Differentiable Manifolds
§6. Smooth Maps on a Manifold

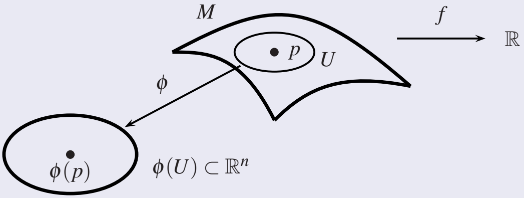
Sichuan University, Fall 2021

Smooth Functions on a Manifold

Definition (Smooth functions)

Let M be a manifold of dimension n.

- A function $f : M \to \mathbb{R}$ is said to be C^∞ or *smooth at a point* $p \in M$ when there is a chart (U, ϕ) about p in M such that the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞ at $\phi(p)$ (here $\phi(U)$ is an open subset of $\mathbb{R}^n$).
- We say that f is C^∞ *on* M when it is C^∞ at every point of M.



Smooth Functions on a Manifold

Remark

- The smoothness condition is independent of the choice of the chart (U, ϕ).
- If (V, ψ) is another chart about p and $f \circ \phi^{-1}$ is C^∞, then $f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$ is C^∞ at p as well, since the transition map $\phi \circ \psi^{-1}$ is a C^∞.

Remark

- If a function $f : M \to \mathbb{R}$ is C^∞ at p, then it is automatically continuous at p.
- If (U, ϕ) is a chart about p and $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$, then $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p, since ϕ is a continuous map.
- Therefore, any C^∞-function on M is continuous.

Smooth Functions on a Manifold

Proposition (Proposition 6.3)

Let $f : M \to \mathbb{R}$ be a function. Then TFAE:

1. *f is C^∞ on M.*
2. *There is a C^∞-atlas $\{(U_\alpha, \phi_\alpha)\}$ on M such that the function $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \supset \phi_\alpha(U_\alpha) \to \mathbb{R}$ is C^∞ for all α.*
3. *For every chart (U, ϕ) on M, the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞.*

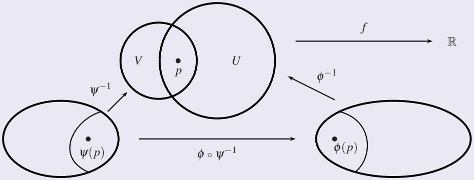
Smooth Maps Between Manifolds

In what follows M is a manifold of dimension m and N is a manifold of dimension n.

Definition (Smooth maps between manifolds)

Let $F : N \to M$ be a continuous map.

- We say that F is C^∞ or *smooth at* $p \in N$ when there are a chart (U, ϕ) about p in N and a chart (V, ψ) about $F(p)$ on N such that the map $\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$ (here $\phi(F^{-1}(V) \cap U)$ is an open set in $\mathbb{R}^n$).
- Then map F is C^∞ on N when it is C^∞ at every point $p \in N$.



Remarks

- We assume $F : N \to M$ to be continuous to ensure that $F^{-1}(V)$ is an open set in N.
- When $M = \mathbb{R}^m$ the continuity assumption can be dropped.

Proposition (Remark 6.6)

A map $F : N \to \mathbb{R}^m$ is C^∞ at p if and only if there is a chart (U, ϕ) about p in N such that the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ at p (here $\phi(U)$ is an open set in $\mathbb{R}^n$).

Smooth Maps Between Manifolds

Proposition (Proposition 6.7)

Suppose that $F : N \to M$ is C^∞ at p. Then, for every chart (U, ϕ) about p in N and every chart (V, ψ) about $F(p)$ in M, the map $\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$.

Proposition (Proposition 6.8)

Let $F : N \to M$ be a continuous map. TFAE:

1. *F is a C^∞ map.*
2. *There are C^∞-atlases (U_α, ϕ_α) for N and (V_β, ψ_β) for M such that the map $\psi_\beta \circ F \circ \phi_\beta^{-1} : \phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to \mathbb{R}^m$ is C^∞ for every α and β.*
3. *For every chart (U, ϕ) on N and every chart (V, ψ) on M, the map $\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^m$ is C^∞.*

Smooth Maps Between Manifolds

Proposition (Proposition 6.9; Composition of C^∞ maps)

If $F : N \to M$ and $G : P \to N$ are C^∞ maps (where P is a manifold), then the composition $F \circ G : P \to M$ is a C^∞ map.

Diffeomorphisms

Definition

We say that a map $F : N \to M$ is a *diffeomorphism* when it is a bijective C^∞ map with C^∞ inverse F^{-1}.

Proposition (Proposition 6.10)

If (U, ϕ) is a chart on M, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^m$ is a diffeomorphism.

Proposition (Proposition 6.11)

Let U be an open subset of M. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^m$, then the pair (U, F) is a chart on M.

Smoothness in Terms of Components

Proposition (Propositions 6.12 & 6.13)

Let $F : N \to \mathbb{R}^m$ be a map with components $F^1, \ldots, F^m : N \to \mathbb{R}$ (so that $F(p) = (F^1(p), \ldots, F^n(p))$. Then TFAE:

1. *F is a C^∞-map.*
2. *For every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*
3. *All the components $F^1, \ldots, F^m : N \to \mathbb{R}$ are C^∞ maps*

Remark

We don't have to assume F to be continuous, since the 2nd and 3rd properties both imply that F is continuous.

Smoothness in Terms of Components

Proposition (Propositions 6.15 & 6.16)

Let $F : N \to M$ be a continuous map. Then TFAE:

1. *F is a C^∞ map.*
2. *For every chart (V, ψ) on M the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.*
3. *For very chart $(V, \psi) = (V, y^1, \ldots, y^n)$ the component functions $y^i \circ F : F^{-1}(V) \to \mathbb{R}^m$ are C^∞.*

Remark

We assume F to be continuous to insure that in the 2nd and 3rd properties $F^{-1}(V)$ is an open subset of $\mathbb{R}^n$.

Examples of Smooth Maps

Example (Example 6.17 + Exercise 6.18)

Let M_1 and M_2 be manifolds.

1. The 1st factor projection $\pi_1 : M_1 \times M_2 \to M_1$, $\pi_1(p_1, p_2) = p_1$ is a C^∞ map. Likewise, the 2nd factor projection $\pi_2 : M_1 \times M_2 \to M_2$ is a smooth map.
2. Given a manifold N, a map $f : N \to M_1 \times M_2$ is C^∞ if and only if the components $\pi_i \circ f : N \to M_i$ are C^∞ maps.

Examples of Smooth Maps

Example (Example 6.19)

Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere. If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a C^∞ function, then the restriction $f_{|\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{R}$ is a C^∞ function on $\mathbb{S}^n$.

Examples of Smooth Maps

Definition (Lie Groups)

A *Lie group* is a group G equipped with a differentiable structure such that:

(i) The multiplication map $\mu : G \times G \to G$, $(x, y) \to xy$ is a C^∞ map.

(ii) The inverse map $\iota : G \to G$, $x \to x^{-1}$ is a C^∞ map.

Examples

1. The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are Lie groups under addition.
2. The set of non-zero complex numbers $\mathbb{C}^\times := \mathbb{C} \setminus 0$ is a Lie group under multiplication.
3. The unit circle $\mathbb{S}^1 \subset \mathbb{C}^\times$ is a Lie group under multiplication.
4. If G_1 and G_2 are Lie groups, then their Cartesian product $G_1 \times G_2$ is again a Lie group.

Examples of Smooth Maps

Example (Example 6.21; see Tu's book)

We saw in Section 5 that the general groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are manifolds. They are also Lie groups under multiplication of matrices.

Remark

Further examples of Lie groups will be presented in Section 15.

Partial Derivatives

In what follows M is a manifold of dimension n.

Reminder

If $(U, \phi) = (U, x^1, \dots, x^n)$ a chart on M, then by definition the components $x^1, \dots, x^n$ of ϕ are given by $x^i = r^i \circ \phi : U \to \mathbb{R}$.

Definition

Let $f : M \to \mathbb{R}$ be a C^∞ function. For $p \in U$ the *partial derivative of f with respect to x^i at p* is

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \phi^{-1})}{\partial r^i}\left(\phi(p)\right).$$

Remark

The partial derivative $\frac{\partial f}{\partial x^i}(p)$ is also denoted $\frac{\partial}{\partial x^i}\Big|_p f$.

Partial Derivatives

Remark

As $\phi^{-1}(\phi(p)) = p$ the equality $\frac{\partial f}{\partial x^i}(p) = \frac{\partial(f\circ\phi^{-1})}{\partial r^i}(\phi(p))$ can be rewritten as

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} (\phi(p)) = \frac{\partial(f \circ \phi^{-1})}{\partial r^i} (\phi(p)) \, .$$

Thus, as functions on $\phi(U)$ we have

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial(f \circ \phi^{-1})}{\partial r^i}.$$

In particular, this shows that $\frac{\partial f}{\partial x^i} : U \to \mathbb{R}$ is C^∞ function on U.

Proposition (Proposition 6.22)

If $(U, x^1, \dots, x^n)$ is a chart on M, then $\frac{\partial x^i}{\partial x^j} = \delta^i_j$.

Partial Derivatives

In what follows M is a manifold of dimension m and N is a manifold of dimension n.

Definition (Jacobian matrices and Jacobian determinants)

Let $F : M \to N$ be a C^∞ map. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on N and $(V, \psi) = (V, y^1, \ldots, y^m)$ a chart on M such that $F(U) \subset V$. Denote $F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$ the i-th component of F in the chart (V, ψ).

1. The matrix $\left[\partial F^i / \partial x^j\right]$ is called the *Jacobian matrix* of F relative to the charts (U, ϕ) and (V, ψ).
2. When $m = n$ the determinant $\det\left[\partial F^i / \partial x^j\right]$ is called the *Jacobian determinant* of F relative to the charts.

Remark

The Jacobian determinant is also denoted $\partial(F^1, \ldots, F^n)/\partial(x^1, \ldots, x^n)$.

Remark

If $N = U$ is an open subset of $\mathbb{R}^n$ and $M = V$ is an open subset of $\mathbb{R}^m$, and we use the charts $(U, r^1, \ldots, r^n)$ and $(V, r^1, \ldots, r^m)$, then the Jacobian matrix $\left[\partial F^i / \partial r^j\right]$ is the usual Jacobian matrix from calculus.

Example (Example 6.24; Jacobian matrix of a transition map)

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be overlapping charts on N. The transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism between open subsets of $\mathbb{R}^n$. Given any $p \in U \cap V$, we have

$$\frac{\partial y^i}{\partial x^j}(p) = \frac{(\psi \circ \phi^{-1})^i}{\partial r^j}\left(\phi(p)\right).$$

The Inverse Function Theorem

In what follows M and N are manifolds of dimension n.

Reminder

By Proposition 6.11, given an open $U \subset M$, any diffeomorphism $F : U \to F(U) \subset \mathbb{R}^n$ defines a coordinate system on U, i.e., (U, F) is a chart on M.

Definition

We say that a C^∞ map $F : N \to M$ is *locally invertible*, or is a *local diffeomorphism*, near $p \in N$ if there is an open neighborhood U of p in N such that $F_{|U} : U \to F(U)$ is a diffeomorphism.

Remark

If $F = (F^1, \ldots, F^n) : N \to \mathbb{R}^n$ is locally invertible near $p \in N$, then it defines a coordinate system about p.

The Inverse Function Theorem

Theorem (Theorem 6.25, Inverse Function Theorem for $\mathbb{R}^n$; see also Appendix B)

Let $F = (F^1, \ldots, F^n) : W \to \mathbb{R}^n$ be a C^∞-map, where W is an open set in $\mathbb{R}^n$. Given any $p \in W$, TFAE:

(i) *F is locally invertible near p.*

(ii) *The Jacobian determinant $\det[\partial F^i/\partial x^j(p)]$ is non-zero.*

The Inverse Function Theorem

Theorem (Theorem 6.26, Inverse Function Theorem for manifolds)

Let $F : N \to M$ be a C^∞-map. Given any $p \in N$, TFAE:

(i) *F is locally invertible near p.*

(ii) *We have a non-zero Jacobian determinant $\det[\partial F^i/\partial x^j(p)]$.*

Remarks

1. In (ii) the Jacobian determinant $\det[\partial F^i/\partial x^j(p)]$ relatively to some chart $(U, x^1, \ldots, x^n)$ about p in N and some chart $(V, y^1, \ldots, y^n)$ about $F(p)$ in M and we have $F^i = y^i \circ F$.
2. The condition $\det[\partial F^i/\partial x^j(p)] \neq 0$ is independent of the choice of the charts.

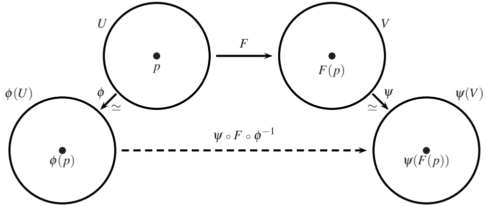


Fig. 6.4. The map F is locally invertible at p because $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$.

Corollary (Corollary 6.27)

Let $F = (F^1, \ldots, F^n) : U \to \mathbb{R}^n$ be C^∞ map on a neighborhood U of a point p in N. TFAE:

1. $F = (F^1, \ldots, F^n)$ *defines a coordinate system near* p.
2. $\det[\partial F^i / \partial x^j (p)] \neq 0$.