Differentiable Manifolds
§15. Lie Groups

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Lie groups were defined in Section 6.

Definition (Lie Groups)

A *Lie group* is a group G equipped equipped with a differentiable structure such that:

(i) The multiplication map $\mu : G \times G \to G$, $(x, y) \to xy$ is a C^∞ map.

(ii) The inverse map $\iota : G \to G$, $x \to x^{-1}$ is a C^∞ map.

Lie Groups and Lie Subgroups. Examples

In Section 6 the following examples of Lie groups were mentioned.

Examples

1. The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Lie groups under addition.
2. The set of non-zero complex numbers $\mathbb{C}^\times := \mathbb{C} \setminus 0$ is a Lie group under multiplication.
3. The unit circle $\mathbb{S}^1 \subset \mathbb{C}^\times$ is a Lie group under multiplication.
4. If G_1 and G_2 are Lie groups, then their Cartesian product $G_1 \times G_2$ is again a Lie group.

Example (Example 6.21)

The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a is a Lie group,

$$\mathrm{GL}(n, \mathbb{R}) = \left\{ A \in \mathbb{R}^{n \times n};\ \det A \neq 0 \right\}.$$

Definition (Left and right multiplication)

Let G be a Lie group

- Given any $a \in G$, we denote by ℓ_a the *left multiplication by* a, i.e., the map,

$$\ell_a : G \longrightarrow G, \quad \ell_a(x) = ax.$$

- We also denote by r_a *right multiplication by* a, i.e.,

$$r_a : G \longrightarrow G, \quad r_a(x) = xa.$$

Proposition (see Exercise 15.2)

For every $a \in G$, the maps $\ell_a : G \to G$ and $r_a : G \to G$ are both diffeomorphisms of G with respective inverses $\ell_{a^{-1}}$ and $r_{a^{-1}}$.

Definition (Lie group homomorphisms)

Let G and H be Lie groups.

- A map $F : H \to G$ is a *Lie group homomorphism* if F is both a smooth map and a group homomorphism.
- It is called a *Lie group isomorphism* if it is a Lie group homomorphism and a diffeomorphism.

Remarks

1. A map $F : H \to G$ is a group homomorphism if and only if
$$F(hx) = F(h)F(x) \qquad \forall h, x \in H.$$
As $F(hx) = F \circ \ell_h(x)$ and $F(h)F(x) = \ell_{F(h)} \circ F(x)$, the above condition is equivalent to
$$F \circ \ell_h = \ell_{F(h)} \circ F \qquad \forall h \in H.$$

2. Denote by e_H and e_G the respective units of H and G. Taking $h = x = e_H$ above gives
$$F(e_H) = F(e_H^2) = F(e_H)^2 \quad \text{which implies that } F(e_H) = e_G.$$
Thus, a group homomorphism always maps the identity to the identity.

Proposition (Theorem 7.5 of Lee's book)

Any Lie group homomorphism $F : H \to G$ has constant rank.

Proof.

- Let $h \in H$. Differentiating at $x = e_H$ the equality $F \circ \ell_h = \ell_{F(h)} \circ F$ gives

$$F_{*,\ell_h(e_H)} \circ (\ell_h)_{*,e_H} = (\ell_{F(h)})_{*,F(e_H)} \circ F_{*,e_H}.$$

  That is,

$$F_{*,h} \circ (\ell_h)_{*,e_H} = (\ell_{F(h)})_{*,e_G} \circ F_{*,e_H}$$

- As ℓ_h and $\ell_{F(h)}$ are diffeomorphisms, their differentials $(\ell_h)_{*,e_H}$ and $(\ell_{F(h)})_{*,e_G}$ are isomorphisms.
- It then follows that $\operatorname{rk} F_{*,h} = \operatorname{rk} F_{*,e_H}$ for all $h \in H$.

This proves the result. □

Lie Groups and Lie Subgroups. Examples

Definition (Lie subgroups)

A *Lie subgroup* of a group G is a subgroup H such that

1. H is an immersed submanifold of H.
2. The multiplication and inverse map of H are smooth maps.

Examples

1. $\mathbb{R}^n$ is a Lie subgroup of $\mathbb{C}^n$ under addition.
2. The circle $\mathbb{S}^1$ is a Lie subgroup of $\mathbb{C}^\times$ under multiplication.
3. Any open subgroup of a Lie group is a Lie subgroup.

Reminder (see Chapter 11)

Let M and N be manifolds and S a regular submanifold in M.

1. If $F : M \to N$ is a smooth map, then the restriction $F_{|S} : S \to N$ is a smooth map (since the inclusion $i : S \to M$ is a smooth map).
2. If $F : N \to M$ is a smooth map taking values in S, then it induces a smooth $F : N \to S$.

Consequence

Let $F : N \to M$ be a smooth map. Assume that S is a regular submanifold of N and R is a regular submanifold of M such that $F(S) \subset R$. Then F induces a smooth map $\overline{F} : S \to R$.

Lie Groups and Lie Subgroups. Examples

Proposition (Proposition 15.11)

If H is a subgroup of a Lie group G and a regular submanifold, then this is an embedded Lie subgroup. In particular, this is a Lie group.

Proof.

We only need to check that the multiplication and inverse maps of H are smooth maps.

- The multiplication $H \times H \to H$ is induced from the multiplication $G \times G \to G$.
- As H and $H \times H$ are regular submanifolds, it follows from the corollary on the previous slide that the multiplication of H is a smooth map.
- Likewise, the inverse map $H \to H$ is smooth, since it is induced from the inverse map $G \to G$.

The proof is complete. □

Lie Groups and Lie Subgroups. Examples

Reminder (Constant Rank Level Set Theorem; see Theorem 11.2)

Let $f : N \to M$ be smooth map of constant rank k. For every $c \in f(N)$ the level set $f^{-1}(c)$ is a regular submanifold of codimension k in N.

Proposition

Let $F : G \to H$ be a Lie group homomorphism. Then $F^{-1}(e_H)$ is an embedded Lie subgroup of G.

Proof.

- $F^{-1}(e_H)$ is a subgroup of G.
- By the proposition on slide 7 the homomorphism F has constant rank, and so by the constant rank theorem the level set $F^{-1}(e_H)$ is a regular submanifold of G.
- It then follows from Proposition 15.11 that $F^{-1}(e_H)$ is an embedded Lie subgroup of G.

The proof is complete. □

Lie Groups and Lie Subgroups. Examples

Example (Special linear group; see also Example 9.13)

The special linear group is

$$\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R});\ \det A = 1\}.$$

- As $\det : \mathrm{GL}(n,\mathbb{R}) \rightarrow \mathbb{R}^{\times}$ is a smooth map and a group homomorphism, this is a Lie group homomorphism.
- It then follows that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ is an embedded Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$, and hence is a Lie group.
- Here the determinant map has constant rank 1, and so $\mathrm{SL}(n,\mathbb{R})$ has codimension 1 in $\mathrm{GL}(n,\mathbb{R})$.

Example (Orthogonal group; Example 15.6)

The orthogonal group is

$$\mathrm{O}(n) = \left\{ A \in \mathrm{GL}(n, \mathbb{R});\ A^T A = I \right\}.$$

- This is a subgroup of $\mathrm{GL}(n, \mathbb{R})$.
- Let S_n be the linear subspace of $\mathbb{R}^{n \times n}$ of symmetric matrices $X^T = X$. This is vector space, and so this is a manifold.
- Define $f : \mathrm{GL}(n, \mathbb{R}) \to S_n$ by $f(A) = A^T A$. This is a smooth map such that $\mathrm{O}(n) = f^{-1}(I)$.
- It can be shown that f is a submersion (see Tu's book). Thus, $\mathrm{O}(n) = f^{-1}(I)$ is a regular submanifold.
- It then follows that $\mathrm{O}(n)$ is an embedded Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, and hence is a Lie group.

Remark

- Set $k = \dim S_n$. As f is a submersion, it has constant rank k, and hence $O(n)$ has codimension k in $GL(n, \mathbb{R})$.
- As $k = \dim S_n = \frac{1}{2}n(n+1)$ and $\dim GL(n, \mathbb{R}) = n^2$, we get

$$\dim O(n) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$$

Example (Special Orthogonal Group; see also Problem 15.11)

The *special orthogonal group* is

$$\mathrm{SO}(n) = \left\{A \in \mathrm{O}(n);\ \det A = 1\right\} = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}).$$

- If $A \in \mathrm{O}(n)$, then $A^T A = I$, and so
$$1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2.$$
Thus, $\det A = \pm 1$.
- It follows that $\mathrm{SO}(n) = (\det_{|\mathrm{O}(n)})^{-1}(1) = (\det_{|\mathrm{O}(n)})^{-1}(\mathbb{R}_+^\times)$, and so $\mathrm{SO}(n)$ is an open set in $\mathrm{O}(n)$.
- Here $\mathrm{SO}(n)$ is an open subgroup of $\mathrm{O}(n)$, and hence this is a Lie subgroup of $\mathrm{O}(n)$ and $\mathrm{GL}(n, \mathbb{R})$.

Remark (Complex Linear Groups)

The complex versions of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ are Lie groups as well. There are the following groups:

- The *complex general linear group*,
$$\mathrm{GL}(n,\mathbb{C}) = \left\{A \in \mathbb{C}^{n\times n};\ \det A \neq 0\right\}.$$
- The *complex special linear group*,
$$\mathrm{SL}(n,\mathbb{C}) = \left\{A \in \mathrm{GL}(n,\mathbb{C});\ \det A = 1\right\}.$$
- Here $\mathrm{SL}(n,\mathbb{C})$ is an embedded Lie subgroup of $\mathrm{GL}(n,\mathbb{C})$ of codimension 2.

Remark (Unitary Groups; see also Problems 15.12 & 15.13)

The complex analogues of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are the following groups:

- The *unitary group*,
$$\mathrm{U}(n) = \left\{A \in \mathrm{GL}(n, \mathbb{C});\ A^*A = I\right\}.$$
- The *special unitary group*,
$$\mathrm{SU}(n) = \left\{A \in \mathrm{U}(n);\ \det A = 1\right\} = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}).$$
- There are both (embedded) Lie subgroups of $\mathrm{GL}(n, \mathbb{C})$.
- Here $\mathrm{U}(n)$ has codimension n^2 in $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$ has codimension 1 in $\mathrm{U}(n)$.

The Matrix Exponential

Definition (Matrix Exponential)

If A is an $n \times n$ matrix with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then its *exponential*, denoted e^A or $\exp(A)$, is

$$\sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \cdots ,$$

where the series converges in $\mathbb{K}^{n \times n}$.

Remark

If A has real entries, then $\exp(A)$ has real entries as well.

The Matrix Exponential

Example (Exponentials of diagonal matrices)

Let D be a diagonal matrix,

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Then its exponential is diagonal,

$$\exp(D) = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}.$$

The Matrix Exponential

Proposition (Main algebraic properties)

The following holds:

$$\exp(0) = I, \qquad \left(e^{A}\right)^{-1} = e^{-A},$$
$$e^{A+B} = e^{A}e^{B} = e^{B}e^{A} \qquad \text{if } AB = BA,$$
$$\exp\left(P^{-1}AP\right) = P^{-1}\exp(A)P \qquad \forall P \in \mathrm{GL}(n, \mathbb{C}).$$

Example (Exponentials of diagonalizable matrices)

Let A be a diagonalizable matrix,

$$A = P^{-1}DP, \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Then

$$\exp(A) = P^{-1}\exp(D)P = P^{-1}\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} P.$$

In particular, $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are the eigenvalues of e^A.

The Matrix Exponential

Proposition (Proposition 15.17)

Let $A \in \mathbb{R}^{n\times n}$*. Then* $\mathbb{R} \ni t \to \exp(tA)$ *is a smooth curve in* $\mathrm{GL}(n,\mathbb{R})$ *such that*

$$\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A, \qquad t \in \mathbb{R}.$$

Remark

It can be shown that $A \to \exp(A)$ is a C^∞ map from $\mathbb{R}^{n\times n}$ to $\mathrm{GL}(n,\mathbb{R})$.

Remark

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$.

- The map $\mathbb{R}\times\mathbb{R}^n \ni (t,p) \to \exp(tA)p \in \mathbb{R}^n$ is the flow of the vector field,
$$X = \sum_{i,j} a_{ij}x^j \frac{\partial}{\partial x^i} \qquad \text{on } \mathbb{R}^n.$$
- Indeed, if $x(t) = (x^1(t),\ldots,x^n(t))$, then
$$\begin{aligned} \frac{dx}{dt} = X_{x(t)} \Longleftrightarrow \dot{x}^i(t) &= \sum_j a_{ij}x^j(t), \quad i = 1,\ldots,n, \\ \Longleftrightarrow \dot{x}(t) &= Ax(t), \\ \Longleftrightarrow x(t) &= e^{tA}x(0). \end{aligned}$$
- Thus, if $x(0) = p$, then $\mathbb{R} \ni t \to e^{tA}p$ is the (maximal) line integral of X that starts at p.

Reminder (Trace of a Matrix)

- If $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, then its *trace* is

$$\begin{aligned} \operatorname{tr}(A) &= a_{11} + \cdots + a_{nn}, \\ &= \lambda_1 + \cdots + \lambda_n, \end{aligned}$$

  where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A counted with multiplicity.

- We have

$$\begin{aligned} \operatorname{tr}(AB) &= \operatorname{tr}(BA), \\ \operatorname{tr}\left(P^{-1}AP\right) &= \operatorname{tr}(A) \quad \forall P \in \mathrm{GL}(n, \mathbb{C}). \end{aligned}$$

The Differential of the Determinant at the Identity

Proposition (Proposition 15.20)

Let $A \in \mathbb{C}^{n\times n}$. Then

$$\det\left[e^A\right] = e^{\mathrm{tr}(A)}.$$

Remark

Let A be diagonalizable and have eigenvalues $\lambda_1, \ldots, \lambda_n$.

- By the example of slide 22 the matrix e^A has eigenvalues $e^{\lambda_1}, \ldots, e^{\lambda_n}$.
- Thus,

$$\det\left[e^A\right] = e^{\lambda_1}\cdots e^{\lambda_n} = e^{\lambda_1+\cdots+\lambda_n} = e^{\mathrm{tr}(A)}.$$

The Differential of the Determinant at the Identity

Facts

- The determinant is a C^∞ map $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$.
- The tangent space of the vector space $\mathbb{R}^{n\times n}$ at I is $T_I(\mathbb{R}^{n\times n}) = \mathbb{R}^{n\times n}$.
- The tangent space of $\mathbb{R}$ at 1 is $T_1(\mathbb{R}) = \mathbb{R}$.
- Thus, the differential $\det_{*,I}$ is a linear map $\mathbb{R}^{n\times n} \to \mathbb{R}$.
- For every $X \in \mathbb{R}^{n\times n}$ the curve $c(t) = e^{tX}$ is a smooth curve such that $c(0) = I$ whose velocity at $t = 0$ is $c'(0) = X$.

Reminder (Proposition 8.18)

Let $F : N \to M$ be a smooth map. Given $p \in N$ and $X \in T_pN$, for any smooth curve $c : I \to N$ starting at p and with velocity vector X at p, we have

$$F_*(X) = (F \circ c)'(0).$$

The Differential of the Determinant at the Identity

Proposition (Proposition 15.21)

We have

$$\det_{*,I}(X) = \mathrm{tr}(X) \qquad \forall X \in \mathbb{R}^{n\times n}$$

Proof.

- Set $c(t) = e^{tX}$. This is a C^∞ curve in $\mathbb{R}^{n\times n}$ such that $c(0) = I$ and $c'(0) = X$.Thus,

$$\det_{*,I}(X) = \frac{d}{dt}\bigg|_{t=0} \det\left(c(t)\right) = \frac{d}{dt}\bigg|_{t=0} \det\left(e^{tX}\right).$$

- As $\det(e^{tX}) = e^{t\,\mathrm{tr}(X)}$, we get

$$\det_{*,I}(X) = \frac{d}{dt}\bigg|_{t=0} e^{t\,\mathrm{tr}(X)} = \mathrm{tr}(X).$$

The result is proved. □

Final Remark

Remark

Let V be a vector space of dimension n.

- V is a smooth manifold of dimension n.
- If $p \in V$, then we have natural map $V \to T_pV$, $v \to D_{p,v}$, where $D_{p,v} \in T_pV$ is defined by
$$D_{p,v}f = \frac{d}{dt}\bigg|_{t=0} f(p+tv), \qquad f \in C_p^\infty(V).$$
- In the same way as with $\mathbb{R}^n$ (*cf.* Chapter 2) it can be shown that the map $v \to D_{p,v}$ yields an isomorphism,
$$T_p(V) \simeq V.$$
- It can be also shown that $V \times V \ni (p,v) \to D_{p,v} \in TV$ is a trivialization of V. Thus,
$$TV \simeq V \times V \qquad \text{as } C^\infty \text{ vector bundles.}$$