class $\mathcal{L}^1$

Lemma

(i) *For all* $S \in \mathcal{L}^1$,

$$\|S\|_1 = \sup_{\|T\|=1} |\mathrm{Tr}(ST)| .$$

(ii) *For all* $T \in \mathcal{L}(\mathcal{H})$,

$$\|T\| = \sup_{\|S\|_1=1} |\mathrm{Tr}(ST)| .$$

Consequence

We isometric linear maps,

$$\mathcal{L}^1 \ni S \longrightarrow (S, \cdot) \in \mathcal{L}(\mathcal{H})',$$
$$\mathcal{L}(\mathcal{H}) \ni T \longrightarrow (\cdot, T) \in (\mathcal{L}^1)'.$$

The Trace-Class $\mathcal{L}^1$

Proposition

The map $T \to (\cdot, T)$ is an isometric isomorphism from $\mathcal{L}(\mathcal{H})$ onto $(\mathcal{L}^1)'$.

Remark

This means that $\mathcal{L}^1$ is the pre-dual of $\mathcal{L}(\mathcal{H})$.

Proposition

The map $S \to (S, \cdot)$ is an isometric isomorphism from $\mathcal{L}^1$ onto $\mathcal{K}'$.

Hilbert-Schmidt Operators

Definition

- For any $T \in \mathcal{L}(\mathcal{H})$, we set

$$\|T\|_2 = \left(\sum \mu_n(T)^2\right)^{\frac{1}{2}}.$$

- We say that T is a Hilbert-Schmidt operator if $\|T\|_2 < \infty$.
- The class of Hilbert-Schmidt operators is denoted $\mathcal{L}^2$.

Remarks

- We have $\mathcal{L}^1 \subset \mathcal{L}^2 \subset \mathcal{K}$.
- If T is compact, then $\mu_n(T)^2 = \mu_n(|T|^2)$, and so

$$\|T\|_2^2 = \sum \mu_n(T)^2 = \sum \mu_n(|T|^2) = \mathrm{Tr}(|T|^2) = \mathrm{Tr}(T^*T).$$

- Thus,

$$T \in \mathcal{L}^2 \Longleftrightarrow \sum \mu_n(|T|^2) < \infty \Longleftrightarrow |T|^2 \in \mathcal{L}^1.$$

Example

Let $\xi, \eta \in \mathcal{H}$. Assume that $(\xi_n)_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}$ such that $\xi_0 = \|\xi\|^{-1}\xi$.

- We have $(|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|$, and so
$$(|\xi\rangle\langle\eta|)^*(|\xi\rangle\langle\eta|) = (|\eta\rangle\langle\xi|)(|\xi\rangle\langle\eta|) = \langle\xi|\xi\rangle\,|\eta\rangle\langle\eta| = \|\xi\|^2|\eta\rangle\langle\eta|.$$
- Thus,
$$\||\xi\rangle\langle\eta|\|_2^2 = \operatorname{Tr}\left[(|\xi\rangle\langle\eta|)^*(|\xi\rangle\langle\eta|)\right] = \|\xi\|^2 \operatorname{Tr}\left[|\eta\rangle\langle\eta|\right] = \|\xi\|^2 \langle\eta|\eta\rangle\,.$$
- That is,
$$\||\xi\rangle\langle\eta|\|_2 = \|\xi\|\|\eta\|.$$

Hilbert-Schmidt Operators

Proposition

The properties of singular values imply the following:

$$\|T\|_2 = \|T^*\|_2 = \||T|\|_2,$$
$$\|\lambda T\|_2 = |\lambda| \|T\|_2, \qquad \lambda \in \mathbb{C},$$
$$\|ATB\|_2 \leq \|A\| \|T\|_2 \|B\|, \qquad A, B \in \mathcal{L}(\mathcal{H}).$$

Remark

The top equalities imply that

$$T \in \mathcal{L}^2 \iff T^* \in \mathcal{L}^2 \iff |T| \in \mathcal{L}^2.$$

Lemma

For every orthonormal basis $(\xi_n)_{n\geq 0}$ *of* $\mathcal{H}$*, we have*

$$\|T\|_2^2 = \sum_{n\geq 0} \|T\xi_n\|^2.$$

Proof.

- We have
$$\|T\|_2^2 = \mathrm{Tr}(|T|^2) = \sum \langle \xi_n | |T|^2 \xi_n \rangle .$$
- Here
$$\langle \xi_n | |T|^2 \xi_n \rangle = \langle \xi_n | T^* T \xi_n \rangle = \langle T\xi_n | T\xi_n \rangle = \|T\xi_n\|^2.$$
- Thus,
$$\|T\|_2^2 = \sum \|T\xi_n\|^2.$$

□

Hilbert-Schmidt Operators

Corollary

We have

$$\|S+T\|_2 \leq \|S\|_2 + \|T\|_2, \qquad S,T \in \mathcal{L}(\mathcal{H}).$$

Proposition

1. $\mathcal{L}^2$ *is a two-sided ideal of* $\mathcal{L}^2$.
2. $\|\cdot\|_2$ *is a norm which respect to which* $\mathcal{L}^2$ *is a Banach space.*

Hilbert-Schmidt Operators

Proposition

Assume $T \in \mathcal{L}^2$*. Let* $T = U|T|$ *be its polar decomposition and* $(\xi_n)_{n\geq 0}$ *an orthonormal family such that* $|T|\xi_n = \mu_n(T)\xi_n$ *for all* n*. Then*

$$T = \sum_{n\geq 0} \mu_n(T)|U\xi_n\rangle\langle\xi_n|,$$

where the series converges in $\mathcal{L}^2$*.*

Corollary

The space of finite-rank operators is dense in $\mathcal{L}^2$*.*

Hilbert-Schmidt Operators

Proposition

If $S, T \in \mathcal{L}^2$, *then* ST *is trace-class, and we have*

$$|\operatorname{Tr}(ST)| \leq \|ST\|_1 \leq \|S\|_2 \|T\|_2,$$
$$\operatorname{Tr}(ST) = \operatorname{Tr}(TS).$$

Corollary

$\mathcal{L}^2$ *is a Hilbert space with respect to the inner product,*

$$\langle S|T\rangle_{\mathcal{L}^2} := \operatorname{Tr}(S^*T), \qquad S, T \in \mathcal{L}^2.$$

Remark

We have

$$\langle T|T\rangle_{\mathcal{L}^2} = \operatorname{Tr}(T^*T) = \operatorname{Tr}(|T|^2) = \|T\|_2^2.$$

Integral Operators

Setup

(X,μ) is a σ-finite measure space s.t. $L^2_\mu(X)$ is separable.

Example

X is a smooth manifold and μ is a Radon measure.

Definition (Integral Operators)

If $K(x,y) \in L^2_{\mu\otimes\mu}(X\times X)$, then $T_K : L^2_\mu(X) \to L^2_\mu(X)$ is given by

$$T_K f(x) := \int_X K(x,y) f(y) d\mu(y), \qquad f \in L^2_\mu(X).$$

Integral Operators

Remark

- By Cauchy-Schwartz's inequality,

$$|T_K f(x)|^2 = \left(\int_X K(x,y) f(y) d\mu(y) \right)^2 \leq \int_X |K(x,y)|^2 d\mu(y) \int_X |f(y)|^2 d\mu(y).$$

- Thus,

$$\int_X |T_K f(x)|^2 d\mu(x) \leq \|f\|_{L^2}^2 \iint_{X \times X} |K(x,y)|^2 f(y) d\mu(y) d\mu(x) < \infty.$$

- That is,

$$T_K f \in L^2_\mu(X) \qquad \text{and} \qquad \|T_K\|_{L^2} \leq \|K\|_{L^2} \|f\|_{L^2}.$$

- It follows that

$$T_K \in \mathcal{L}(L^2_{\mu \otimes \mu}(X \times X)) \qquad \text{and} \qquad \|T_K\| \leq \|K\|_{L^2}.$$

Example

Let $\varphi, \psi \in L^2_\mu(X)$, and set $K(x,y) = \varphi(x)\overline{\varphi(y)} \in L^2_{\mu\otimes\mu}(X \times X)$.

- If $f \in L^2_\mu(X)$, then

$$T_K f(x) = \int_X \varphi(x)\overline{\varphi(y)} f(y) d\mu(y) = f(x) \langle \psi | f \rangle .$$

- Thus,

$$T_K = |\varphi \rangle\langle \psi|$$

- It follows that every rank 1 operator on $L^2_\mu(X)$ is an integral operator.
- By linearity every finite-rank operator is an integral operator

Proposition

For $K(x,y)$ *and* $K'(x,y)$ *in* $L^2_{\mu\otimes\mu}(X\times X)$*, we have*

$$T_K^* = T_{K^*} \qquad \text{and} \qquad T_K T_{K'} = T_{K*K'},$$

where we have set

$$K^*(x,y) = \overline{K(y,x)},$$

$$K*K'(x,y) = \int_X K(x,z)K'(z,y)d\mu(z).$$

Proposition

1. *If* $K(x,y) \in L^2_{\mu\otimes\mu}(X\times X)$*, then* T_K *is Hilbert-Schmidt.*
2. *The linear map* $K \to T_K$ *is an isometric isomorphism from* $L^2_{\mu\otimes\mu}(X\times X)$ *onto* $\mathcal{L}^2$*.*

Proof.

Let $(\varphi_n)_{n\geq 0}$ be an orthonormal basis of $L^2_\mu(X)$.

- If $K(x,y) \in L^2_{\mu\otimes\mu}(X\times X)$, then

$$\|T_K\|_2^2 = \sum_{n\geq 0} \|T_K\varphi_n\|^2 = \sum_{n,m} |\langle \varphi_m | T_K\varphi_n\rangle|^2.$$

- Here

$$\langle \varphi_m | T_K\varphi_n\rangle = \iint \overline{\varphi_m(x)} K(x,y)\varphi_n(y)d\mu(y) = \langle \varphi_m \otimes \overline{\varphi_n} | K\rangle .$$

- As $(\varphi_m \otimes \overline{\varphi_n})_{m,n\geq 0}$ is an orthonormal basis of $L^2_\mu(X\times X)$, we get

$$\|T_K\|_2^2 = \sum_{n,m} \left|\langle \varphi_m \otimes \overline{\varphi_n} | K\rangle\right|^2 = \|K\|_{L^2} < \infty.$$

- Thus, $T_K \in \mathcal{L}^2$ and we have isometric linear map $K \to T_K$ from $L^2_{\mu\otimes\mu}(X\times X)$ to $\mathcal{L}^2$. In particular, it has closed range.
- As the range contains finite-rank operators, which are dense in $\mathcal{L}^2$, the map is onto, and hence is an isometric isomorphism. □

Assumptions

- $X =$ separable metrizable locally compact Hausdorff space (e.g., X is a smooth manifold).
- $\mu =$ Radon measure.

Remark

Under those assumptions $L^2_\mu(X)$ is a separable Hilbert space.

Trace Theorems

Theorem (Duflo)

Assume that

- $K(x,y) \in L^2_{\mu\otimes\mu}(X \times X) \cap C(X \times X)$.
- T_K *is trace-class.*

Then $K(x,x)$ *is in* $L^1_\mu(X)$, *and we have*

$$\mathrm{Tr}(T_K) = \int_X K(x,x)d\mu(x).$$

Remark

In the above generality we need to assume T_K to be trace-class, since otherwise the trace formula may fail.

Trace Theorems

If $T_K \geq 0$, then we don't need to assume T_K is trace-class.

Theorem

Assume that

- $K(x, y) \in L^2_{\mu \otimes \mu}(X \times X) \cap C(X \times X)$.
- T_K *is positive.*

Then:

1. $K(x, x) \geq 0$ *for all* $x \in X$.
2. *We have*
$$\mathrm{Tr}(T_K) = \int_X K(x, x) d\mu(x).$$
3. *In particular,*
$$T_K \in \mathcal{L}^1 \iff K(x, x) \in L^1_\mu(X).$$

Banach Ideals

Definition

A Banach ideal is a two-sided ideal $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ together with a norm $\|.\|_{\mathcal{I}}$ such that

(i) $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a Banach space.

(ii) We have

$$\|ATB\|_{\mathcal{I}} \leq \|A\| \|T\|_{\mathcal{I}} \|B\| \qquad \forall T \in \mathcal{I} \quad \forall A, B \in \mathcal{L}(\mathcal{H}).$$

Examples

The following are Banach ideals:

- The ideal of compact operators $\mathcal{K}$.
- The trace-class $\mathcal{L}^1$.
- The ideal of Hilbert-Schmidt operators $\mathcal{L}^2$.

Banach Ideals

Remark

If $\mathcal{I}$ is a two-sided ideal, then

$$T \in \mathcal{I} \iff |T| \in \mathcal{I} \iff T^* \in \mathcal{I}.$$

Proof.

If $T = U|T|$, then $|T| = U^*T$ and $T^* = U^*TU^*$ and $T = UT^*U$. □

Proposition

Assume $(\mathcal{I}, \|\cdot\|)$ is a Banach ideal. If $T \in \mathcal{I}$, then:

$$\|T\|_{\mathcal{I}} = \||T|\|_{\mathcal{I}} = \|T^*\|_{\mathcal{I}},$$
$$\|U^*TU\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \qquad \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary}.$$

Corollary

If $T \in \mathcal{I}$ and $\mu_n(S) = \mu_n(T)$ for all $n \geq 0$, then

$$S \in \mathcal{I} \qquad \text{and} \qquad \|S\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}.$$

Thus, $\|T\|_{\mathcal{I}}$ only depends on the singular values $\mu_n(T)$.

Proof.

- Let $(\xi_n)_{n\geq 0}$ and $(\eta_n)_{n\geq 0}$ be orthonormal bases such that
$$|T|\xi_n = \mu_n(T)\xi_n \qquad \text{and} \qquad |S|\eta_n = \mu_n(T)\eta_n.$$
- Let U be the unitary operator s.t. $U\xi_n = \eta_n$. Then
$$U|T|\xi_n = \mu_n(T)U\xi_n = \mu_n(T)\eta_n = |S|\eta_n = |S|U\xi_n.$$
- That is, $U|T| = |S|U$, i.e., $U|T|U^* = |S|$. Thus,
$$T \in \mathcal{I} \Longleftrightarrow |T| \in \mathcal{I} \Longleftrightarrow U|T|U^* \in \mathcal{I} \Longleftrightarrow |S| \in \mathcal{I} \Longleftrightarrow S \in \mathcal{I}.$$
Moreover, $\|S\|_{\mathcal{I}} = \||S|\|_{\mathcal{I}} = \|U|T|U^*\|_{\mathcal{I}} = \||T|\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}.$ □

Remark

If $\mathcal{I}$ is a two-sided ideal, then, we always have

$$\{\text{finite-rank operators}\} \subset \mathcal{I} \subset \mathcal{K}$$

Definition

$\mathcal{I}^0$ is the closure in $\mathcal{I}$ of the ideal of finite-rank operators.

Remarks

- $(\mathcal{I}^0, \|\cdot\|_{\mathcal{I}})$ is a Banach ideal itself.
- As a Banach space $\mathcal{I}_0$ is separable (i.e., it contains a dense countable subset).

Lemma

Let $T \in \mathcal{L}(\mathcal{H})$. *TFAE:*

1. $T \in \mathcal{I}$.
2. *Any Schmidt series* $T = \sum \mu_n(T)|U\xi_n\rangle\langle\xi_n|$ *converges in* $\mathcal{I}$.

Proposition

TFAE:

1. $\mathcal{I}$ *is separable.*
2. $\mathcal{I}^0 = \mathcal{I}$, *i.e., finite-rank operators are dense in* $\mathcal{I}$.

Proposition

$\mathcal{I}^0$ always contains the trace-class $\mathcal{L}^1$ and we can choose $\|\cdot\|_{\mathcal{I}}$ so that $\|\cdot\|_{\mathcal{I}} \leq \|\cdot\|_1$ on $\mathcal{L}^1$.

Remark

In other words $\mathcal{L}^1$ is the smallest Banach ideal.

Definition

A quasi-norm on a vector space E is a function $\|\cdot\| : E \to [0, \infty)$ such that:

1. $\|x\| = 0 \Leftrightarrow x = 0$.
2. $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in E$.
3. There is $C > 0$ such that

$$\|x + y\| \leq C\left(\|x\| + \|y\|\right) \qquad \forall x, y \in E.$$

A vector space equipped with a quasi-norm is called quasi-normed.

Quasi-Banach Ideals

Facts

1. A quasi-norm generates a topological vector space topology.
2. A basis of neighbourhoods of the origin is given by the "quasi-balls",
$$B(0,1/n) = \{x \in E;\ \|x\| < 1/n\}, \qquad n \geq 1.$$
3. As we have a countable basis, the topology is metrizable.
4. In particular, $x_n \to x$ in E iff $\|x_n - x\| \to 0$.

Proposition

Let $T : E \to F$ be a linear map between quasi-normed spaces. TFAE:

1. *T is continuous.*
2. *T is continuous at 0.*
3. *There is $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in E$.*

Quasi-Banach Ideals

Definition

A quasi-Banach space is a vector space E together with a complete quasi-norm (i.e., every Cauchy sequence is convergent).

Remark

1. A quasi-Banach space need not be locally convex.
2. In particular, Hahn-Banach Theorem need not hold.

Example

Let (X, μ) be a measure space.

1. For $0 < p < 1$ the space $L^p_\mu(X)$ is a quasi-Banach space with quasi-norm,
$$\|f\|_{L^p} := \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$
2. The triangular inequality is replaced by
$$\|f+g\|^p_{L^p} \leq \|f\|^p_{L^p} + \|g\|^p_{L^p}.$$
In particular $d(f,g) = \|f-g\|^p_{L^p}$ is a metric.
3. Thanks to the convexity of $t \to t^{1/p}$ this gives the inequality,
$$\|f+g\|_{L^p} \leq 2^{\frac{1-p}{p}} \left(\|f\|_{L^p} + \|g\|_{L^p} \right).$$

Quasi-Banach Ideals

Definition

A quasi-Banach ideal is a two-sided ideal $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ together with a quasi-norm $\|.\|_{\mathcal{I}}$ such that

(i) $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a quasi-Banach space.

(ii) We have

$$\|ATB\|_{\mathcal{I}} \leq \|A\|\|T\|_{\mathcal{I}}\|B\| \qquad \forall T \in \mathcal{I} \quad \forall A, B \in \mathcal{L}(\mathcal{H}).$$

Remark

We will see that the Schatten classes $\mathcal{L}^p$, $0 < p < 1$, and the weak Schatten classes $\mathcal{L}^{p,\infty}$, $0 < p \leq 1$, are instances of quasi-Banach spaces.

Quasi-Banach Ideals

Proposition

Assume $(\mathcal{I}, \|\cdot\|)$ *is a Banach ideal. If* $T \in \mathcal{I}$, *then:*

$$\|T\|_{\mathcal{I}} = \||T|\|_{\mathcal{I}} = \|T^*\|_{\mathcal{I}},$$

$$\|U^*TU\|_{\mathcal{I}} = \|T\|_{\mathcal{I}} \qquad \forall U \in \mathcal{L}(\mathcal{H}),\ U \text{ unitary}.$$

Corollary

If $T \in \mathcal{I}$ *and* $\mu_n(S) = \mu_n(T)$ *for all* $n \geq 0$, *then*

$$S \in \mathcal{I} \qquad \text{and} \qquad \|S\|_{\mathcal{I}} = \|T\|_{\mathcal{I}}.$$

Thus, $\|T\|_{\mathcal{I}}$ *only depends on the singular values* $\mu_n(T)$.

Quasi-Banach Ideals

Definition

$\mathcal{I}^0$ is the closure in $\mathcal{I}$ of the ideal of finite-rank operators.

Proposition

TFAE:

1. $\mathcal{I}$ *is separable.*
2. $\mathcal{I}^0 = \mathcal{I}$, *i.e., finite-rank operators are dense in* $\mathcal{I}$.

Definition (Schatten Classes)

Let $p \in (0, \infty)$.

- For $T \in \mathcal{L}(\mathcal{H})$, set

$$\|T\|_p = \left(\sum \mu_n(T)^p\right)^{\frac{1}{p}}.$$

- The Schatten class $\mathcal{L}^p$ consists of all operators T such that $\|T\|_p < \infty$.

Remarks

- We have $\mathcal{L}^p \subset \mathcal{L}^q \subset \mathcal{K}$ for $0 < p < q$.
- As $\mu_n(T)^p = \mu_n(|T|^p)$, we have

$$T \in \mathcal{L}^p \Longleftrightarrow \sum \mu_n(|T|^p) < \infty \Longleftrightarrow |T|^p \in \mathcal{L}^1.$$

Proposition

1. *We have*

$$\|T\|_p = \|T^*\|_p = \||T|\|_p,$$
$$\|\lambda T\|_p = |\lambda|\|T\|_p, \qquad \lambda \in \mathbb{C},$$
$$\|ATB\|_p \leq \|A\|\|T\|_p\|B\|, \qquad A, B \in \mathcal{L}(\mathcal{H}).$$

2. *If* $p \geq 1$, *then*

$$\|S + T\|_p \leq \|S\|_p + \|T\|_p.$$

3. *If* $0 < p < 1$, *then*

$$\|S + T\|_p^p \leq \|S\|_p^p + \|T\|_p^p.$$

Proposition

1. $\mathcal{L}^p$ *is a two-sided ideal of* $\mathcal{L}(\mathcal{H})$.
2. *If* $p \geq 1$, *then* $\|\cdot\|_p$ *is a norm which respect to which* $\mathcal{L}^p$ *is a Banach ideal.*
3. *If* $p < 1$, *then* $\|\cdot\|_p$ *is a quasi-norm which respect to which* $\mathcal{L}^p$ *is a quasi-Banach ideal.*

Proposition

1. *If* $T \in \mathcal{L}^p$, *then all its Schmidt series converge in* $\mathcal{L}^p$.
2. *Finite rank operators are dense in* $\mathcal{L}^p$.

Proposition (Hölder Inequality)

Assume $p^{-1} + q^{-1} = r^{-1}$. *If* $S \in \mathcal{L}^p$ *and* $T \in \mathcal{L}^q$, *then* $ST \in \mathcal{L}^r$, *and we have*

$$\|ST\|_r \leq \|S\|_p \|T\|_q.$$

Remark

If $p_1^{-1} + \cdots + p_k^{-1} = r^{-1}$ and $T_i \in \mathcal{L}^{p_i}$, then $T_1 \cdots T_k \in \mathcal{L}^r$, and we have

$$\|T_1 \cdots T_k\|_r \leq \|T_1\|_{p_1} \cdots \|T_k\|_{p_k}.$$

Corollary

Assume $p^{-1} + (p')^{-1} = 1$. *If* $S \in \mathcal{L}^p$ *and* $T \in \mathcal{L}^{p'}$, *then* ST *is trace-class, and we have*

$$|\mathrm{Tr}(ST)| \leq \|ST\|_1 \leq \|S\|_p \|T\|_{p'}.$$

Notation

If $S \in \mathcal{L}^p$ and $T \in \mathcal{L}^{p'}$, then $(S, T) := \mathrm{Tr}(ST)$.

Remark

The above corollary implies that $T \to (\cdot, T)$ is a continuous linear map from $\mathcal{L}^{p'}$ to the (topological) dual $(\mathcal{L}^p)'$.

Proposition

The map $T \to (\cdot, T)$ *is an isometric linear isomorphism from* $\mathcal{L}^{p'}$ *onto* $(\mathcal{L}^p)'$.

Weak Schatten Classes

Definition (Weak Schatten Classes $\mathcal{L}^{p,\infty}$)

Let $p \in (0, \infty)$.

1. The weak Schatten class $\mathcal{L}^{p,\infty}$ consists of all $T \in \mathcal{L}(\mathcal{H})$ such that
$$\mu_n(T) = \mathrm{O}\left(n^{-\frac{1}{p}}\right) \qquad \text{as } n \to \infty.$$
2. For $T \in \mathcal{L}(\mathcal{H})$, we set
$$\|T\|_{p,\infty} := \sup_{n \geq 0} (n+1)^{\frac{1}{p}} \mu_n(T).$$

Remark

For $0 < p < q$ we have strict inclusions,

$$\mathcal{L}^p \subsetneq \mathcal{L}^{p,\infty} \subsetneq \mathcal{L}^q \subsetneq \mathcal{K}.$$

Proposition

We have

$$\|T\|_{p,\infty} = \|T^*\|_{p,\infty} = \||T|\|_{p,\infty},$$
$$\|\lambda T\|_{p,\infty} = |\lambda|\|T\|_{p,\infty}, \qquad \lambda \in \mathbb{C},$$
$$\|ATB\|_{p,\infty} \leq \|A\|\|T\|_{p,\infty}\|B\|, \qquad A, B \in \mathcal{L}(\mathcal{H}),$$
$$\|S+T\|_{p,\infty} \leq 2^{\frac{1}{p}}\left(\|S\|_{p,\infty} + \|T\|_{p,\infty}\right).$$

Weak Schatten Classes

Proposition

1. $\mathcal{L}^{p,\infty}$ *is a two-sided ideal of* $\mathcal{L}(\mathcal{H})$.
2. $\|\cdot\|_{p,\infty}$ *is a quasi-norm which respect to which* $\mathcal{L}^{p,\infty}$ *is a quasi-Banach ideal.*

Proposition

1. *If* $p > 1$, *then* $\|\cdot\|_{p,\infty}$ *is equivalent to the norm,*
$$\|T\|'_{p,\infty} = \sup_{N\geq 1}\left\{N^{-1+\frac{1}{p}}\sum_{j<N}\mu_j(T)\right\}, \qquad T \in \mathcal{L}^{p,\infty}.$$
2. *In this case,* $\mathcal{L}^{p,\infty}$ *is a Banach ideal (w.r.t. that norm).*

Weak Schatten Classes

Notation

$\mathcal{L}_0^{p,\infty}$ is the closure in $\mathcal{L}^{p,\infty}$ of the ideal of finite-rank operators.

Proposition

1. *We have*

$$\mathcal{L}_0^{p,\infty} = \left\{ T \in \mathcal{L}(\mathcal{H});\ \mu_n(T) = \mathrm{o}\left(n^{-\frac{1}{p}}\right) \right\}.$$

2. *We have a strict inclusion* $\mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty}$.
3. *In particular,* $\mathcal{L}^{p,\infty}$ *is not separable.*

Remark

For $0 < p < q$ we have strict inclusions,

$$\mathcal{L}^p \subsetneq \mathcal{L}_0^{p,\infty} \subsetneq \mathcal{L}^{p,\infty} \subsetneq \mathcal{L}^q.$$

Weak Schatten Classes

Proposition (Hölder's Inequality)

Assume that $p^{-1} + q^{-1} = r^{-1}$.

1. *There is* $C_{pqr} > 0$ *such that*

$$\|ST\|_{r,\infty} \leq C_{pqr} \|S\|_{p,\infty} \|T\|_{q,\infty}.$$

2. *In particular,*

$$(S \in \mathcal{L}^{p,\infty} \text{ and } T \in \mathcal{L}^{q,\infty}) \Longrightarrow ST \in \mathcal{L}^{r,\infty}.$$

3. *If* $S \in \mathcal{L}_0^{p,\infty}$ *or* $T \in \mathcal{L}_0^{q,\infty}$, *then* $ST \in \mathcal{L}_0^{r,\infty}$.

Remark (Sukochev-Zanin '21)

The best constant C_{pqr} is equal to $p^{-\frac{1}{q}} q^{-\frac{1}{p}} (p+q)^{\frac{1}{r}}$.

Weak Schatten Classes

Corollary

Suppose that $p_1^{-1} + \cdots + p_k^{-1} = r^{-1}$.

1. *There is* $C = C(p_1, \ldots, q_k, r) > 0$ *such that*
$$\|T_1 \cdots T_k\|_{r,\infty} \leq C\|T_1\|_{p_1,\infty} \cdots \|T_k\|_{p_k,\infty}.$$
2. *If* $T_i \in \mathcal{L}^{p_i,\infty}$ *for* $i = 1, \ldots, k$, *then* $T_1 \cdots T_k \in \mathcal{L}^{r,\infty}$.
3. *If in addition one of the* T_i *is in* $\mathcal{L}_0^{p_i,\infty}$, *then* $T_1 \cdots T_k \in \mathcal{L}_0^{r,\infty}$.

The Dixmier-Macaev Ideal

Definition

The Dixmier-Macaev ideal is

$$\mathcal{M}^{1,\infty} := \left\{ T \in \mathcal{L}(\mathcal{H});\ \sum_{j<N} \mu_j(T) = \mathrm{O}(\log N) \right\}.$$

Proposition

1. $\mathcal{M}^{1,\infty}$ *is a Banach ideal with respect to the norm,*

$$\|T\|'_{1,\infty} = \sup_{N\geq 1} \left\{ \frac{1}{\log(N+1)} \sum_{j<N} \mu_j(T) \right\}, \qquad T \in \mathcal{M}^{1,\infty}.$$

2. *We have a strict inclusion* $\mathcal{L}^{1,\infty} \subsetneq \mathcal{M}^{1,\infty}$.

Remark

In Connes' book (as well as elsewhere) the Dixmier-Macaev ideal is denoted $\mathcal{L}^{1,\infty}$.

The Dixmier-Macaev Ideal

Notation

$\mathcal{M}_0^{1,\infty}$ is the closure in $\mathcal{M}^{1,\infty}$ of the ideal of finite-rank operators.

Proposition

1. *We have*

$$\mathcal{M}_0^{1,\infty} = \left\{ T \in \mathcal{L}(\mathcal{H});\ \sum_{j<N} \mu_n(T) = \mathrm{o}(\log N) \right\}.$$

2. *There is a strict inclusion* $\mathcal{M}_0^{1,\infty} \subsetneq \mathcal{M}^{1,\infty}$.
3. *In particular,* $\mathcal{M}^{1,\infty}$ *is not separable.*