Differentiable Manifolds
§13. Bump Functions and Partitions of Unity

Sichuan University, Fall 2021

C^∞ Bump Functions

Reminder (Support of a function)

If $f : X \to \mathbb{R}$ is a (continuous) function on a topological space X, then its *support*, denoted $\text{supp}(f)$, is the closure in X of the points at which f is non-zero. That is,

$$\text{supp}(f) = \overline{\{x \in X;\ f(x) \neq 0\}}.$$

We say that f is *compactly supported* when $\text{supp}(f)$ is compact.

Remark

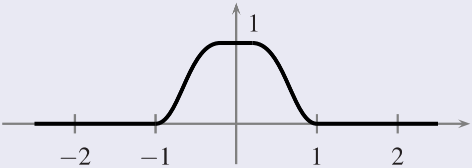
- The complement $X \setminus \text{supp}(f)$ is the interior of the zero-set of f. That is, $x \notin \text{supp}(f)$ if and only if $f = 0$ near x.
- Thus, if $f = 0$ on an open set U, then $U \subset X \setminus \text{supp}(f)$, and hence $\text{supp}(f) \subset X \setminus U$.

C^{∞} Bump Functions

Definition

Given a point q in a manifold M and an open neighborhood U of q in M a bump function at q supported in U is any continuous function $\rho : M \to \mathbb{R}$ such that:

$$0 \le \rho \le 1, \qquad \rho = 1 \text{ near } q, \qquad \operatorname{supp}(\rho) \subseteq U.$$



C^∞ Bump Functions

We are more especially interested in *smooth* bump functions. Our aim is to establish the following result:

Proposition (Exercise 13.1)

For every point $q \in M$ and any neighborhood U of p in M, there exists a C^∞ bump function at q supported in U.

Remark

We shall first construct a C^∞ bump functions on $\mathbb{R}$, and then we are going to extend the construction to $\mathbb{R}^n$ and manifolds.

C^∞ Bump Functions

Step 1 (Example 1.3 and Problem 1.2)

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} e^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

By definition $0 \leq f \leq 1$ and $f(t) = 0$ if and only if $t \leq 0$. It can also be shown that f is smooth (see Problem 1.2).

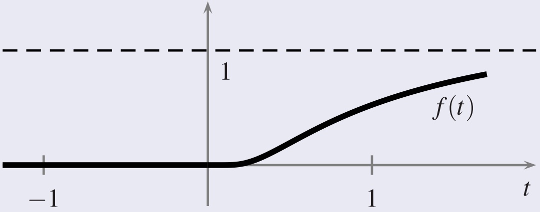


Fig. 13.3. The graph of $f(t)$.

C^∞ Bump Functions

Step 2

Let $g : \mathbb{R} \to \mathbb{R}$ be given by

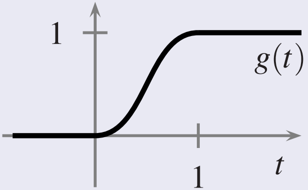
$$g(t) = \frac{f(t)}{f(t) + f(1-t)}, \qquad t \in \mathbb{R}.$$

- Here $g(t)$ is well defined, since $f(t) + f(1-t) > 0$. Indeed:
  - $f \geq 0$, and so $f(t) + f(1-t) \geq \max\{f(t), f(1-t)\}$.
  - $f(t) > 0$ for $t > 0$, and $f(1-t) > 0$ for $1 - t > 0$, i.e., $t < 1$.
  - Thus, $f(t) + f(1-t) > 0$ on $(0, \infty) \cup (-\infty, 1) = \mathbb{R}$.
- The function g is C^∞, since f is C^∞ and $f(t) + f(1-t)$ is C^∞ and > 0.
- As $f \geq 0$, we have $0 \leq g \leq 1$. Furthermore:

$$g(t) = 0 \Longleftrightarrow f(t) = 0 \Longleftrightarrow t \leq 0,$$
$$g(t) = 1 \Longleftrightarrow f(1-t) = 0 \Longleftrightarrow 1 - t \leq 0 \Longleftrightarrow t \geq 1.$$

C^∞ Bump Functions

Step 2



We have constructed a C^∞ function $g : \mathbb{R} \to \mathbb{R}$ such that

$$0 \leq g \leq 1,$$

$$g^{-1}(0) = (-\infty, 0], \qquad g^{-1}(1) = [1, \infty).$$

Step 3

Let $0 < a < b$, and define $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2}\right), \qquad x \in \mathbb{R}.$$

- $\rho(x)$ is a C^∞ function, since g is C^∞.
- As $0 \leq g \leq 1$, and hence $0 \leq 1 - g \leq 1$, we have $0 \leq \rho \leq 1$.
- We have

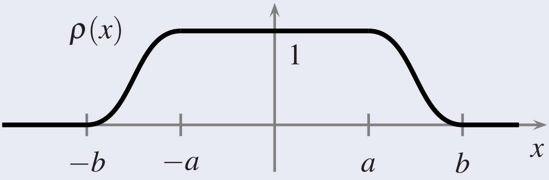
$$\rho(x) = 1 \Leftrightarrow g\left(\frac{x^2 - a^2}{b^2 - a^2}\right) = 0 \Leftrightarrow \frac{x^2 - a^2}{b^2 - a^2} \leq 0 \Leftrightarrow x^2 \leq a^2,$$
$$\rho(x) = 0 \Leftrightarrow g\left(\frac{x^2 - a^2}{b^2 - a^2}\right) = 1 \Leftrightarrow \frac{x^2 - a^2}{b^2 - a^2} \geq 1 \Leftrightarrow x^2 \geq b^2.$$

Thus,

$$\rho(x) = 1 \Leftrightarrow |x| \leq a, \qquad \rho(x) = 0 \Leftrightarrow |x| \geq b.$$

C^∞ Bump Functions

Step 3



We have constructed a C^∞ function $\rho : \mathbb{R} \to \mathbb{R}$ such that

$$0 \leq \rho \leq 1,$$

$$\rho^{-1}(1) = [-a, a], \qquad \rho^{-1}(0) = (-\infty, -b] \cup [b, \infty).$$

In particular, ρ is a C^∞ bump function at the origin on $\mathbb{R}$.

C^∞ Bump Functions

Notation

If $q \in \mathbb{R}^n$ and $r > 0$, then $B(q, r)$ is the open ball about q with radius r in $\mathbb{R}^n$ and $\overline{B}(q, r)$ is its closure.

Step 4

Let $q \in \mathbb{R}^n$, and define $\sigma : \mathbb{R}^n \to \mathbb{R}$ by

$$\sigma(x) = \rho(\|x - q\|) = 1 - g\left(\frac{\|x - q\|^2 - a^2}{b^2 - a^2}\right), \qquad x \in \mathbb{R}^n.$$

- As g and $x \to \|x - q\|^2$ are C^∞ functions, σ is C^∞ as well.
- As $0 \leq \rho \leq 1$, we have $0 \leq \sigma \leq 1$.
- As $\rho^{-1}(1) = [-a, a]$ and $\rho^{-1}(0) = (-\infty, -b] \cup [b, \infty)$, we get

$$\sigma^{-1}(1) = \overline{B}(q, a), \qquad \sigma^{-1}(0) = \mathbb{R}^n \setminus B(q, b).$$

Thus, σ is a C^∞ bump function at q supported in $\overline{B}(q, b)$.

C^∞ Bump Functions

Step 5

Suppose that M is a manifold. Let U be an open set in M and $f : U \to \mathbb{R}$ a smooth function with compact support. Define $\tilde{f} : M \to \mathbb{R}$ by

$$\tilde{f} = f \text{ on } U \qquad \text{and} \qquad \tilde{f} = 0 \text{ on } M \setminus U.$$

- As $\tilde{f} = f$ on U, we see that $\tilde{f}$ is C^∞ on U.
- Set $K = \operatorname{supp}(f)$. This is a compact in U. As the inclusion of U into M is continuous, this is a compact in M as well.
- Here $\tilde{f} = 0$ on $M \setminus U$ and $\tilde{f} = f = 0$ on $U \setminus K$. Thus, $\tilde{f} = 0$ on $(M \setminus U) \cup (U \setminus K) = M \setminus K$. In particular, $\tilde{f}$ is C^∞ on the open set $M \setminus K$.
- As $\tilde{f}$ is C^∞ on the open sets U and $M \setminus K$, it is C^∞ on $U \cup (M \setminus K) = M$.

C^∞ Bump Functions

Step 5 (continued)

- As $\tilde{f} = 0$ on the open set $M \setminus K$, it follows that $\operatorname{supp} \tilde{f} \subset K$ (*cf.* remark on slide 2).
- If $p \in K = \operatorname{supp}(f) \subset U$, then $\tilde{f} = f$ near p and p is not an interior point of $f^{-1}(0) = U \cap \tilde{f}^{-1}(0)$. Thus p is in the interior of $\tilde{f}^{-1}(0)$, and hence $p \in \operatorname{supp}(\tilde{f})$. Thus $K \subset \operatorname{supp}(\tilde{f})$
- It follows that $\operatorname{supp}(\tilde{f}) = K$.

Therefore, we have proved the following result:

Lemma

Let U be an open set in M and $f : U \rightarrow \mathbb{R}$ a smooth function with compact support. Then f uniquely extends to a smooth function $\tilde{f} : M \rightarrow \mathbb{R}$ such that

$$\tilde{f} = 0 \text{ on } M \setminus U \qquad \text{and} \qquad \operatorname{supp}(\tilde{f}) = \operatorname{supp}(f).$$

C^∞ Bump Functions

Step 6

Let $p \in M$ and U an open in M containing p. Set $n = \dim M$.

- Let (V, ϕ) be chart near p. Possibly by replacing (V, ϕ) by $(V \cap U, \phi_{|V \cap U})$ we may assume that $V \subset U$.
- Set $q = \phi(p)$. Here $\phi : V \to \phi(V)$ is a homeomorphism. In particular, $\phi(V)$ is an open in $\mathbb{R}^n$ containing q.
- Let $0 < a < b$ be such that $\overline{B}(q, b) \subset \phi(V)$. Define $\sigma : \mathbb{R}^n \to \mathbb{R}$ as in Step 4. Thus, σ is a C^∞ function such that $0 \leq \sigma \leq 1$, $\sigma^{-1}(1) = \overline{B}(q, a)$ and $\sigma^{-1}(0) = \mathbb{R}^n \setminus B(0, b)$.
- Set $\psi = \sigma \circ \phi : V \to \mathbb{R}$. Then ψ is C^∞ and $0 \leq \psi \leq 1$.
- We have

  $$\psi^{-1}(1) = (\sigma \circ \phi)^{-1}(1) = \phi^{-1}\left(\sigma^{-1}(1)\right) = \phi^{-1}\left(\overline{B}(q, a)\right).$$

  Thus, $\psi = 1$ on $\phi^{-1}\left(B(q, a)\right)$, which is an open containing $\phi^{-1}(q) = p$. Hence $\psi = 1$ near p.

C^∞ Bump Functions

Step 6 (continued)

- The zero set $\psi^{-1}(0) = (\sigma \circ \phi)^{-1}(0)$ is equal to
$$\phi^{-1}\big(\sigma^{-1}(0)\big) = \phi^{-1}\big(\mathbb{R}^n \setminus B(q,b)\big) = V \setminus \phi^{-1}\big(B(q,b)\big).$$
- Set $K = \phi^{-1}(\overline{B}(q,b))$. This is a compact subset of V since $\overline{B}(q,b)$ is compact and ϕ is a homeomorphism.
- As $V \setminus K \subset V \setminus \phi^{-1}\big(B(q,b)\big) = \psi^{-1}(0)$, we see that $\psi = 0$ on the open set $V \setminus K$, and hence $\operatorname{supp}(\psi) \subset K$.
- By Step 5 we can extends ψ to a smooth function $\tilde{\psi} : M \to \mathbb{R}$ such that $\tilde{\psi} = 0$ on $M \setminus V$ and $\operatorname{supp}(\tilde{\psi}) = K$. In particular:
$$0 \leq \tilde{\psi} \leq 1, \quad \tilde{\psi} = \psi = 1 \text{ near } p, \quad \operatorname{supp}(\tilde{\psi}) \subset K \subset V \subset U.$$
Thus, $\tilde{\psi}$ is a C^∞ bump function at p with support in U.
- Note also that $\tilde{\psi}$ has compact support, since $\operatorname{supp}(\tilde{\psi})$ is a closed subset of the compact K.

C^∞ Bump Functions

To sum up we have proved:

Proposition

Suppose that M is a smooth manifold. For any $p \in M$ and any open neighborhood U of p in M there exists a C^∞ bump function at p with compact support in U.

C^∞ Bump Functions

Remark

Let $f : U \to \mathbb{R}$ be a C^∞ function on an open U of a manifold M.

- If f is constant outside some compact set (or even some closed set in M contained in U), then in a similar way as in Step 5 we may extend f into a C^∞ on the whole manifold M.
- In general this is not possible. However, we still have the following result:

Proposition (Proposition 13.2; Extension of smooth functions)

Let $p \in M$ and let U be an open neighborhood of p. Then, for every smooth function $f : U \to \mathbb{R}$, there exists a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f} = f$ near p and $\operatorname{supp}(\tilde{f}) \subset U$.

C^∞ Bump Functions

Proof of Proposition 13.2.

Let $\psi : M \to \mathbb{R}$ be a C^∞ bump function at p whose support is compact and contained in U. Then:

- The product ψf is a smooth function on U whose support is contained in $\operatorname{supp}(\psi)$, and hence is compact.
- By Step 5 it extends to a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\operatorname{supp}(\tilde{f}) = \operatorname{supp}(\psi f) \subset U$.
- As $\tilde{f} = \psi f$ on $U \ni p$ and $\psi = 1$ near p, it follows that $\tilde{f} = \psi f = f$ near p.

This proves the result. $\square$

Definition

A family of subsets $(Y_\alpha)_{\alpha \in A}$ of a topological space X is called *locally finite* when, for every $p \in X$, there is a neighborhood V of p such that meets at most finitely many of the Y_α, i.e., the index subset $\{\alpha \in A;\ Y_\alpha \cap V \neq \emptyset\}$ is finite.

Lemma (see Problem 13.7)

Let $(Y_\alpha)_{\alpha \in A}$ be a locally finite family in a topological space X. Then

$$\overline{\bigcup Y_\alpha} \subset \bigcup \overline{Y_\alpha}.$$

Facts

Let $(f_\alpha)_{\alpha\in A}$ be a family of smooth (real-valued) functions on a manifold M such that the family of supports $(\mathrm{supp}(f_\alpha))_{\alpha\in A}$ is locally finite.

- Let $p\in M$ and V an open neighborhood of p such that $J:=\{\alpha;\, V\cap\mathrm{supp}(f_\alpha)\neq\emptyset\}$ is finite.
- Then $J(p):=\{\alpha;\, p\in\mathrm{supp}(f_\alpha)\}\subset J$ is finite. In particular, $f_\alpha(p)=0$ for $\alpha\notin J(p)$.
- We then set

$$\sum_{\alpha\in A} f_\alpha(p) := \sum_{\alpha\in J(p)} f_\alpha(p).$$

  This defines a function $\sum_{\alpha\in A} f_\alpha : M\rightarrow\mathbb{R}$. Such a function is called a *locally finite sum*.

Partitions of Unity

Facts (continued)

- If $q \in V$, then $J(q) \subset J$ and $f_\alpha(q) = 0$ if $\alpha \in J \setminus J(q)$. Thus,
$$\sum_{\alpha \in A} f_\alpha(q) = \sum_{\alpha \in J(q)} f_\alpha(q) = \sum_{\alpha \in J} f_\alpha(q).$$
That is, $\sum_{\alpha \in A} f_\alpha = \sum_{\alpha \in J} f_\alpha$ on V.
- As $\sum_{\alpha \in J} f_\alpha$ is a finite sum of C^∞ function, we deduce that $\sum_{\alpha \in A} f_\alpha$ is C^∞ on V, and hence is C^∞ near p.
- As this is true for every $p \in M$, it follows that $\sum_{\alpha \in A} f_\alpha$ is C^∞ function on M.

Partitions of Unity

Lemma (see also Lemma 13.5)

Let $\sum f_\alpha$ be a locally finite sum of C^∞ functions on M. Then

$$\operatorname{supp}\left(\sum f_\alpha\right) \subset \bigcup \operatorname{supp}(f_\alpha).$$

Proof of the lemma.

Set $f = \sum f_\alpha$. Recall that $\text{supp}(f) = \overline{f^{-1}(\mathbb{R} \setminus 0)}$.

- If $f_\alpha(p) = 0$ for all α, then $f(p) = \sum f_\alpha(p) = 0$. Thus, if $f(p) \neq 0$, then $f_\alpha(p) \neq 0$ for some α. That is,

$$f^{-1}(\mathbb{R} \setminus 0) \subset \bigcup f_\alpha^{-1}(\mathbb{R} \setminus 0).$$

- By assumption the family of subsets $\{\text{supp}(f_\alpha)\}$ is locally finite in M, and so $\{f_\alpha^{-1}(\mathbb{R} \setminus 0)\}$ is locally finite as well.
- Thus, by the lemma of slide 18 we have

$$\overline{f^{-1}(\mathbb{R} \setminus 0)} \subset \overline{\bigcup f_\alpha^{-1}(\mathbb{R} \setminus 0)} \subset \bigcup \overline{f_\alpha^{-1}(\mathbb{R} \setminus 0)}.$$

That is,

$$\text{supp}(f) \subset \bigcup \text{supp}(f_\alpha)$$

This proves the lemma. □

Partitions of Unity

Reminder

An *open cover of* M is a family of open sets $(U_\alpha)_{\alpha \in A}$ such that $M = \cup_{\alpha \in A} U_\alpha$.

Definition

A C^∞ *partition of unity* on M is a family $(\rho_\alpha)_{\alpha \in A}$ of real-valued functions on M such that

(i) $\rho_\alpha \geq 0$ for all $\alpha \in A$.

(ii) The family of supports $\{\mathrm{supp}(\rho_\alpha)\}_{\alpha \in A}$ is locally finite.

(iii) $\sum \rho_\alpha = 1$ on M.

If $(U_\alpha)_{\alpha \in A}$ is an open cover of M such that $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$, then we say that the partition of unity is *subordinate to* $(U_\alpha)_{\alpha \in A}$.

Partitions of Unity

Proposition (Proposition 13.6)

Suppose that M is a compact manifold, and let $(U_\alpha)_{\alpha\in A}$ be an open cover of M. Then there exists a (finite) C^∞ partition of unity $(\rho_\alpha)_{\alpha\in A}$ subordinate to $(U_\alpha)_{\alpha\in A}$.

Proof of Proposition 13.6 (Part 1)

- Given $q \in M$, let $\alpha \in A$ be such that $q \in U_\alpha$. Let ψ_q be a C^∞ bump function at q supported in U_α. In particular, $0 \leq \psi_q \leq 1$ and $\psi_q = 1$ near q.
- For each $q \in M$ set $W_q = \{\psi_q > 0\}$. Then W_q is an open neighborhood of q, and we have $M = \cup_{q \in M} W_q$.
- As M is compact there are $q_1, \ldots, q_m$ in M such that $M = \cup_{i=1}^m W_{q_i}$.
- Set $\psi = \sum_{i=1}^m \psi_{q_i}$. This is a C^∞ function on M.
- If $q \in M$, then there i such that $q \in W_{q_i} = \{\psi_{q_i} > 0\}$. Thus,

$$\psi(q) = \sum \psi_{q_j}(q) \geq \psi_{q_i}(q) > 0.$$

Hence $\psi > 0$ on M.

Partitions of Unity

Proof of Proposition 13.6 (Part 1, continued)

- For $i = 1, \ldots, m$, set
$$\varphi_i = \frac{\psi_{q_i}}{\psi} = \frac{\psi_{q_i}}{\sum_{j=1}^{m} \psi_{q_j}}.$$
This is a well defined C^∞ function on M since $\psi > 0$.
- We have $\varphi_i \geq 0$, and
$$\sum_{1 \leq i \leq m} \varphi_i = \frac{\sum_{i=1}^{m} \psi_{q_i}}{\sum_{j=1}^{m} \psi_{q_j}} = 1.$$
- For each i there is $\tau(i) \in A$ such $\operatorname{supp} \psi_{q_i} \subset U_{\tau(i)}$, and hence
$$\operatorname{supp}(\varphi_i) \subset \operatorname{supp}(\psi_{q_i}) \subset U_{\tau(i)}.$$
Thus, $(\varphi_i)_{i=1}^{m}$ is a (finite) C^∞ partition of unity such that, for each i there is $\alpha = \tau(i)$ in A so that $\operatorname{supp} \varphi_i \subset U_\alpha$.

Proof of Proposition 13.6 (Part 2)

- Set $I = \{1, \ldots, m\}$. We have a map $\tau : i \to \tau(i)$ from I to A such that $\operatorname{supp} \varphi_i \subset U_{\tau(i)}$. If $\alpha \in \tau(I)$, set
$$\rho_\alpha = \sum_{i \in \tau^{-1}(\alpha)} \varphi_i.$$
Otherwise, set $\rho_\alpha = 0$. In any case $\rho_\alpha \in C^\infty(M)$ and $\rho_\alpha \geq 0$.
- Here $\operatorname{supp} \rho_\alpha \neq \emptyset \Leftrightarrow \rho_\alpha \neq 0 \Leftrightarrow \alpha \in \tau(I)$. As $\tau(I)$ is finite, this means that all but finitely of the supports $\operatorname{supp} \rho_\alpha$ are empty, and so the family $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in A}$ is locally finite.
- As $I = \cup_{\alpha \in \tau(I)} \tau^{-1}(\alpha)$ and $\rho_\alpha = 0$ if $\alpha \notin \tau(I)$, we have
$$\sum_{\alpha \in A} \rho_\alpha = \sum_{\alpha \in \tau(I)} \rho_\alpha = \sum_{\alpha \in \tau(I)} \sum_{i \in \tau^{-1}(\alpha)} \varphi_i = \sum_{1 \leq i \leq m} \varphi_i = 1.$$

Proof of Proposition 13.6 (Part 2, continued).

- If $\alpha \notin \tau(I)$, then $\rho_\alpha = 0$, and hence $\operatorname{supp} \rho_\alpha = \emptyset \subset U_\alpha$.
- If $\alpha \in \tau(I)$ and $i \in \tau^{-1}(i)$, then $\operatorname{supp} \varphi_i \subset U_{\tau(i)} = U_\alpha$. Thus, by using the lemma of slide 21 we get

$$\operatorname{supp} \rho_\alpha = \operatorname{supp} \Big(\sum_{i \in \tau^{-1}(\alpha)} \varphi_i \Big) \subset \bigcup_{i \in \tau^{-1}(\alpha)} \operatorname{supp}(\varphi_i) = U_\alpha.$$

Therefore, $\{\rho_\alpha\}_{\alpha \in A}$ is a C^∞-partition of the unity subordinate to $\{U_\alpha\}_{\alpha \in A}$. The proof is complete. □

Partitions of Unity

In general, we have the following result (see Appendix C of Tu's book for a proof).

Theorem (Theorem 13.7; Existence of C^∞ partition of unity)

Let $\{U\}_{\alpha\in A}$ be an open cover of M.

(i) *There is a C^∞ partition of unity $(\varphi_k)_{k\geq 1}$ such that, for each integer $k \geq 1$, the support of φ_k is compact and contained in some U_α.*

(ii) *If we do not require compact support, then there is a C^∞-partition of the unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$.*

Remarks

1. Part (i) is the substitute for Part 1 of the proof of Proposition 13.6 in the noncompact case.
2. We deduce part (ii) from part (i) in a similar way as in Part 2 of Proposition 13.6 by using locally finite sums instead of finite sums.

Partitions of Unity

We shall use partitions of unity throughout the rest of the course. Here is a couple of direct applications.

Proposition (Smooth Urysohn Lemma; Problem 13.3(a))

If A and B are disjoint closed sets in M, then there is $f \in C^\infty(M)$ such that $f = 1$ on A and $f = 0$ on B.

Theorem

Every closed set in M is the zero set of some non-negative smooth function on M.

Remark

A proof of the above theorem can be found in Chapter 2 of J.M. Lee's book "Introduction to Smooth Manifolds".