Commutative Algebra
Chapter 7: Noetherian Rings

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Reminder from Chapter 6

Definition

A module M over a ring A is *Noetherian* if it satisfies one of the following equivalent conditions:

(i) *Ascending chain condition (a.c.c.):* Every ascending sequence of submodules $M_1 \subseteq M_2 \subseteq \cdots$ is sationnary.

(ii) *Maximal condition:* Every non-empty set of submodules of M has a maximal element.

Definition

A ring A is *Noetherian* if it is Noetherian as a module over itself, i.e., it satisfies a.c.c. and the maximal condition on ideals.

Examples

1. Any field k is Noetherian.
2. The ring $\mathbb{Z}$ is Noetherian.
3. Any principal ideal domain is Noetherian (this follows from Proposition 6.2).

Reminder from Chapter 6

Proposition (Proposition 6.2)

Let M be a module over A. TFAE:

(i) *M is Noetherian.*

(ii) *Every submodule of M is finitely generated.*

In particular, for $M = A$ we get:

Corollary

A ring A is Noetherian if and only if every ideal of A is finitely generated (as an A-module).

Reminder from Chapter 6

Proposition (Proposition 6.5)

Let A be a Noetherian ring and M a finitely generated A-module. Then M is Noetherian.

Proposition (Proposition 6.6)

Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. Then $A/\mathfrak{a}$ is a Noetherian ring.

Noetherian Rings

Proposition (Proposition 7.1)

Let $\phi : A \to B$ be a surjective ring homomorphism. If A is Noetherian, then so is B.

Proof.

- If $\mathfrak{a} = \ker \phi$, then $A/\mathfrak{a}$ is Noetherian by Proposition 6.6.
- As $B \simeq A/\mathfrak{a}$ it follows that B is Noetherian as well. □

Noetherian Rings

Proposition (Proposition 7.2)

Let $A \subseteq B$ be rings such that A is Noetherian and B is finitely generated as an A-module. Then B is Noetherian (as a ring).

Proof.

- By Proposition 6.5 B is Noetherian as an A-module.
- Any ideal of B is an A-module.
- Thus any ascending chain of ideals of B is stationary, and hence B is Noetherian as a B-module. □

Example

The ring of Gaussian integer $B = \mathbb{Z}[i]$ is Noetherian.

Reminder (see Proposition 3.11)

Let A be a ring and S a multiplicatively closed subset of A.

- If $\mathfrak{a}$ is an ideal of A such that $\mathfrak{a} \cap S = \emptyset$, then its extension in $S^{-1}A$ is $S^{-1}\mathfrak{a}$.
- If $\mathfrak{b}$ is an ideal of $S^{-1}A$, then its contraction in A is

$$\mathfrak{b}^c = \{x \in A; x/1 \in \mathfrak{b}\}.$$

- This gives a one-to-one correspondence $\mathfrak{a} \to S^{-1}\mathfrak{a}$ with inverse $\mathfrak{b} \to \mathfrak{b}^c$ between ideals of A that do not meet S and ideals of $S^{-1}A$.

Remark

The above correspondance is order-preserving, i.e.,

$$\mathfrak{a} \subseteq \mathfrak{a}' \Longrightarrow S^{-1}\mathfrak{a} \subseteq S^{-1}\mathfrak{a}'.$$

Noetherian Rings

Proposition (Proposition 7.3)

If A is a Noetherian ring and $S \subseteq A$ is closed under multiplication, then the fraction ring $S^{-1}A$ is Noetherian.

Proof.

- We have an order-preserving one-to-one correspondence between ideals of A which do not meet S and ideals of $S^{-1}A$.
- Thus, we have a one-to-one correspondence between ascending chains of ideals of A which do not meet S and ascending chains of ideals of $S^{-1}A$.
- As A is Noetherian, it follows that ascending chains of ideals of $S^{-1}A$ is stationary, and hence $S^{-1}A$ is Noetherian. $\square$

Corollary (Corollary 7.4)

If A is a Noetherian ring and $\mathfrak{p}$ is a prime ideal of A, then the local ring $A_{\mathfrak{p}}$ is Noetherian.

Noetherian Rings

Theorem (Hilbert's Basis Theorem; Theorem 7.5)

If A is a Noetherian ring, then the polynomial ring $A[x]$ is Noetherian as well.

Proof.

- By Proposition 6.2 it is enough to show that every ideal $\mathfrak{a}$ of $A[x]$ is finitely generated.
- Let $\mathfrak{i}$ consist of leading coefficients of polynomials in $\mathfrak{a}$, i.e.,

$$\mathfrak{i} = \{a \in A;\ \exists f \in \mathfrak{a} \text{ s.t. } f = ax^r + (\text{lower terms})\}\,.$$

Claim

$\mathfrak{i}$ is an ideal of A.

Proof of the Claim.

- Let $a \in \mathfrak{i}$ and $f \in \mathfrak{a}$ be s.t. $f = ax^p + (\text{lower terms})$.
- If $y \in A$, then $yf \in \mathfrak{a}$ and $yf = yax^p + (\text{lower terms})$, and hence $ya \in \mathfrak{i}$.
- Let $b \in \mathfrak{i}$ and $g \in \mathfrak{a}$ be s.t. $g = bx^q + (\text{lower terms})$. Then $x^q f + x^p g \in \mathfrak{a}$ and $x^q f + x^p g = (a+b)x^{p+q} + (\text{lower terms})$. Thus, $a + b \in \mathfrak{i}$.

□

Proof of Theorem 7.5; Continued.

- As A is Noetherian, the ideal $\mathfrak{i}$ is finitely generated by $a_1, \ldots, a_n$.
- For each i there is $f_i \in \mathfrak{a}$ s.t. $f_i = a_i x^{r_i} + (\text{lower terms})$.
- Let $\mathfrak{a}'$ be the ideal of $A[x]$ generated by $f_1, \ldots, f_n$. Then $\mathfrak{a}' \subseteq \mathfrak{a}$.
- Set $r = \max(r_1, \ldots, r_n)$ and $M = A + Ax + \cdots + Ax^r$.

Claim

$\mathfrak{a} = M \cap \mathfrak{a} + \mathfrak{a}'$.

Proof of the Claim.

- If $f \in \mathfrak{a}$ has degree $\leq r$, then $f \in M \cap \mathfrak{a} \subseteq M \cap \mathfrak{a} + \mathfrak{a}'$.
- We shall show by induction that if $f \in \mathfrak{a}$ has degree $m \geq r$, then $f \in M \cap \mathfrak{a} + \mathfrak{a}'$.
- This is true for $m = r$.

□

Proof of the Claim; Continued.

- Assume the assertion is true for $m-1$ (with $m \geq r+1$).
- Let $f \in \mathfrak{a}$ have degree m, i.e., $f = ax^m + (\text{lower terms})$.
- As $a \in \mathfrak{i}$, we may write $a = u_1a_1 + \cdots + u_na_n$, $u_i \in A$.
- Set $g = u_1f_1x^{m-r_1} + \cdots + u_nf_nx^{m-r_n}$. Then $g \in \mathfrak{a}'$, and

$$\begin{aligned} g &= (u_1a_1 + \cdots + u_na_n)x^m + (\text{lower terms}) \\ &= ax^m + (\text{lower terms}) \\ &= f + (\text{lower terms}). \end{aligned}$$

- Thus, $f - g$ is in $\mathfrak{a}$ and has degree $\leq m-1$, and hence is in $M \cap \mathfrak{a} + \mathfrak{a}'$.
- Thus, $f = g + (f-g)$ is in $M \cap \mathfrak{a} + \mathfrak{a}'$, and hence the assertion is true for m.

This proves the claim. □

Back to the Proof of Theorem 7.5.

- By definition $\mathfrak{a}'$ is generated by $f_1, \ldots f_n$.
- $M = A + Ax + \cdots + Ax^r$ is a finitely A-generated module.
- As A is Noetherian, by Proposition 6.5 M is Noetherian.
- By Proposition 6.2 $M \cap \mathfrak{a}$ is finitely generated as an A-module.
- Thus, there are $g_1, \ldots, g_k$ in $M \cap \mathfrak{a}$ such that

$$M \cap \mathfrak{a} = Ag_1 + \cdots + Ag_k \subseteq g_1 A[x] + \cdots g_k A[x].$$

- Thus, $\mathfrak{a} = M \cap \mathfrak{a} + \mathfrak{a}'$ is generated over $A[x]$ by $f_1, \ldots, f_n$ and $g_1, \ldots, g_k$.
- Therefore, every ideal of $A[x]$ is finitely generated, and hence $A[x]$ is Noetherian by Proposition 6.2.

The proof is complete. □

Noetherian Rings

Remark

If A is a Noetherian ring, then it can be also shown that the ring of formal power series $A[[x]]$ is Noetherian (see Carlson's notes; see also Corollary 10.27).

Corollary (Corollary 7.6)

If A is a Noetherian ring, then the ring $A[x_1, \ldots, x_n]$ is Noetherian.

Proof.

- By induction on n,
- For $n = 1$ this is Theorem 7.5.
- If $B = A[x_1, \ldots, x_{n-1}]$ is Noetherian, then by Theorem 7.5 again the ring $A[x_1, \ldots, x_n] = B[x_n]$ is Noetherian . □

Corollary (Corollary 7.7)

If A is Noetherian, then every finitely generated A-algebra is a Noetherian ring. In particular, every finitely generated ring and every finitely generated algebra over a field are Noetherian rings.

Proof.

- By assumption $B = A[b_1, \ldots, b_n]$, $b_i \in B$.
- Let $\phi : A[x_1, \ldots, x_n] \to B$ be the homomorphism defined by

$$\phi\left(\sum a_k x_1^{k_1} \cdots x_n^{k_n}\right) = \sum a_k b_1^{k_1} \cdots b_n^{k_n}.$$

- As ϕ is onto, B is Noetherian by Proposition 7.1. □

Noetherian Rings

Reminder (see Proposition 5.1)

Let $A \subseteq B$ be rings and $x \in B$. TFAE:

(i) x is integral over A.

(ii) $A[x]$ is a finitely generated A-module.

(iii) $A[x]$ is contained in a subring C of B such that C is a finitely generated A-module.

Reminder (Corollary 5.2)

Let $x_1, \ldots, x_n$ be elements of B that are integral over A. Then the ring $A[x_1, \ldots, x_n]$ is a finitely generated A-module.

Proposition (Proposition 7.8)

Let $A \subseteq B \subseteq C$ be rings. Suppose that A is Noetherian and C is finitely generated as an A-algebra. Assume further that one of the following two conditions holds:

(i) *C is finitely generated as a B-module.*

(ii) *C is integral over B, i.e., $C = C * B$.*

Then B is finitely generated as an A-algebra.

Remark

In this situation the conditions (i) and (ii) are equivalent.

Proof of (i)⇒(ii).

- If $x \in C$, then $B[x] \subseteq C$.
- As C is a finitely generated B-module, by Prop. 5.1(iii) $x \in C * B$, and hence $C = C * B$. □

Proof of (ii)⇒(i).

- Suppose that $C = C * B$.
- As C is a finitely generated A-algebra, $C = A[x_1, \ldots, x_n]$, $x_i \in C$, and hence $C = B[x_1, \ldots, x_n]$
- As $x_1, \ldots, x_n \in C * B$, by Corollary 5.2 $C = B[x_1, \ldots, x_n]$ is a finitely generated B-module. □

Proof of Proposition 7.8.

- Let $x_1, \ldots, x_m$ generate C as an A-algebra, and let $y_1, \ldots, y_n$ generate C as a B-module. Then:

$$(*) \qquad x_i = \sum b_{ij} y_j, \quad b_{ij} \in B,$$

$$(**) \qquad y_i y_j = \sum b_{ijk} y_k, \quad b_{ijk} \in B.$$

- Let B_0 be the A-algebra generated by the b_{ij} and b_{ijk}. We have $A \subseteq B_0 \subseteq B \subseteq C$.
- As A is Noetherian, so is B_0 by Corollary 7.7.
- Let C' be the B_0-module generated by $y_1, \ldots, y_n$.
- By using $(**)$ it can be checked by induction that $y_{i_1} \cdots y_{i_p} \in C'$, and hence C' is an algebra over B_0.
- As the x_i are in B_0 by $(*)$ and generate C as an A-algebra, it follows that $C = C'$.
- Thus, C is a finitely generated B_0-module.

□

Proof of Proposition 7.8; Continued.

- As B_0 is Noetherian, C is Noetherian by Corollary 7.7.
- As B is a submodule of C, by Proposition 6.2 B is finitely generated as a B_0-module.
- As B_0 is finitely generated as an A-algebra, B is finitely generated as an A-algebra as well.

The proof is complete. □

Proposition (Proposition 7.9)

Let k be a field and E a finitely generated k-algebra. If E is a field, then this is a finite algebraic extension of k.

Proof.

- By assumption $E = k[x_1, \ldots, x_n]$, $x_i \in E$.
- Suppose that E is not algebraic over k.
- We may assume that $x_1, \ldots, x_r$ are algebraically independent and $x_{r+1}, \ldots, x_n$ are algebraic over the field $F = k(x_1, \ldots, x_r)$.
- Thus, E is a finite algebraic extension of F, and hence is a finite dimensional vector space over F, i.e., a finitely generated F-module.

□

Proof of Proposition 7.9; Continued.

- We have $k \subseteq F \subseteq E$, where:
  - k is Noetherian, since this is a field.
  - E is finitely generated as k-algebra and as an F-module.
- Thus, by Proposition 7.8 F is a finitely generated k-algebra.
- That is, $F = k[f_1/g_1, \ldots, f_s/g_s]$, with $f_j, g_j \in k[x_1, \ldots, x_r]$.
- The polynomial $1 + g_1 \cdots g_s$ is prime with each of the g_j.
- Thus, if h is an irreducible component of $1 + g_1 \cdots g_s$, then h is prime with each of the g_j as well.
- $h \in k[x_1, \ldots, x_r] \subseteq F$, and so $h^{-1} \in F = k[f_1/g_1, \ldots, f_s/g_s]$.
- Thus, there are $m_1, \ldots, m_s$ such that
$$g_1^{m_1} \cdots g_s^{m_s} h^{-1} \in k[f_1, \ldots, f_s, g_1, \ldots, g_s] \subseteq k[x_1, \ldots, x_r].$$
- Thus, h divides $g_1^{m_1} \cdots g_s^{m_s}$ in $k[x_1, \ldots, x_r]$, and hence divides at least one of the g_j since it is irreducible (contradiction). □

Corollary (Weak Nullstellensatz; Corollary 7.10)

Let k be a field, A a finitely generated k-algebra, and $\mathfrak{m}$ a maximal ideal of A. Then the field $A/\mathfrak{m}$ is a finite algebraic extension of k. In particular, if k is algebraically closed, then $A/\mathfrak{m} \simeq k$.

Remark

The weak Nullstellensatz allows us to get the strong form of Hilbert's Nullstellensatz (see Problem 7.14 and Carlson's notes).

Primary Decomposition in Noetherian Rings

Definition

An ideal $\mathfrak{a}$ of a ring A is called *irreducible* if

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies \left(\mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c}\right).$$

Lemma (Lemma 7.11)

In A is a Noetherian ring A, every ideal is finite intersection of irreducible ideals.

Lemma (Lemma 7.12)

In a Noetherian ring A, every irreducible ideal is primary.

Primary Decomposition in Noetherian Rings

From the previous two lemmas we immediately obtain:

Theorem (Theorem 7.13)

In a Noetherian ring, every ideal has a primary decomposition.

Consequence

All the results of Chapter 4 apply to Noetherian rings.

Primary Decomposition in Noetherian Rings

Proposition (Proposition 7.14)

In a Noetherian ring A, every ideal $\mathfrak{a}$ contains a power of its radical $r(\mathfrak{a})$.

Proof.

- As A is Noetherian any ideal of A is finitely generated by Proposition 6.2.
- Let $x_1, \ldots, x_n$ be generators of $r(\mathfrak{a})$, i.e., $x_i^{n_i} \in \mathfrak{a}$.
- Set $m = 1 + \sum(n_i - 1)$. Then $r(\mathfrak{a})^m$ is generated by monomials $x_1^{r_1} \cdots x_n^{r_n}$ with $\sum r_i = m$.
- If $r_i \leq n_i - 1$ for all i, then $\sum r_i \leq \sum(n_i - 1) = m - 1$ (contradiction).
- Thus, $r_i \geq n_i$ for some i, and hence $x_i^{r_i} = x_i^{r_i - n_i} x_i^{n_i} \in \mathfrak{a}$.
- Thus, each monomial $x_1^{r_1} \cdots x_n^{r_n}$ is in $\mathfrak{a}$, and hence $r(\mathfrak{a})^m \subseteq \mathfrak{a}$.

This proves the result. □

Corollary (Corollary 7.15)

In a Noetherian ring the nilradical $\mathfrak{N}$ is nilpotent.

Proof.

- By definition $\mathfrak{N} = r(\mathfrak{a})$ with $\mathfrak{a} = (0)$.
- By Proposition 7.14 $\mathfrak{N}^m \subseteq (0)$, and hence $\mathfrak{N}^m = (0)$, i.e., $\mathfrak{N}$ is nilpotent.

□

Primary Decomposition in Noetherian Rings

Reminder (Proposition 4.2)

If $\mathfrak{a}$ is an ideal in A whose radical $r(\mathfrak{a})$ is maximal, then $\mathfrak{a}$ is primary. In particular, every power of a maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-primary.

Primary Decomposition in Noetherian Rings

Corollary (Corollary 7.16)

Suppose that A is a Noetherian ring. Let $\mathfrak{m}$ be a maximal ideal of A and let $\mathfrak{q}$ be any ideal. TFAE:

(i) $\mathfrak{q}$ *is* $\mathfrak{m}$*-primary.*

(ii) $r(\mathfrak{q}) = \mathfrak{m}$.

(iii) $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ *for some* $n \geq 1$.

Proof.

- (i)$\Rightarrow$(ii) is immediate.
- (ii)$\Rightarrow$(i) is the contents of Proposition 4.2.
- (ii)$\Rightarrow$(iii) follows from Proposition 7.14.
- If $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, then $r(\mathfrak{m}^n) \subseteq r(\mathfrak{q}) \subseteq r(\mathfrak{m})$.
- As $r(\mathfrak{m}^n) = r(\mathfrak{m}) = \mathfrak{m}$, since $\mathfrak{m}$ is prime, it follows that $r(\mathfrak{q}) = \mathfrak{m}$. Thus, (iii)$\Rightarrow$(ii).

The proof is complete. □

Primary Decomposition in Noetherian Rings

Reminder (1st Uniqueness Theorem; Theorem 4.5)

Let $\mathfrak{a}$ be a decomposable ideal and $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$ a primary decomposition. Set $p_i = r(\mathfrak{q}_i)$, $i = 1, \ldots, n$. Then the $\mathfrak{p}_i$ are exactly the prime ideals of the form $r(\mathfrak{a} : x)$, $x \in A$. In particular, they don't depend on the primary decomposition of $\mathfrak{a}$.

Remark

The proof of Theorem 4.5 shows that, for all $x \neq 0$,

$$r(\mathfrak{a} : x) = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j.$$

Proposition (Proposition 7.17)

Let A be a Noetherian ring and $\mathfrak{a} \subsetneq A$ an ideal of A. Then the prime ideals which belong to $\mathfrak{a}$ are exactly the prime ideals of the form $(\mathfrak{a} : x)$, $x \in A$.

Proof of Proposition 7.17.

- Let $\mathfrak{a} = \cap \mathfrak{q}_i$ be a min. primary decomposition. Set $\mathfrak{p}_i = r(\mathfrak{q}_i)$.
- Set $\mathfrak{a}_i = \cap_{j \neq i} \mathfrak{q}_j$. If $x \in \mathfrak{a}_i \setminus 0$, then by the proof of Thm. 4.5,
$$r(\mathfrak{a} : x) = \cap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j = \mathfrak{p}_i.$$
- Thus $(\mathfrak{a} : x) \subseteq r(\mathfrak{a} : x) = \mathfrak{p}_i$.
- As $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, by Proposition 7.14 $\mathfrak{p}_i^m \subseteq \mathfrak{q}_i$ for some m, and hence
$$\mathfrak{a}_i \mathfrak{p}_i^m \subseteq \mathfrak{a}_i \cap \mathfrak{p}_i^m \subseteq \mathfrak{a}_i \cap \mathfrak{q}_i = \mathfrak{a}.$$
- Let m be the smallest integer s.t. $\mathfrak{a}_i \mathfrak{p}^m \subseteq \mathfrak{a}$.
- If $x \in \mathfrak{a}_i \mathfrak{p}_i^{m-1}$, $x \neq 0$, then $\mathfrak{p}_i x \in \mathfrak{a}_i \mathfrak{p}_i^m \subseteq \mathfrak{a}$.
- Thus, $\mathfrak{p}_i \subseteq (\mathfrak{a} : x)$, and hence $(\mathfrak{a} : x) = \mathfrak{p}_i$.
- Conversely, if $(\mathfrak{a} : x)$ is a prime $\mathfrak{p}$, then $r(\mathfrak{a} : x) = r(\mathfrak{p}) = \mathfrak{p}$, and hence $\mathfrak{p}$ belongs to $\mathfrak{a}$ by Thm. 4.5.

The proof is complete. □