Differentiable Manifolds
§21. Orientations

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Orientations of a Vector Space

Example (Orientations of $\mathbb{R}$)

On $\mathbb{R}$ an orientation is one of two directions:

The orientations of a line.

Two (nonzero) vectors u and v define the same direction if and only if $u = av$ with $a > 0$.

Orientations of a Vector Space

Example (Orientations of $\mathbb{R}^2$)

On $\mathbb{R}^2$ an orientation is either direct (counterclockwise) or indirect (clockwise).

The orientations of a plane.

- An ordered basis (v_1, v_2) defines the direct (resp., indirect) orientation if the angle θ from v_1 to v_2 is > 0 (resp., < 0).
- As $\det(v_1, v_2) = |v_1||v_2| \sin \theta$, we see that

$$(v_1, v_2) \text{ is direct} \Longleftrightarrow \det(v_1, v_2) > 0,$$
$$(v_1, v_2) \text{ is indirect} \Longleftrightarrow \det(v_1, v_2) < 0.$$

Orientations of a Vector Space

Example (The orientations of a plane, continued)

- Let (u_1, u_2) and (v_1, v_2) be ordered bases. Write $u_i = \sum a_i^j v_j$. The matrix $A = [a_i^j]$ is called the *change-of-basis matrix*. We have
$$\det(u_1, u_2) = \det(A) \det(v_1, v_2).$$
- Thus, (u_1, u_2) and (v_1, v_2) defines the same orientation if and only if $\det(A) > 0$.

Definition

Two bases (u_1, u_2) and (v_1, v_2) are called *equivalent* if the change-of-basis matrix has positive determinant.

- This defines an equivalence relation on order bases.
- We have a one-to-one correspondance:
$$\{\text{orientations}\} \longleftrightarrow \{\text{equivalence classes of bases}\}.$$

Orientations of a Vector Space

Definition

Let V be a vector space of dimension n. Two bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are said to be *equivalent*, and we write $(u_1, \ldots, u_n) \sim (v_1, \ldots, v_n)$ if we can go from one to the other by a change-of-base matrix with positive determinant.

Remark

This defines an equivalence relation on bases of V.

Definition

An *orientation* of V is a choice of an equivalence class of bases.

Remarks

- A vector space has exactly two orientations.
- We denote by $[(v_1, \ldots, v_n)]$ the class of $(v_1, \ldots, v_n)$.

Remark

Let $(v_1, \ldots, v_n)$ be a basis of a vector space V. Let $(\alpha^1, \ldots, \alpha^n)$ be the dual basis of V^*. Then, for any n-covector $\beta \in \Lambda^n(V^*)$, we have

$$\beta = \beta(v_1, \ldots, v_n)\alpha^1 \wedge \cdots \wedge \alpha^n.$$

In particular, $\beta \neq 0$ if and only if $\beta(v_1, \ldots, v_n) \neq 0$.

Orientations and Covectors

Lemma (Lemma 21.1)

Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in V such that $u_i = \sum a_i^j v_j$ for some matrix $A = [a_i^j]$. For any n-covector β we have

$$\beta(u_1, \ldots, u_n) = (\det A)\beta(v_1, \ldots, v_n).$$

Consequence

Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be bases and $\beta \neq 0$. Then $\beta(u_1, \ldots, u_n)$ and $\beta(v_1, \ldots, v_n)$ have same sign if and only if $\det A > 0$, i.e., $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ define the same orientation.

Orientations and Covectors

Definition

We say that an n-covector β on V *specifies* the orientation $[(v_1, \dots, v_n)]$ if $\beta(v_1, \dots, v_n) > 0$.

Remark

Let $(v_1, \dots, v_n)$ be a basis of a vector space V. Let $(\alpha^1, \dots, \alpha^n)$ be the dual basis of V^*. By the remark on slide 6, we have

$$\beta = \beta(v_1, \dots, v_n)\alpha^1 \wedge \cdots \wedge \alpha^n$$

Thus, β *specifies* the orientation $[(v_1, \dots, v_n)]$ if and only if β is a positive scalar multiple of $\alpha^1 \wedge \cdots \wedge \alpha^n$.

Orientations and Covectors

Definition

We say that two non-zero n-covectors β and β' are equivalent if $\beta' = a\beta$ with $a > 0$.

Remark

This defines an equivalence relation on $\Lambda^n(V^*) \setminus 0$.

Fact

We have a one-to-one correspondence:

$$\{\text{orientations of } V\} \longleftrightarrow \{\text{equivalence classes of } n\text{-covectors} \neq 0\}.$$

Orientations of a Manifold

Fact

Let M be a smooth manifold of dimension n. If $(X_1, \dots, X_n)$ is a frame of TM over U and $p \in U$, then $(X_{1,p}, \dots, X_{n,p})$ is a basis of T_pM, and hence it defines an orientation of T_pM.

Remark

We say that a frame $(X_1, \dots, X_n)$ of TM over an open U is continuous, if, for each i, the vector field X_i is continuous as a map from U to TM.

Orientations of a Manifold

Definition (Pointwise orientation)

- A *pointwise orientation* of M assigns to each $p \in M$ an orientation of T_pM, i.e., an equivalence class $\mu_p = [(X_{1,p}, \ldots, X_{n,p})]$ of (ordered) bases of T_pM.
- We say that a pointwise orientation is *continuous* at $p \in M$ if there is an open U containing p and a continuous tangent frame $(Y_1, \ldots, Y_n)$ over U such that $(Y_{1,q}, \ldots, Y_{n,q})$ defines the orientation of T_qM for every $q \in U$.

Orientations of a Manifold

Definition (Orientations)

- An *orientation* of M is a pointwise orientation which is continuous at every $p \in M$.
- We say that M is *orientable* when it admits an orientation.
- We say that M is *oriented* when it is equipped with an orientation.

Remarks

- Any continuous (or even smooth) global frame $(X_1, \ldots, X_n)$ of TM over M defines an orientation.
- The converse does not hold. For instance, the even-dimensional spheres $\mathbb{S}^{2n}$, $n \geq 1$, do not admit global tangent frames; yet they are orientable.

Orientations of a Manifold

Example

$\mathbb{R}^n$ is oriented by the global frame $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. More generally, any vector space is orientable.

Example (see also Problem 21.7)

If G is a Lie group, then G admits a global tangent frame consisting of left-invariant vector fields, and so G is orientable.

Example (Möbius Band; Example 21.2)

The Möbius band is the quotient of the rectangle $R = [0, 1] \times [-1, 1]$ by the equivalence relation,

$$(x, y) \sim (x, y), \qquad 0 < x < 1, \ -1 \leq y \leq 1,$$
$$(0, y) \sim (1, -y), \qquad -1 \leq y \leq 1.$$

This is a non-orientable surface (see Tu's book).

Proposition (Proposition 21.3)

If an orientable manifold is connected, then it has exactly two possible orientations.

Lemma (see Lemma 21.4)

Let μ be a pointwise orientation of M. TFAE:

(i) *μ is continuous on M.*

(ii) *For every $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ near p such that the orientation of T_pM is defined by $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$.*

(ii) *For every $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ near p such that the orientation of T_pM is specified by $dx^1 \wedge \cdots \wedge dx^n$.*

Theorem (Theorem 21.5)

A manifold M of dimension n is orientable if and only if there exists a smooth nowhere-vanishing n-form on M.

Remark

Let ω be a nowhere vanishing n-form on M. Then ω defines an orientation of M as follows:

- For every $p \in M$, there is a chart $(U, x^1, \ldots, x^n)$ near p such that $\omega(\partial/\partial x^1, \ldots, \partial/\partial x^n) > 0$ on U.
- The orientation of T_pM is the class of $(\partial/\partial x^1\big|_p, \ldots, \partial/\partial x^n\big|_p)$.
- As the frames $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ are continuous (since they are smooth), we get a continuous pointwise orientation on M, i.e., an orientation of M.

Example

Suppose that 0 is a regular value of some smooth function $f(x, y, z)$ on $\mathbb{R}^3$.

- By the regular level set theorem, the zero set $S = f^{-1}(0)$ is a regular submanifold of $\mathbb{R}^3$, and hence is manifold.
- By Problem 19.11 it admits a smooth nowhere-vanishing 2-form.
- Thus, by Theorem 21.5 the manifold S is orientable.

For instance, the 2-sphere $\mathbb{S}^2$ is orientable.

Orientations and Differential Forms

Definition

We say that two C^∞ nowhere-vanishing n-forms ω and ω' on M are *equivalent*, and we write $\omega \sim \omega'$, if there is $f \in C^\infty(M)$, $f > 0$, such that $\omega' = f\omega$.

Remark

This defines an equivalence relation on C^∞ nowhere-vanishing n-forms on M.

Proposition

We have a one-to-one correspondence:

$$\{\textit{orientations of } M\} \longleftrightarrow \left\{ \begin{array}{c} \textit{equivalence classes of} \\ C^\infty \ \textit{nowhere-vanishing } n\textit{-forms} \end{array} \right\}$$

Definition

If ω is a C^∞ nowhere-vanishing n-form that specifies the orientation at every point, then we say that ω is an *orientation form*.

Example

The (standard) orientation of $\mathbb{R}^n$ is specified by the n-form $dx^1 \wedge \cdots \wedge dx^n$.

Remark

An oriented manifold is often represented as $(M, [\omega])$, where $[\omega]$ is a class of orientation forms.

Orientations and Differential Forms

Definition

A diffeomorphism $F : (N, [\omega_N]) \to (M, [\omega_M])$ between oriented manifolds is called *orientation-preserving* if $[F^*\omega_M] = [\omega_N]$. It is called *orientation-reversing* if $[F^*\omega_M] = [-\omega_N]$.

Proposition (Proposition 21.8)

Let U and V be open sets in $\mathbb{R}^n$ equipped with orientations inherited from $\mathbb{R}^n$. A diffeomorphism $F : U \to V$ is orientation-preserving if and only if the Jacobian determinant $\det[\partial F^i / \partial x^j]$ is everywhere positive on U.

Orientations and Atlases

Definition (Definition 21.9)

An atlas of M is called *oriented* if given two overlapping charts $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ the transition map is orientation-preserving, i.e., the Jacobian determinant $\det[\partial y^i / \partial x^j]$ is everywhere positive on $U \cap V$.

Theorem (Theorem 21.10)

A manifold M is orientable if and only if it admits an oriented atlas.

Orientations and Atlases

Remark

An oriented atlas defines an orientation of M as follows:

- Given $p \in M$ and a chart $(U, x^1, \ldots, x^n)$, the orientation of T_pM is the class of $(\partial/\partial x^1\big|_p, \ldots, \partial/\partial x^n\big|_p)$.
- The orientation of T_pM does not depend on the choice of the chart, since the atlas is oriented.
- As the frames $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ are continuous, we get a continuous pointwise orientation on M, i.e., an orientation of M.

Orientations and Atlases

Definition (Definition 21.11)

Two oriented atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ on M are said to be *equivalent* if the transition functions

$$\phi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap V_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap V_\beta)$$

have positive Jacobian determinants for all α, β.

Remark

This defines an equivalence relation on oriented atlases.

Proposition

We have a one-to-one correspondence:

$$\{\text{orientations of } M\} \longleftrightarrow \left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of oriented atlases} \end{array} \right\}$$

Orientations and Atlases

Summary

If M is an orientable manifold of dimension n, there are 3 equivalent ways to define an orientation:

1. By using a continuous pointwise orientation.
2. By using a smooth nowhere-vanishing n-form.
3. By using an oriented atlas.