Commutative Algebra
Chapter 8: Artinian Rings

Sichuan University, Fall 2021

Reminder from Chapter 6

Definition

A module M over a ring A is *Artinian* if it satisfies one of the following equivalent conditions:

(i) *Descending chain condition (d.c.c.):* Every descending sequence of submodules $M_1 \supseteq M_2 \supseteq \cdots$ is sationnary.

(ii) *Minimal condition:* Every non-empty set of submodules of M has a minimal element.

Definition

An *Artin ring* is a ring which is Artinian as a module over itself, i.e., it satisfies d.c.c. and the minimal condition on ideals.

Reminder from Chapter 6

Examples (see Chapter 6)

1. Every field is an Artin ring.
2. The rings $\mathbb{Z}$ and $k[x]$ (k field) are Noetherian, but are not Artin rings.

Proposition (Corollary 6.6)

Let A be an Artin ring and $\mathfrak{a}$ an ideal of A. Then $A/\mathfrak{a}$ is an Artin ring as well.

Proposition

In an Artin ring A every prime ideal is maximal.

Proof.

- Let $\mathfrak{p}$ be a prime ideal and set $B = A/\mathfrak{p}$.
- As $\mathfrak{p}$ is prime B is an integral domain. As A is an Artin ring, by Corollary 6.6 B is an Artin ring.
- Let $x \in B \setminus 0$. By (d.c.c) the chain of ideals $(x) \supseteq (x^2) \supseteq \cdots$ is stationary, so $(x^n) = (x^{n+1})$ for some n.
- Thus, $x^n = x^{n+1}y$ for some $y \in B$, i.e., $x^n(xy - 1) = 0$.
- As B is an integral domain and $x \neq 0$, $xy = 1$, i.e., x is a unit.
- Thus, every $x \in B \setminus 0$ is a unit, so $B = A/\mathfrak{p}$ is a field, and hence $\mathfrak{p}$ is maximal.

The proof is complete. □

Reminder (see Chapter 1)

Let A be a ring.

1. The nilradical of A is the intersection of its maximal ideals.
2. The Jacobson radical of A is the intersection of its prime ideals.

Corollary (Corollary 8.2)

In an Artin ring A its nilradical and its Jacobson radical agree.

Proof.

- Let $\mathfrak{N}$ be the nilradical of A and $\mathfrak{R}$ its Jacobson radical.
- By Proposition 8.1 we have

$$\mathfrak{N} = \bigcap \{\text{maximal ideals of } A\}$$
$$= \bigcap \{\text{prime ideals of } A\} = \mathfrak{R}.$$

The result is proved. □

Artin Rings

Reminder (see Proposition 1.11)

If $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are ideals of a ring A such that $\cap \mathfrak{a}_i$ is a prime ideal $\mathfrak{p}$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i.

Proposition (Proposition 8.3)

If A is an Artin ring, then it has only finitely many maximal ideals.

Proof of Proposition 8.3.

- Let Σ be the set of ideals of the form,

$$\mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n, \qquad \mathfrak{m}_i \text{ maximal.}$$

- As A is an Artin ring, Σ has a minimal element $\mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$.
- Let $\mathfrak{m}$ be a maximal ideal. Then $\mathfrak{m} \cap \mathfrak{a} \in \Sigma$ and $\mathfrak{m} \cap \mathfrak{a} \subseteq \mathfrak{a}$.
- As $\mathfrak{a}$ is minimal, $\mathfrak{a} \cap \mathfrak{m} = \mathfrak{m}$, and hence $\mathfrak{m} \subseteq \mathfrak{a}$.
- As $\mathfrak{m}$ is maximal, $\mathfrak{m} = \mathfrak{a} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$.
- As $\mathfrak{m}$ is prime, by Proposition 1.11 $\mathfrak{m} = \mathfrak{a}_i$ for some i.
- Therefore, $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are the only maximal ideals of A.

The proof is complete. □

Proposition (Proposition 8.4)

If A is an Artin ring, then its nilradical $\mathfrak{N}$ is nilpotent.

Proof.

- $\mathfrak{N} \supseteq \mathfrak{N}^2 \supseteq \cdots$ is a descending chain of ideals, and hence is stationary since A is an Artin ring.
- Thus, there is k such that $\mathfrak{N}^k = \mathfrak{N}^j$ for all $j \geq k$.
- Assume $\mathfrak{N}^k \neq 0$, and set $\mathfrak{a} = \mathfrak{N}^k$.
- Let Σ be the set of ideals $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} \neq 0$.
- Note that $\mathfrak{a} \in \Sigma$, since $\mathfrak{a}^2 = (\mathfrak{N}^k)^2 = \mathfrak{N}^{2k} = \mathfrak{N}^k = \mathfrak{a} \neq 0$, and hence $\Sigma \neq \emptyset$.
- As A is an Artin ring, Σ has a minimal element $\mathfrak{c}$. □

Proof of Proposition 8.4; Continued.

- Let $x \in \mathfrak{c} \setminus 0$ be such that $x\mathfrak{a} \neq 0$. Then $(x) \in \Sigma$
- As $(x) \subseteq \mathfrak{c}$ and $\mathfrak{c}$ is minimal, $\mathfrak{c} = (x)$.
- We have $(x\mathfrak{a})\mathfrak{a} = x\mathfrak{a}^2 = x\mathfrak{a} \neq 0$, and hence $x\mathfrak{a} \in \Sigma$.
- As $x\mathfrak{a} \subseteq (x)$ and $(x) = \mathfrak{c}$ is minimal, $x\mathfrak{a} = (x)$.
- Thus $xy = x$ for some $y \in \mathfrak{a}$.
- By induction $x = xy^j$ for all $j \geq 0$.
- Note that $y \in \mathfrak{a} = \mathfrak{N}^k \subseteq \mathfrak{N}$, and hence y is nilpotent, i.e., $y^m = 0$ for some $m \geq 1$.
- Thus, $x = xy^m = 0$ (contradiction), and hence $\mathfrak{N}^k = 0$, i.e., $\mathfrak{N}$ is nilpotent.

The proof is complete. □

Artin Rings

Definition

Let A be a ring, $A \neq 0$.

1. A *chain* of prime ideals is a finite strictly increasing sequence,

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

2. The *length* of such a chain is n.
3. The *dimension* $\dim A$ is the supremum of the lengths of all the chains of prime ideals.

Example

If k is a field, then $\dim k = 0$.

Proof.

The only prime ideal of k is □

More generally, we have the following:

Lemma 8.4*

Let A be a non-zero ring. TFAE:

(i) $\dim A = 0$.

(ii) *Every prime ideal is maximal.*

Reminder (Corollary 1.4)

Every ideal $\mathfrak{a}$ of a ring A is contained in a maximal ideal.

Proof of Lemma 8.4* (i)⇒(ii).

- Let $\mathfrak{p}$ be a non-maximal prime ideal and $\mathfrak{m}$ a maximal containing $\mathfrak{p}$.
- As $\mathfrak{p} \subsetneq \mathfrak{m}$, since $\mathfrak{p}$ is not maximal, we have a chain of prime ideals of length 1.
- Thus, $\dim A \geq 1$, i.e., $\dim A \neq 0$.
- By contraposition (i)⇒(ii). □

Proof of Lemma 8.4* (ii)⇒(i).

- If $\dim A \neq 0$, then there is a chain of prime ideals of length ≥ 1.
- Thus, we can find prime ideals $\mathfrak{p}_0$ and $\mathfrak{p}_1$ s.t. $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$.
- In particular, $\mathfrak{p}_0$ is not maximal.
- By contraposition (ii)⇒(i). □

Example

If A is an Artin ring, then $\dim A = 0$.

Proof.

- By Proposition 8.1 every prime ideal of A is maximal.
- Thus, $\dim A = 0$ by Lemma 8.4*. □

Example

The ring $\mathbb{Z}$ has dimension 1.

Proof.

- The only prime ideals of $\mathbb{Z}$ are
$$(0), \qquad (p), \quad p \text{ prime number}$$
- Each ideal (p), p prime, is maximal, since $\mathbb{Z}/(p) = \mathbb{Z}/p\mathbb{Z}$ is a field.
- As (0) is not maximal, $\dim A \geq 1$ by Lemma 8.4*.
- If $\dim \mathbb{Z} \geq 2$, then there are primes p_1 and p_2 such that $(p_1) \subsetneq (p_2)$, and hence (p_1) is not maximal (contradiction).
- Thus, $\dim \mathbb{Z} \leq 1$, and hence $\dim \mathbb{Z} = 1$.

□

Artin Rings

Reminder (Corollary 6.11)

Let A be a ring such that there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ so that $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$. Then A is Noetherian if and only if it is Artinian.

Theorem (Theorem 8.5)

Let A be a non-zero ring. TFAE:

1. *A is an Artin ring.*
2. *A is Noetherian and $\dim A = 0$.*

Proof of Theorem 8.5 (i)⇒(ii).

- Suppose that A is an Artin ring.
- By Lemma 8.4* $\dim A = 0$.
- By Proposition 8.3 A has finitely many (distinct) maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$.
- By Proposition 8.4 the nilradical $\mathfrak{N}$ is nilpotent, i.e., $\mathfrak{N}^k = 0$ for some $k \geq 1$.
- Thus, $\mathfrak{m}_1^k \cdots \mathfrak{m}_n^k \subseteq (\cap \mathfrak{m}_i)^k = \mathfrak{N}^k = 0$.
- By Corollary 6.11 A is Noetherian.

□

Artin Rings

Reminder (Theorem 7.13)

In a Noetherian ring every ideal admits a primary decomposition.

Reminder (see Corollary 7.16)

Suppose that A is Noetherian and $\mathfrak{m}$ is a maximal ideal. If $\mathfrak{q}$ is $\mathfrak{m}$-primary, then $\mathfrak{q} \supseteq \mathfrak{m}^k$ for some $k \geq 1$.

Proof of Theorem 8.5 (ii)⇒(i).

- Suppose that A is Noetherian and $\dim A = 0$.
- By Lemma 8.4* every prime ideal of A is maximal.
- By Theorem 7.13 $(0) = \cap_{i=1}^{n} \mathfrak{q}_i$, where $\mathfrak{q}_i$ is primary.
- Set $\mathfrak{m}_i = r(\mathfrak{q}_i)$. This is a prime ideal, and hence $\mathfrak{m}_i$ is maximal.
- By Corollary 7.16 $\mathfrak{m}_i^{k_i} \subseteq \mathfrak{q}_i$ for some $k_i \geq 1$.
- Then $\mathfrak{m}_1^{k_1} \cdots \mathfrak{m}_n^{k_n} \subseteq \mathfrak{q}_1 \cdots \mathfrak{q}_n \subseteq \cap \mathfrak{q}_i = (0)$.
- By Corollary 6.11 A is an Artin ring.

The proof is complete. □

Corollary 8.5*

Let A be a Noetherian ring. TFAE:

(i) *A has a unique prime ideal.*

(ii) *A is an Artin local ring.*

Proof of (i)⇒(ii).

- As A has a unique prime ideal, it has a unique maximal ideal, and hence is a local ring.
- This also implies that $\dim A = 0$.
- As A is Noetherian, A is an Artin ring by Theorem 8.5. □

Proof of (ii)⇒(i).

- As A is a local ring, it has a unique maximal ideal.
- As A is an Artin ring, every prime ideal is maximal.
- Thus, A has a unique prime ideal. □

Example

Let p be a prime and $n \geq 1$, then $\mathbb{Z}/p^n\mathbb{Z}$ is an Artin local ring.

Proof.

- We have a one-to-one correspondance between prime ideals of $\mathbb{Z}$ containing (p^n) and ideals of $\mathbb{Z}/p^n\mathbb{Z}$.
- (p) is the unique prime ideal containing (p^n).
- Thus, its image in $\mathbb{Z}/p^n\mathbb{Z}$ is the unique prime ideal.
- By Lemma 8.5* $\mathbb{Z}/p^n\mathbb{Z}$ is an Artin local ring. □

Artin Rings

Facts

Let A be an Artin local ring with maximal ideal $\mathfrak{m}$. Then:

- $\mathfrak{m}$ is the unique prime ideal of A, and so $\mathfrak{m}$ agrees with the nilradical.
- Thus, every element of $\mathfrak{m}$ is nilpotent.
- In addition, $\mathfrak{m}$ is nilpotent by Proposition 8.4.
- Every non-unit is contained in $\mathfrak{m}$, and hence is nilpotent.
- Thus, any $x \in A$ is either a unit or is nilpotent.

Reminder (Nakayama's Lemma; Proposition 2.6)

Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal contained in the Jacobson radical of A. Then $\mathfrak{a}M = 0 \Rightarrow M = 0$.

Corollary

If $\mathfrak{a}$ *be an ideal of* A *and* $\mathfrak{m}$ *a maximal ideal, then* $\mathfrak{m}\mathfrak{a} = 0 \Rightarrow \mathfrak{a} = 0$.

Proposition (Proposition 8.6)

Let A be a Noetherian local ring and $\mathfrak{m}$ its maximal ideal. Then only one of the following statements hold:

(i) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ *for all* n.

(ii) $\mathfrak{m}^n = 0$ *for some* n, *in which case* A *is an Artin local ring.*

Proof.

- Suppose that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n, i.e., $\mathfrak{m} \cdot \mathfrak{m}^n = \mathfrak{m}^n$.
- As $\mathfrak{m}$ is maximal, by the corollary of Nakayama's lemma $\mathfrak{m}^n = 0$.
- Let $\mathfrak{p}$ be a prime ideal. As $\mathfrak{m}^n = 0 \subseteq \mathfrak{p}$, by taking radicals $\mathfrak{m} = r(\mathfrak{m}^n) \subseteq r(\mathfrak{p}) = \mathfrak{p}$.
- $\mathfrak{m}$ is maximal, $\mathfrak{p} = \mathfrak{m}$, and hence A has a unique prime ideal.
- By Lemma 8.5* A is an Artin local ring.

The proof is complete. □

Reminder (see Chapter 1)

Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A are called *coprime* if $\mathfrak{a} + \mathfrak{b} = (1)$.

Reminder (Proposition 1.16)

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals such that $r(\mathfrak{a})$ and $r(\mathfrak{b})$ are coprime, then $\mathfrak{a}$ and $\mathfrak{b}$ are coprime as well.

Reminder (see Proposition 1.10)

Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals of A such that $\mathfrak{a}_i$ and $\mathfrak{a}_j$ are coprime for $i \neq j$. Define

$$\phi : A \to \prod_{i=1}^{n} (A/\mathfrak{a}_i), \qquad x \to (x + \mathfrak{a}_1, \ldots, x + \mathfrak{a}_n).$$

1. $\prod \mathfrak{a}_i = \cap \mathfrak{a}_i$
2. ϕ is surjective.
3. If $\cap \mathfrak{a}_i = (0)$, then ϕ is injective, and hence is an isomorphism.

Artin Rings

Fact

If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are distinct maximal ideals, then $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are coprime.

Proof.

- Set $k_1 = A/\mathfrak{m}_1$ and let $f : A \to k$ be the canonical homomorphism.
- As $\mathfrak{m}_1$ is maximal, k is a field.
- As $\mathfrak{m}_2$ is maximal, $\mathfrak{m}_2 \not\subseteq \mathfrak{m}_1$, i.e., $\exists x \in \mathfrak{m}_2$ s.t. $x \notin \mathfrak{m}_1$.
- As $x \notin \mathfrak{m}_1$ its image $\overline{x}$ in k is non-zero, and hence $\overline{x}$ is a unit, since k is a field.
- Thus, $f(\mathfrak{m}_2)$ contains a unit. As this is an ideal since f is onto, $f(m_2) = k$.
- Therefore, $A = f^{-1}(k) = f^{-1}(f(\mathfrak{m}_2)) = \mathfrak{m}_2 + \mathfrak{m}_1$, i.e., $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are coprime. □

Theorem (Theorem 8.7; Structure Theorem for Artin Rings)

An Artin ring A is uniquely (up to isomorphism) a finite direct product of Artin local rings.

Proof.

- Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the (distinct) maximal ideals of A. They are the only prime ideals of A.
- By the proof of Thm. 8.5 there is $k \geq 1$ s.t. $\mathfrak{m}_1^k \cdots \mathfrak{m}_n^k = (0)$.
- If $i \neq j$, then $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are coprime, since they are distinct maximal ideals.
- Here $r(m_i^k) = \mathfrak{m}_i$ and $r(\mathfrak{m}_j^k) = \mathfrak{m}_j$, and so $\mathfrak{m}_i^k$ and $\mathfrak{m}_j^k$ are coprime by Proposition 1.16.
- By Proposition 1.10 $\cap \mathfrak{m}_i^k = \prod \mathfrak{m}_i^k = (0)$, and so the canonical homomorphism $\phi : A \to \prod(A/\mathfrak{m}_i^k)$ is an isomorphism. □

Proof of Theorem 8.7; Continued.

- As $\mathfrak{m}_i$ is the only prime ideal containing $\mathfrak{m}_i^k$, its image in $A/\mathfrak{m}_i^k$ is the unique prime ideal of $A/\mathfrak{m}_i^k$.
- Here $A/\mathfrak{m}_i^k$ is Noetherian, since it is a quotient of a Noetherian ring.
- Thus, by Lemma 8.5* $A/\mathfrak{m}_i^k$ is an Artin local ring.
- This shows that A is isomorphic to the direct product of the Artin local rings $A/\mathfrak{m}_i^k$.

This completes the proof of the product decomposition. □

Remark

For the proof of the uniqueness; see Atiyah-MacDonald's book.

Remark

A (local) ring a with a unique prime ideal need not be an Artin ring. It need not even be a Noetherian ring.

Example

Let $A = k[x_1, x_2, \dots, x_n, \dots]$ (k field; infinitely many variables) and let $\mathfrak{a} = (x_1, x_2^2, \dots, x_n^n, \dots)$. Set $B = A/\mathfrak{a}$ and denote by $\overline{x}_j$ the image of x_j in B. Then:

- It can be shown that $(\overline{x}_1, \overline{x}_2, \dots)$ is the unique prime ideal, and hence B is a local ring and has dimension 0 (see Carlson's notes).
- However, B is not Noetherian (and hence is not an Artin ring), since $(\overline{x}_1) \subsetneq (\overline{x}_1, \overline{x}_2) \subsetneq \cdots$ is a non-stationary infinite chain.

Artin Rings

Remark

Suppose that A is a local ring. Let $\mathfrak{m}$ be its maximal ideal and $k = A/\mathfrak{m}$ its residue field.

- The A-module $\mathfrak{m}/\mathfrak{m}^2$ is annihilated by $\mathfrak{m}$, and hence it is a k-vector space.
- If $\mathfrak{m}$ is finitely generated (e.g., if A is Noetherian), then $\mathfrak{m}/\mathfrak{m}^2$ is finitely generated as k-vector space by Proposition 2.8.
- Thus, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) < \infty$.

Proposition (Proposition 8.8)

Let A be an Artin local ring. TFAE:

(i) *Every ideal of A is principal.*

(ii) *The maximal ideal of A is principal.*

(iii) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 1$.