

Commutative Algebra

Chapter 2: Modules

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Modules and Module Homomorphisms

Notation

Throughout this chapter A is a *ring* (which is commutative and has an identity element).

Definition (Modules)

An A -*module* is an Abelian group M on which A acts linearly. That is, there is a map $A \times M \ni (a, x) \rightarrow ax \in M$ such that

$$a(x + y) = ax + ay,$$

$$(a + b)x = ax + bx,$$

$$(ab)x = a(bx),$$

$$1x = x, \quad a, b \in A, \quad x, y \in M.$$

Remark

Equivalently, M is an Abelian group together with a ring homomorphism $A \rightarrow E(M)$, where $E(M)$ is the ring of homomorphisms of M .

Modules and Module Homomorphisms

Examples

- 1 Any ideal $\mathfrak{a}$ of A is an A -module. In particular, A itself is an A -module.
- 2 If A is a field k , then an A -module is exactly a vector space of over k .
- 3 If $A = \mathbb{Z}$, then a $\mathbb{Z}$ -module is just an Abelian group.
- 4 If $A = k[x]$, k field, then an A -module is a k -vector space together with a linear transformation (which corresponds to the action of x).
- 5 If A is the group ring kG of a group G over a field k , then an A -module is exactly a k -representation of G , i.e., a k -vector space together with a group morphism $G \rightarrow \text{End}(V)$.

Modules and Module Homomorphisms

Definition (Module Homomorphisms)

Given modules M and M' , an map $f : M \rightarrow M'$ is an A -module homomorphism (or is A -linear) if

$$\begin{aligned}f(x + y) &= f(x) + f(y), \\f(ax) &= af(x), \quad a \in A, \ x, y \in M.\end{aligned}$$

Remark

In other words, f is a homomorphism of Abelian groups that commutes with the action of A .

Example

If A is a field k , then a k -module homomorphism is nothing but a linear transformation between k -vector spaces.

Modules and Module Homomorphisms

Definition

The set of all A -module homomorphisms $f : M \rightarrow M'$ is denoted $\text{Hom}_A(M, N)$ (or simply $\text{Hom}(M, N)$ when there is no ambiguity on the ground ring).

Fact

$\text{Hom}_A(M, N)$ is an A -module. Given $f, g \in \text{Hom}_A(M, N)$ and $a \in A$ we define $f + g$ and af by

$$\begin{aligned}(f + g)(x) &= f(x) + g(x), \\ (af)(x) &= af(x), \quad x \in M.\end{aligned}$$

Modules and Module Homomorphisms

Remarks

- ① The composition of A -module homomorphisms is again an A -module homomorphism.
- ② Given homomorphisms $u : M \rightarrow M'$ and $v : N \rightarrow N'$ we get maps,

$$\begin{aligned}\tilde{u} : \operatorname{Hom}_A(M, N) &\rightarrow \operatorname{Hom}_A(M', N), & \tilde{u}(f) &= f \circ u, \\ \tilde{v} : \operatorname{Hom}_A(M, N) &\rightarrow \operatorname{Hom}_A(M, N'), & \tilde{v}(f) &= v \circ f.\end{aligned}$$

These maps are A -module homomorphisms.

- ③ For any module M we have a natural isomorphism,

$$\operatorname{Hom}_A(A, M) \simeq M.$$

Any $f \in \operatorname{Hom}_A(A, M)$ is uniquely determined by $f(1)$.

Submodules and Quotient Modules

Definition (Submodules)

A *submodule* M' of a module M is a subgroup that is closed under the action of A .

Fact

If M' is a submodule of M , then the Abelian group M/M' inherits an A -module structure given by

$$a(x + M') = ax + M', \quad x \in M.$$

Definition (Quotient Modules)

M/M' is called the *quotient* of M by M' .

Facts

- 1 The canonical map $M \rightarrow M/M'$ is an A -module homomorphism.
- 2 There is a one-to-one correspondence between submodules of M that contains M' and submodules of M/M' .

Submodules and Quotient Modules

Definition

Let $f : M \rightarrow N$ be an A -module homomorphism.

- ① The *kernel* of f is

$$\ker(f) = \{x \in M; f(x) = 0\}.$$

This is a submodule of M .

- ② The *image* of f is

$$\operatorname{im}(f) = f(M).$$

This is a submodule of N .

- ③ The *cokernel* of f is

$$\operatorname{coker}(f) = N / \operatorname{im}(f).$$

This is a quotient module of N .

Submodules and Quotient Modules

Facts

Let $f : M \rightarrow N$ be an A -module homomorphism and M' a submodule of M such that $M' \subseteq \ker(f)$.

- f gives rise to a homomorphism $\bar{f} : M/M' \rightarrow N$ defined by

$$\bar{f}(\bar{x}) = f(x),$$

where $\bar{x} \in M/M'$ is the image of $x \in M$.

- The kernel of $\bar{f}$ is $\ker(f)/M'$.

Definition (Induced Homomorphisms)

The homomorphism $\bar{f}$ is said to be *induced* by f .

Remark

For $M' = \ker(f)$ we get an isomorphism,

$$M / \ker(f) \simeq \operatorname{im}(f).$$

Operations on Modules

Definition

Let M be an A -module, and $(M_i)_{i \in I}$ be a family of sub-modules of M . The *sum* $\sum M_i$ consists of all finite sums $\sum x_i$, where $x_i \in M_i$.

Remark

$\sum M_i$ is the smallest sub-module that contains all the M_i .

Facts

- 1 The intersection $\cap M_i$ is again a submodule of M .
- 2 The submodules form a lattice with respect to inclusion.

Proposition (Proposition 2.1)

- ① If $L \supseteq M \supseteq N$ are A -modules, then

$$(L/M)/(M/N) \simeq L/N.$$

- ② If M_1 and M_2 are submodules of M , then

$$(M_1 + M_2)/M_1 \simeq M_2/(M_1 \cap M_2).$$

Definition

If $\mathfrak{a}$ is an ideal of A and M is an A -module, then $\mathfrak{a}M$ consists of all finite sums $\sum a_i x_i$ with $a_i \in \mathfrak{a}$ and $x_i \in M$.

Remarks

- 1 $\mathfrak{a}M$ is a submodule of M .
- 2 In general we cannot define the product of two submodules.

Definition

- ① If N and P are submodules of M , then $(N : P)$ is the set of all $a \in A$ such that $aP \subseteq N$.
- ② $(0 : M)$ is called the *annihilator* of M and is denoted by $\text{Ann}(M)$.

Remarks

- ① $(N : P)$ is an ideal of A .
- ② $\text{Ann}(M)$ consists of all $a \in A$ such that $aM = 0$.

Operations on Modules

Fact

If $\mathfrak{a}$ is an ideal contained in $\text{Ann}(M)$, then we may regard M as an $A/\mathfrak{a}$ -module as follows: if $m \in M$ and $\bar{x} \in A/\mathfrak{a}$ is the class of $x \in A$, then

$$\bar{x}m = xm.$$

This definition makes sense since $\mathfrak{a}M = 0$.

Definition (Faithful Modules)

We say that an A -module M is *faithful* if $\text{Ann}(M) = 0$.

Remark

If $\mathfrak{a} = \text{Ann}(M)$, then M is always faithful as an $A/\mathfrak{a}$ -module.

Exercise (Exercise 2.2)

- (i) $\text{Ann}(M + N) = \text{Ann}(M) \cap \text{Ann}(N)$.
- (ii) $(N : P) = \text{Ann}((N + P)/N)$.

Operations on Modules

Fact

If $x \in M$, the set of all multiples ax with $a \in A$ is a submodule of M denoted by Ax or (x) .

Definition

- If $M = \sum Ax_i$, then we say that the x_i form a *set of generators* of M .
- We say that M is *finitely generated* if it admits a finite set of generators.

Remark

- That the x_i form a set of generators of M means that every $x \in M$ is a finite linear combination $\sum a_i x_i$ with $a_i \in A$.
- This linear combination need not be unique.

Definition (Direct Sum)

- If M and N are A -modules, their *direct sum* $M \oplus N$ consist of all pairs (x, y) with $x \in M$ and $y \in N$.
- This is an A -module with respect to the following addition and scalar multiplication,

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2), \\ a(x, y) &= (ax, ay).\end{aligned}$$

Direct Sum and Product

Definition (Direct Sum and Direct Product)

Let $(M_i)_{i \in I}$ be a family of A -modules.

- 1 The *direct sum* $\bigoplus M_i$ consists of all families $(x_i)_{i \in I}$ where all but finitely many of the x_i are zero.
- 2 The *direct product* $\prod M_i$ consists of all families $(x_i)_{i \in I}$.

Remarks

- 1 The direct sum $\bigoplus M_i$ and the direct product $\prod M_i$ are both A -modules.
- 2 They agree when the index set I is finite.

Facts

Suppose that the ring A is a direct product $\prod_{i=1}^n A_i$.

- 1 Let $\mathfrak{a}_i$ be the set of all elements of $\mathfrak{a}$ of the form

$$(0, \dots, 0, a_i, 0, \dots, 0), \quad a_i \in A_i.$$

This is an ideal of A .

- 2 The ring A , considered as an A -module, agrees with the direct sum $\mathfrak{a}_1 \oplus \dots \oplus \mathfrak{a}_n$.

Facts

Conversely, suppose we have a module decomposition,

$$A = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n,$$

where $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are ideals.

- 1 We have

$$A \simeq \prod_{i=1}^n (A/\mathfrak{b}_i), \quad \text{where } \mathfrak{b}_i = \bigoplus_{j \neq i} \mathfrak{a}_j.$$

- 2 Each ideal $\mathfrak{a}_i$ is a ring isomorphic to $A/\mathfrak{b}_i$.
- 3 The identity element e_i of $\mathfrak{a}_i$ is an idempotent in A and $\mathfrak{a}_i = (e_i)$.

Finitely Generated Modules

Definition (Free Modules)

A *free A -module* is an A -module of the form $\bigoplus_{i \in I} M_i$, where $M_i \simeq A$,

Example

- The direct sum $A^n = A \oplus \cdots \oplus A$ (n summands) is a free module.
- By convention A^0 is the zero module, denoted by 0 .

Fact

Any finitely generated free module is isomorphic to A^n for some n .

Proposition (Proposition 2.3)

An A -module M is finitely generated if and only if it is a quotient of A^n for some $n \geq 1$.

Finitely Generated Modules

Proposition (Cayley-Hamilton Theorem; Proposition 2.4)

Suppose that M is a finitely generated A -module and $\mathfrak{a}$ is ideal of A . Let $\phi : M \rightarrow M$ be an A -module endomorphism such that $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form,

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0, \quad a_i \in \mathfrak{a}.$$

Remarks

- We identify A with its image in $\text{End}_A(M) = \text{Hom}_A(M, M)$.
- The above equality holds in $\text{End}_A(M)$, which is an A -module.

Proof of Proposition 2.4.

- Let $x_1, \dots, x_n$ be generators of M . Then:

$$(*) \quad \phi x_j = a_{1j}x_1 + \cdots + a_{nj}x_n, \quad a_{ij} \in \mathfrak{a}.$$

- Let B be the sub-ring of $\text{End}_A(M)$ generated by ϕ and A . This is a commutative ring.
- Set $a = [a_{ij}] \in M_n(\mathfrak{a})$ and $b = \phi I_n - a \in M_n(B)$. Note that $M_n(B)$ acts on M^n . Then $(*)$ means that

$$bx = 0 \quad \text{with } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Finitely Generated Modules

Proof of Proposition 2.4 (continued).

- Let c be the cofactor matrix of b . As B is a commutative ring, we have

$$cb = \det(b)I_n.$$

- As $bx = 0$, we get

$$0 = cbx = \det(b)x = \begin{bmatrix} \det(b)x_1 \\ \vdots \\ \det(b)x_n \end{bmatrix}.$$

- As $x_1, \dots, x_n$ generate M , this gives $\det(b) = 0$ in $\text{End}_A(M)$.
- Here $b = \phi I_n - a$ with $a \in M_n(\mathfrak{a})$. Expanding the equation $\det(\phi I_n - a) = 0$ shows there are $a_1, \dots, a_n$ in $\mathfrak{a}$ such that

$$\phi^n + a_1\phi^{n-1} + \dots + a_n = 0.$$

- Equivalently, if $P(\lambda) = \det(\lambda I_n - a)$ is the characteristic polynomial of $a = [a_{ij}]$, then $P(\phi) = 0$.



Finitely Generated Modules

Corollary (Corollary 2.5)

Let M be a finitely generated A -module and $\mathfrak{a}$ an ideal of A such that $\mathfrak{a}M = M$. Then there is $x \equiv 1 \pmod{\mathfrak{a}}$ such that $xM = 0$.

Proof.

- Apply Prop. 2.4 to $\phi = \text{id}_M$. There are $a_1, \dots, a_n$ in $\mathfrak{a}$ such that

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0.$$

That is,

$$(1 + a_1 + \cdots + a_n)\text{id}_M = 0.$$

- Thus, if we set $x = 1 + a_1 + \cdots + a_n$, then $x \equiv 1 \pmod{\mathfrak{a}}$, and

$$xy = 0 \quad \forall y \in M.$$

That is, $xM = 0$.

The proof is complete. □

Finitely Generated Modules

Reminder (Proposition 1.9)

If $\mathfrak{R}$ the Jacobson radical of A , then $x \in \mathfrak{R}$ if and only if $1 - xy$ is unit for all $y \in A$.

Proposition (Nakayama's Lemma; Proposition 2.6)

Let M be a finitely generated A module and $\mathfrak{a}$ an ideal of A contained in $\mathfrak{R}$. Then $\mathfrak{a}M = M$ implies that $M = 0$.

Proof.

- By Corollary 2.5 there is $x \equiv 1 \pmod{\mathfrak{R}}$ such that $xM = 0$.
- As $1 - x \in \mathfrak{R}$, by Proposition 1.9 $x = 1 - (1 - x)1$ is a unit.
- Thus,

$$M = x^{-1}(xM) = 0.$$

The proof is complete. □

Finitely Generated Modules

Corollary (Corollary 2.7)

Let M be a finitely generated A module, N a submodule of A , and $\mathfrak{a}$ an ideal of A contained in $\mathfrak{R}$. Then $M = \mathfrak{a}M + N \Rightarrow M = N$.

Proof.

- If $M = \mathfrak{a}M + N$, then

$$\mathfrak{a}(M/N) = (\mathfrak{a}M + N)/N = M/N.$$

- Nakayama's lemma then implies that $M/N = 0$, i.e., $M = N$.

The result is proved. □

Fact

Let A be a local ring, $\mathfrak{m}$ its maximal ideal, and $k = A/\mathfrak{m}$ its residue field. Let M be a finitely generated A -module M . Then:

- The quotient module $V = M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, and hence this is an $A/\mathfrak{m}$ -module, i.e., a vector space over k .
- This vector space has finite dimension.

Finitely Generated Modules

Proposition (Proposition 2.8)

Let $x_1, \dots, x_n$ be elements in M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then $x_1, \dots, x_n$ generate M .

Proof.

- Let N be the module generated by $x_1, \dots, x_n$, and let $\phi : M \rightarrow M/\mathfrak{m}M$ be the canonical homomorphism.
- As $\{\phi(x_1), \dots, \phi(x_n)\}$ is a basis of $M/\mathfrak{m}M$, we see that $\phi(N) = M/\mathfrak{m}M$.
- Thus,

$$M = \phi^{-1}(M/\mathfrak{m}M) = \phi^{-1}(\phi(N)) = N + \mathfrak{m}M.$$

- Corollary 2.7 then implies that $M = N$, i.e., $x_1, \dots, x_n$ generate M .

The result is proved. □

Exact Sequences

Definition (Exact Sequences)

A sequence of A -modules and A -homomorphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is said to be *exact at M_i* if $\text{im}(f_i) = \ker(f_{i+1})$. It is called an *exact sequence* if it is exact at each M_i .

Examples

- ❶ A sequence $0 \rightarrow M' \xrightarrow{f} M$ is exact if and only if f is injective.
- ❷ A sequence $M \xrightarrow{g} M'' \rightarrow 0$ is exact if and only if g is surjective.
- ❸ A sequence $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact if and only if f is injective, g is surjective, and g induces an isomorphism of $\text{coker}(f) = M/f(M')$ onto M'' . Such a sequence is called a *short exact sequence*.

Remark

- Any long exact sequence,

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

can be split up into short exact sequences.

- If we set $N_i = \operatorname{im}(f_i) = \ker(f_{i+1})$, then, for each i , we have a short exact sequence,

$$0 \longrightarrow N_i \longrightarrow M_i \xrightarrow{f_i} N_{i+1} \longrightarrow 0.$$

Proposition (Proposition 2.9; see Carlson)

- ① A sequence of the form

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact if and only if, for every A -module N , the sequence

$$0 \longrightarrow \operatorname{Hom}(M'', N) \xrightarrow{\tilde{v}} \operatorname{Hom}(M, N) \xrightarrow{\tilde{u}} \operatorname{Hom}(M', N)$$

is exact.

- ② A sequence of the form

$$0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N''$$

is exact if and only if, for every A -module M , the sequence

$$0 \longrightarrow \operatorname{Hom}(M, N') \xrightarrow{\tilde{u}} \operatorname{Hom}(M, N) \xrightarrow{\tilde{v}} \operatorname{Hom}(M, N'')$$

is exact.

Exact Sequences

Proposition (Snake Lemma, Proposition 2.10; see Carlson)

Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of A -modules and A -homomorphisms with exact rows. Then there exists an exact sequence,

$$\begin{aligned} 0 \longrightarrow \ker(f') &\xrightarrow{\bar{u}} \ker(f) \xrightarrow{\bar{v}} \ker(f'') \xrightarrow{d} \\ &\text{coker}(f') \xrightarrow{\bar{u}'} \text{coker}(f) \xrightarrow{\bar{v}'} \text{coker}(f'') \longrightarrow 0, \end{aligned}$$

where $\bar{u}, \bar{v}$ are restrictions of u, v , and $\bar{u}', \bar{v}'$ are induced by u', v' .

Remark

The map $d : \ker(f'') \rightarrow \operatorname{coker}(f')$ is called *boundary homomorphism*. It is constructed as follows:

- If $x'' \in \ker(f'')$, we have $x'' = v(x)$ for some $x \in M$ (since v is surjective).
- We have $v'(f(x)) = f''(v(x)) = f''(x'') = 0$, and so $f(x) \in \ker(v') = \operatorname{ran}(u')$.
- As u' is injective, there is a unique $y' \in N'$ such that $f(x) = u'(y')$.
- We define dx'' to be the class of y' in $\operatorname{coker}(f') = N' / \operatorname{im}(f')$.
- It can be shown that the class of y' does not depend on the choice of x , and dx'' is well defined.

Exact Sequences

Definition

Let $\mathcal{C}$ be a class of A -modules. A function $\lambda : \mathcal{C} \rightarrow \mathbb{Z}$ is *additive* if, for every short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in $\mathcal{C}$, we have

$$\lambda(M') - \lambda(M) + \lambda(M'') = 0.$$

Example

Let A be a field k and $\mathcal{C}$ be the class of finite-dimension vector spaces over k . Then the function $V \rightarrow \dim V$ is additive.

Proposition (Proposition 2.11)

Let $0 \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow 0$ be an exact sequence in $\mathcal{C}$. Then, for any additive function λ on $\mathcal{C}$, we have

$$\sum_{0 \leq i \leq n} (-1)^i \lambda(M_i) = 0.$$

Tensor Product of Modules

Definition (A -Bilinear Maps)

Given A -modules M , N , and P , an A -bilinear map is any map $f : M \times N \rightarrow P$ such that

- (i) For every $x \in M$, the map $N \ni y \rightarrow f(x, y) \in P$ is A -linear.
- (ii) For every $y \in N$, the map $M \ni x \rightarrow f(x, y) \in P$ is A -linear.

Fact

Given A -modules M and N , their tensor product $M \otimes_A N$ is an A -module such that A -bilinear maps $M \times N \rightarrow P$ are in one-to-one correspondence with A -linear maps $M \otimes_A N \rightarrow P$.

Tensor Product of Modules

Proposition (Proposition 2.12)

Let M, N be A -modules.

- 1 There exist an A -module $M \otimes_A N$ and an A -bilinear map $\otimes : M \times N \rightarrow M \otimes_A N$ satisfying the following universal property:

For any A -module P and A -bilinear map $f : M \times N \rightarrow P$ there is a unique A -linear map $f' : M \otimes_A N \rightarrow P$ such that

$$f(x, y) = f'(x \otimes y) \quad \text{for all } x \in M \text{ and } y \in N.$$

- 2 If $(M \otimes' N, \otimes')$ is another pair satisfying the above universal property, then there is a unique isomorphism $j : M \otimes_A N \rightarrow M \otimes' N$ such that

$$j(x \otimes y) = x \otimes' y \quad \text{for all } x \in M \text{ and } y \in N.$$

Tensor Product of Modules

Remark

The A -module $M \otimes_A N$ is constructed as follows:

- Let C be the free A -module $\bigoplus_{(x,y) \in M \times N} A$ generated by all pairs $(x, y) \in M \times N$. It consists of finite formal linear combinations $\sum a_j(x_j, y_j)$ with $a_j \in A$ and $(x_j, y_j) \in M \times N$.
- Let D be the submodule generated by elements of the form,
$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),$$
$$(ax, y) - a(x, y), \quad (x, ay) - a(x, y).$$
- The A -module $M \otimes_A N$ is the quotient module C/D . If $(x, y) \in M \times N$, we denote by $x \otimes y$ the class of (x, y) in $M \otimes_A N$. From the definition we have

$$(x + x') \otimes y = x \otimes y + x' \otimes y, \quad x \otimes (y + y') = x \otimes y + x \otimes y',$$
$$(ax) \otimes y = a(x \otimes y), \quad x \otimes (ay) = a(x \otimes y).$$

That is, $\otimes : M \times N \rightarrow M \otimes_A N$ is an A -bilinear map.

Remarks

- ① We often denote $M \otimes_A N$ by $M \otimes N$ when the ring A is understood from context.
- ② In practice, we will not need the construction of the tensor product. What is essential is to keep in mind its universal property.
- ③ If $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are generator sets of M and N , respectively, then the elements $x_i \otimes y_j$ generate $M \otimes N$. In particular, if M and N are finitely generated, then $M \otimes N$ is finitely generated as well.

Tensor Product of Modules

Remark

- Let M' and N' be submodules of M and N , respectively. If $x \in M'$ and $y \in N'$, then it may happen that $x \otimes y$ is zero as an element of $M \otimes N$, but is not zero as an element of $M' \otimes N'$.
- Take $A = M = \mathbb{Z}$, $N = N' = \mathbb{Z}/2\mathbb{Z}$ and $M' = 2\mathbb{Z}$. Let x be the non-zero element of $\mathbb{Z}/2\mathbb{Z}$. Then $2 \otimes x$ is not zero in $M' \otimes N'$ since it generates $M' \otimes N'$. However, it is zero in $M \otimes N$, since we have

$$2 \otimes x = 2(1 \otimes x) = 1 \otimes (2x) = 0.$$

- Nevertheless, we have the following result:

Corollary (Corollary 2.13)

Let $x_i \in M$ and $y_i \in N$ be such that $\sum x_i \otimes y_i = 0$ in $M \otimes N$. Then there are finitely generated submodules M_0 of M and N_0 of N such that $\sum x_i \otimes y_i = 0$ in $M_0 \otimes N_0$.

Tensor Product of Modules

Remark

- We can also define multi-tensor products $M_1 \otimes \cdots \otimes M_r$ by using multilinear maps instead of
- A map $M_1 \times \cdots \times M_r \rightarrow P$ is multilinear if it is linear with respect to each argument.

Proposition (Proposition 2.12*; see Carlson)

- ① *There exist an A -module $M_1 \otimes \cdots \otimes M_r$ and an multilinear map $\otimes \cdots \otimes : M_1 \times \cdots \times M_r \rightarrow M_1 \otimes \cdots \otimes M_r$ satisfying the following universal property:*

For any A -module P and multilinear map $f : M_1 \times \cdots \times M_r \rightarrow P$ there is a unique A -linear map $f' : M_1 \otimes \cdots \otimes M_r \rightarrow P$ such that

$$f(x_1, \dots, x_r) = f'(x_1 \otimes \cdots \otimes x_r) \quad \text{for all } x_i \in M_i.$$

- ② *The pair $(M_1 \otimes \cdots \otimes M_r, \otimes \cdots \otimes)$ is unique up to isomorphism.*

Tensor Product of Modules

Proposition (Proposition 2.14; see Atiyah-MacDonald and Carlson)

Let M, N, P be A -modules. Then we have canonical isomorphisms:

- (i) $M \otimes N \simeq N \otimes M$, where $x \otimes y \rightarrow y \otimes x$.
- (ii) $(M \otimes N) \otimes P \simeq M \otimes (N \otimes P) \simeq M \otimes N \otimes P$, where
$$(x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z) \longrightarrow x \otimes y \otimes z.$$
- (iii) $(M \oplus N) \otimes P \simeq (M \otimes P) \oplus (N \otimes P)$, where
$$(x + y) \otimes z \longrightarrow (x \otimes z) + (y \otimes z).$$
- (iv) $A \otimes M \simeq M$, where $a \otimes x \rightarrow ax$.

Tensor Product of Modules

Definition (Bimodules)

Given rings A and B , an (A, B) -bimodule is an Abelian group N which is both an A -module and a B -module and the two structures are compatible in the sense that

$$a(xb) = (ax)b, \quad a \in A, x \in M, b \in B.$$

Exercise (Exercise 2.15; see Carlson)

Suppose that M is A -module, P is an B -module, and N is an (A, B) -bimodule. Then:

- $M \otimes_A N$ is naturally a B -module.
- $N \otimes_B P$ is naturally an A -module.
- We have a natural isomorphism of (A, B) -bimodules,

$$(M \otimes_A N) \otimes_B P \simeq M \otimes_A (N \otimes_B P).$$

Tensor Product of Modules

Facts

- If $f : M \rightarrow M'$ and $g : N \rightarrow N'$ are A -linear maps, then $M \times N \ni (x, y) \rightarrow f(x) \otimes g(y) \in M' \otimes N'$ is an A -bilinear map.
- Therefore, there is a unique A -linear map $f \otimes g : M \otimes N \rightarrow M' \otimes N'$ such that

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y) \quad \text{for all } x \in M \text{ and } y \in N.$$

- If $f' : M' \rightarrow M''$ and $g' : N' \rightarrow N''$ are A -linear maps, then

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g)$$

Fact

Let $f : A \rightarrow B$ be a ring homomorphism.

- Any B -module N can be turned into an A -module as follows: if $a \in A$ and $x \in N$, then ax is defined to be $f(a)x$.
- This A -module is said to be obtained from N by *restriction of scalars*.
- In particular, B is a module over A this way,

Restrictions and Extensions of Scalars

Proposition (Proposition 2.16)

If N is finitely generated as a B -module and B is finitely generated as an A -module, then N is finitely generated as an A -module.

Proof.

- Let $y_1, \dots, y_n$ generate N over B , and let $x_1, \dots, x_m$ generate B over A .
- If $y \in N$, then $y = \sum b_j y_j$ with $b_j \in B$.
- Write $b_j = \sum a_{ij} x_i$ with $a_{ij} \in A$. Then

$$y = \sum_{j=1}^n b_j y_j = \sum_{j=1}^n \sum_{i=1}^m a_{ij} x_i y_j.$$

- Therefore N is generated by the $x_i y_j$ over A , and hence is finitely generated over A .

The proof is complete



Restrictions and Extensions of Scalars

Fact

Let M be an A -module.

- As B is an A -module, we can form the tensor product $M_B = B \otimes_A M$.
- In fact, M_B is a B -module such that

$$b(b' \otimes x) = (bb') \otimes x, \quad b, b' \in B, x \in M.$$

- We say that M_B is obtained from M by *extensions of the scalars*.

Proposition (Proposition 2.17)

If M is finitely generated as an A -module, then M_B is finitely generated as a B -module.

Proof.

If $x_1, \dots, x_n$ generate M over A , then $1 \otimes x_1, \dots, 1 \otimes x_n$ generate M_B over B . □

Fact

Let S be the set of all A -bilinear maps $f : M \times N \rightarrow P$.

- S is an A -module.
- If $f : M \times N \rightarrow P$ is bilinear, then, for every $x \in M$, we have an A -linear map $y \rightarrow f(x, y)$. It depends linearly on x and f , and so we get an A -linear map $S \rightarrow \text{Hom}_A(M, \text{Hom}_A(N, P))$.
- Conversely, any $\phi \in \text{Hom}_A(M, \text{Hom}_A(N, P))$ gives rise an A -bilinear map $(x, y) \rightarrow (\phi(x))(y)$.
- Therefore, we have a canonical isomorphism,

$$S \simeq \text{Hom}_A(M, \text{Hom}_A(N, P)).$$

Exactness Properties of the Tensor Product

Fact

Thanks to the defining property of the tensor product we also have a canonical isomorphism,

$$S \simeq \operatorname{Hom}_A(M \otimes N, P).$$

Consequence

We have a canonical isomorphism,

$$\operatorname{Hom}_A(M \otimes N, P) \simeq \operatorname{Hom}_A(M, \operatorname{Hom}_A(N, P)).$$

Remark

In the language of functors on the category of A -modules, the above result means that the functor $- \otimes_A N : M \rightarrow M \otimes N$ is the left adjoint of the functor $\operatorname{Hom}_A(N, -) : P \rightarrow \operatorname{Hom}_A(N, P)$ (and hence $\operatorname{Hom}_A(N, -)$ is the right adjoint of $- \otimes_A N$).

Exactness Properties of the Tensor Product

Proposition (Proposition 2.18)

Suppose we are given an exact sequence of A -modules and homomorphisms of the form,

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

Then, for every A -module N , we have an exact sequence,

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \longrightarrow 0,$$

where we have denoted by 1 the identity map of N .

Exactness Properties of the Tensor Product

Proof of Proposition 2.18 (Sketch).

Let P be any A -module.

- As $M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, by Proposition 2.9 we get an exact sequence,

$$0 \rightarrow \operatorname{Hom}(M'', \operatorname{Hom}(N, P)) \rightarrow \operatorname{Hom}(M, \operatorname{Hom}(N, P)) \rightarrow \operatorname{Hom}(M', \operatorname{Hom}(N, P)).$$

- As $\operatorname{Hom}(Q, \operatorname{Hom}(N, P)) \simeq \operatorname{Hom}(Q \otimes N, P)$, we get an exact sequence,

$$0 \longrightarrow \operatorname{Hom}(M'' \otimes N, P) \longrightarrow \operatorname{Hom}(M \otimes N, P) \longrightarrow \operatorname{Hom}(M' \otimes N, P).$$

- As this is true for any A -module P , by Proposition 2.9 again we get an exact sequence,

$$M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \longrightarrow 0.$$

This gives the result. □

Remarks

- ① Proposition 2.18 means that, for every A -module N , the functor $- \otimes_A N$ is *left exact*.
- ② More generally, it can be shown that any functor that has a left adjoint is left exact.
- ③ Likewise, any functor that has a right adjoint is right exact.

Exactness Properties of the Tensor Product

Remark

The functor $- \otimes_A N$ need not be exact, i.e., if $M' \rightarrow M \rightarrow M''$ is an exact sequence, then the sequence

$M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N$ may fail to be exact.

Example

Take $A = \mathbb{Z}$ and consider the exact sequence,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f}, \quad f(x) = 2x.$$

- If we take tensor products with $N = \mathbb{Z}/2\mathbb{Z}$, then the sequence

$$0 \rightarrow \mathbb{Z} \otimes N \xrightarrow{f \otimes 1} \mathbb{Z} \otimes N \text{ is not exact.}$$

- Indeed, given any $x \in \mathbb{Z}$ and $y \in \mathbb{Z}/2\mathbb{Z}$, we have

$$(f \otimes 1)(x \otimes y) = f(x) \otimes y = 2x \otimes y = x \otimes (2y) = 0.$$

- This implies that $(f \otimes 1) = 0$, i.e., $\text{ran}(f \otimes 1) = 0 \neq \mathbb{Z} \otimes N$. Therefore, the sequence is not exact.

Exactness Properties of the Tensor Product

Definition (Flat Module)

We say that an A -module is *flat* if the functor $- \otimes_A N$ is exact, i.e., for every exact sequence $M' \rightarrow M \rightarrow M''$, the sequence $M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N$ is again exact.

Proposition (Proposition 2.19; see also Gaillard)

Let N be an A -module. Then TFAE:

- (i) N is flat.
- (ii) If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence, then so is the tensored sequence $0 \rightarrow M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0$.
- (iii) If $f : M' \rightarrow M$ is injective, then so is $f \otimes 1 : M' \otimes N \rightarrow M \otimes N$.
- (iv) If $f : M' \rightarrow M$ is injective and M and M' are finitely generated, then $f \otimes 1 : M' \otimes N \rightarrow M \otimes N$ is again injective.

Exercise (Exercise 2.20; see Carlson)

If $A \rightarrow B$ is a ring homomorphism, and M is a flat A -module, then $M_B = B \otimes_A M$ is a flat B -module.

Hint: Use the isomorphisms from Proposition 2.14 and Exercise 2.15.

Definition

An A -algebra is an A -module B together with a multiplication $(b, b') \rightarrow bb'$ which is A -bilinear and with respect to which B is a ring.

Remark

The A -bilinearity of the multiplication means that

$$a(bb') = (ab)b' = b(ab') \quad a \in A, b, b' \in B.$$

This accounts for the compatibility of the module and ring structures of B .

Example

Let B be a ring and let $f : A \rightarrow B$ be a ring homomorphism.

- By restriction B is an A -module with $ab = f(a)b$.
- If B is commutative or the image of f is contained in the center of B , then B is an A -algebra, since

$$f(a)(bb') = (f(a)b)b' = b(f(a)b'), \quad a \in A, \quad b, b' \in B.$$

Fact

Let B an A -algebra with an identity element 1_B , and define $f : A \rightarrow B$ by

$$f(a) = a1_B, \quad a \in A.$$

Then f is a ring homomorphism whose image is contained in the center of A , since

$$\begin{aligned} (a1_B)(a'1_B) &= a[1_B(a'1_B)] = a[a'(1_B1_B)] = (aa')1_B, \\ (a1_B)b &= a(1_Bb) = a(b1_B) = b(a1_B), \quad a, a' \in A, \quad b \in B. \end{aligned}$$

Consequences

- An A -algebra with identity is exactly a ring B with identity together with a ring homomorphism $f : A \rightarrow B$ whose image is contained in the center of B .
- A commutative A -algebra with identity is exactly a commutative ring B with identity together with a ring homomorphism $f : A \rightarrow B$.

Remark

- In Atiyah-MacDonald's book rings are assumed to be commutative and have an identity.
- Therefore, A -algebras are assumed to be commutative and to have an identity element.
- Atiyah-MacDonald then defines an A -algebra as a ring B together a ring homomorphism $f : A \rightarrow B$.

Remark

Suppose that A is a field k and B is a k -algebra with identity element.

- As k is a field, the ring homomorphism $k \ni \lambda \rightarrow \lambda 1_B$ must be injective (see Proposition 1.2).
- Thus, k can be identified as a subring of B .

Therefore, a k algebra (with identity) is a ring that contains k as a subring.

Examples

- ① $A = k$ and $B = k[x_1, \dots, x_n]$ (polynomials with n variables).
- ② $A = k$ and $B = kG$ (group ring of a group G). This is not a commutative algebra unless G is Abelian.

Remark

If B is a ring with identity, then $\mathbb{Z} \ni m \rightarrow m1 \in B$ is a ring homomorphism. Therefore, any such ring is automatically a $\mathbb{Z}$ -algebra.

Definition (Algebra Homomorphisms)

Given A -algebras B and C , a map $h : B \rightarrow C$ is called an A -algebra homomorphism if it is both a ring homomorphism and an A -module homomorphism.

Remark

Suppose that B and C have identities with ring homomorphisms $f : A \rightarrow B$ and $g : A \rightarrow B$. Then, for every A -algebra homomorphism $h : B \rightarrow C$, we have $g = h \circ f$.

Definition (Finite Algebras)

A *finite A -algebra* is an A -algebra which is finitely generated as an A -module.

Definition (Finitely Generated Algebras)

Let B be a commutative A -algebra with identity. We say that B is a *finitely generated A -algebra* if there is a finite set of elements $x_1, \dots, x_n$ such that every element of B is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$.

Remark

The above condition means that we have surjective A -algebra homomorphism $A[X_1, \dots, X_n] \rightarrow B$ that maps X_i to x_i . Every $b \in B$ is a polynomial in the generators $x_1, \dots, x_n$.

Definition (Finitely Generated Rings)

A ring A is said to be *finitely generated* if it is finitely generated as a $\mathbb{Z}$ -algebra.

Tensor Product of Algebras

Proposition

Let B and C be A -algebras. Then their tensor product $B \otimes_A C$ is an A -algebra whose product is such that

$$(b \otimes c)(b' \otimes c') = (bb') \otimes (cc'), \quad b, b' \in B, \quad c, c' \in C.$$

Remark

In general, we have

$$\left(\sum_i b_i \otimes c_i \right) \left(\sum_j b'_j \otimes c'_j \right) = \sum_{i,j} (b_i b'_j) \otimes (c_i c'_j).$$

Tensor Product of Algebras

Remark

The product of $B \otimes_A C$ is well defined:

- Consider the multilinear map,

$$B \times C \times B \times C \ni (b, c, b', c') \rightarrow (bb') \otimes (cc') \in B \otimes C.$$

- It gives rise to an A -module homomorphism,

$$B \otimes C \otimes B \otimes C \longrightarrow B \otimes C.$$

- As $B \otimes C \otimes B \otimes C \simeq (B \otimes C) \otimes (B \otimes C)$, we get an A -module homomorphism,

$$(B \otimes C) \otimes (B \otimes C) \longrightarrow B \otimes C.$$

- This then gives rise to an A -bilinear map,

$$(B \otimes C) \times (B \otimes C) \longrightarrow B \otimes C,$$

which is our product.

Remark

Suppose that B and C are commutative algebras with identity elements 1_B and 1_C .

- The algebra $B \otimes C$ is commutative and has $1_B \otimes 1_C$ as identity element.
- We have natural algebra homomorphisms,

$$\text{id}_B \otimes 1_C : B \longrightarrow B \otimes C, \quad b \mapsto b \otimes 1_C,$$

$$1_B \otimes \text{id}_C : C \longrightarrow B \otimes C, \quad c \mapsto 1_B \otimes c.$$

Remark (Continued)

We actually have a commutative diagram,

