

Differentiable Manifolds

§3. The Exterior Algebra of Multivectors

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Dual Space

Definition

- 1 If V and W are vector spaces we denote by $\text{Hom}(V, W)$ the vector space of all linear maps $f : V \rightarrow W$.
- 2 The *dual space* $\text{Hom}(V, \mathbb{R})$ is denoted by V^\vee . It consists of all linear forms $f : V \rightarrow \mathbb{R}$.
- 3 The elements of V^\vee are called *covectors*.

Convention

We assume that V is a *finite-dimensional* space with basis e_1, \dots, e_n .

Fact

Every $v \in V$ can be uniquely written as $v = \sum v^i e_i$ with $v^i \in \mathbb{R}$.

Definition

The i -coordinate function $\alpha^i : V \rightarrow \mathbb{R}$ is defined by

$$\alpha^i(v) = v^i \quad \text{if } v = \sum v^j e_j.$$

Remark

We have

$$\alpha^i(e_j) = \delta_j^i = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Here δ_j^i is called the *Kronecker symbol*.

Proposition (Proposition 3.1)

The coordinate functions $\alpha^1, \dots, \alpha^n$ form a basis of V^\vee .

Definition

The basis $\alpha^1, \dots, \alpha^n$ is called the *dual basis* of the basis e_1, \dots, e_n .

Corollary (Corollary 3.2)

The dual space V^\vee has the same dimension as V , and hence has finite dimension.

Permutations

Definition (Permutations)

- ① A *permutation* is any bijection of $\{1, \dots, k\}$ onto itself.
- ② The set of all permutations of $\{1, \dots, k\}$ is denoted by S_k .

Remarks

- ① In other words a permutation $\sigma \in S_k$ is a *reordering* of the set $\{1, \dots, k\}$.
- ② $|S_k| = k!$ (number of elements of S_k).
- ③ It is sometimes convenient to represent a permutation $\sigma \in S_k$ by its matrix,

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

Facts

- ① The composition of maps induces a group law on S_k ,

$$(\sigma\tau) = \sigma \circ \tau, \quad \sigma, \tau \in S_k$$

- ② The identity permutation is the identity element of S_k .

Definition

S_k is called the *symmetric group* of degree k .

Definition (Transposition)

The *transposition* $\tau = (a, b)$, $a \neq b$, is the permutation that exchanges a and b and leaves all other elements unchanged, i.e.,

$$\tau(i) = \begin{cases} b & \text{for } i = a, \\ a & \text{for } i = b, \\ i & \text{otherwise.} \end{cases}$$

Example

The transposition $(1, 2) \in S_3$ has matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}.$$

Permutations

Definition (r -Cycles)

An r -cycle $\sigma = (a_1, \dots, a_r)$, where the a_i are distinct, is the permutation such that

- 1 $\sigma(a_i) = a_{i+1}$ for $i = 1, \dots, r - 1$.
- 2 $\sigma(a_r) = a_1$.
- 3 It leaves all other elements unchanged.

Example

The 3-cycle $(123) \in S_4$ has matrix,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}.$$

Remarks

- 1 Any 1-cycle (a) is the identity permutation.
- 2 2-cycles are just transpositions.

Example (The symmetric group S_2)

The symmetric group S_2 consists of

- The identity (1) .
- The transposition $(1, 2)$.

Example (The symmetric group S_3)

The symmetric group S_3 consists of

- The identity (1) .
- Transpositions: $(1, 2)$, $(1, 3)$, and $(2, 3)$.
- 3-cycles: (123) and (132) .

Example (The symmetric group S_4)

The symmetric group S_4 consists of

- The identity (1) .
- Transpositions: $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$.
- 3-cycles: $(123), (124), (134), (234), (132), (142), (143), (243)$.
- 4-cycles: $(1234), (1324), (1432), (1243), (1342), (1423)$.
- Products of 2-cycles: $(12)(34), (13)(24), (14)(23)$.

Fact

Any permutation $\sigma \in S_k$ can be written as a product of *disjoint* cycles.

Facts

- ① Any permutation $\sigma \in S_k$ can be written as a product of transpositions,

$$\sigma = \tau_1 \tau_2 \cdots \tau_n.$$

- ② The decomposition is not unique, but the parity of n depends only on σ .

Definition

We say that σ is *even* (resp., *odd*) when n is even (resp., odd).

Example

We have

$$\begin{aligned}(1234) &= (14)(13)(12) \\ &= (12)(24)(23) \\ &= (23)(14)(12)(13)(23).\end{aligned}$$

Fact

Any r -cycle (a_1, \dots, a_r) can be decomposed as the product of $r - 1$ transpositions,

$$(a_1 a_r)(a_1 a_{r-1}) \cdots (a_1 a_2).$$

Therefore, it has the opposite parity of r .

Permutations

Definition (Sign of a Permutation)

The *sign* of a permutation $\sigma \in S_k$ is

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Fact

The map $\operatorname{sgn} : S_k \rightarrow \{\pm 1\}$ is a morphism of groups, i.e.,

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau), \quad \operatorname{sgn}((1)) = 1.$$

Consequence

If $\sigma \in S_k$ is the product of p cycles of even length and q cycles of odd length, then

$$\operatorname{sgn}(\sigma) = (-1)^p.$$

Multilinear Functions

Definition

Set $V^k = V \times \cdots \times V$, where V is a vector space.

- 1 A function $f : V^k \rightarrow \mathbb{R}$ is k -linear when it is linear in each of its arguments, i.e.,

$$\begin{aligned} f(v_1, \dots, \lambda v_i + \mu w_i, \dots, v_n) \\ = \lambda f(v_1, \dots, v_i, \dots, v_n) + \mu f(v_1, \dots, w_i, \dots, v_n). \end{aligned}$$

- 2 The space of k -linear maps on V is denoted by $L_k(V)$.

Remarks

- 1 A 2-linear function is called *bilinear*.
- 2 A k -linear map is also called a k -tensor or *tensor of degree k* .

Multilinear Functions

Example

The dot product on \mathbb{R}^n is bilinear,

$$v \cdot w = \sum v^j w^j, \quad \text{where } v = \sum v^j e_j, \quad w = \sum w^j e_j.$$

Here e_1, \dots, e_n is the canonical basis of \mathbb{R}^n .

Example

The determinant of vectors in \mathbb{R}^n is n -linear,

$$(v_1, \dots, v_n) \longrightarrow \det[v_1 \cdots v_n].$$

Multilinear Functions

Definition

Let $f : V^k \rightarrow \mathbb{R}$ be a k -linear function.

- ① We say that f is *symmetric* when

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(v_1, \dots, v_k) \quad \text{for all } \sigma \in S_k.$$

- ② We say that f is *alternating* when

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sgn}(\sigma) f(v_1, \dots, v_k) \quad \text{for all } \sigma \in S_k.$$

Remark

If $f : V^2 \rightarrow \mathbb{R}$ is bilinear, then

$$\begin{aligned} f \text{ is symmetric} &\iff f(v, w) = f(w, v) \quad \forall v, w \in V, \\ f \text{ is alternating} &\iff f(v, w) = -f(w, v) \quad \forall v, w \in V. \end{aligned}$$

Multilinear Functions

Fact

Let $f : V^k \rightarrow \mathbb{R}$ be k -linear. TFAE:

- (i) f is symmetric.
- (ii) $f(v_1, \dots, v_n)$ remains unchanged whenever two arguments are interchanged, i.e.,

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = f(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

Fact (Problem 3.4 + Problem 3.5)

Let $f : V^k \rightarrow \mathbb{R}$ be k -linear. TFAE:

- (i) f is alternating.
- (ii) $f(v_1, \dots, v_n)$ changes sign whenever two arguments are interchanged, i.e.,

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -f(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

- (iii) $f(v_1, \dots, v_n) = 0$ whenever two arguments agree.

Multilinear Functions

Fact

More generally, it can be shown that if $f : V^k \rightarrow \mathbb{R}$ is an alternating k -linear function, then

$$f(v_1, \dots, v_n) = 0 \quad \text{whenever } v_1, \dots, v_n \text{ are linearly dependent.}$$

Consequence

Any alternating k -linear function on V with $k > \dim V$ must be zero.

Examples

- 1 The dot product $v \cdot w$ on \mathbb{R}^n is symmetric.
- 2 The determinant of vectors $\det[v_1 \cdots v_n]$ on \mathbb{R}^n is alternating.
- 3 The cross product $v \times w$ on \mathbb{R}^3 is alternating.

Example

Let $f, g : V \rightarrow \mathbb{R}$ be linear maps, and define their wedge product $f \wedge g : V^2 \rightarrow \mathbb{R}$ by

$$(f \wedge g)(u, v) = f(u)g(v) - f(v)g(u).$$

Then $f \wedge g$ is an alternating bilinear map.

Multilinear Functions

Definition

Let V be a vector space.

- ① $A_k(V)$ is the space of all alternating k -linear maps $f : V^k \rightarrow \mathbb{R}$.
- ② Elements of $A_k(V)$ are called *alternating k -tensors*, *k -covectors*, or *multicovectors of degree k* .

Remarks

- ① By convention a 0-covector is a scalar, so that $A_0(V) = \mathbb{R}$.
- ② A 1-covector is just a covector, and so $A_1(V) = V^\vee$.
- ③ $A_k(V) = \{0\}$ for $k > \dim V$.

Permutation Action on Multilinear Functions

Definition

Given a k -linear function on V and a permutation $\sigma \in S_k$, we define the k -linear function σf by

$$(\sigma f)(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)}), \quad v_i \in V.$$

Remark

$$\begin{aligned} f \text{ is symmetric} &\iff \sigma f = f \quad \forall \sigma \in S_k, \\ f \text{ is alternating} &\iff \sigma f = \text{sgn}(\sigma) f \quad \forall \sigma \in S_k. \end{aligned}$$

Permutation Action on Multilinear Functions

Lemma (Lemma 3.11)

Let $f : V^k \rightarrow \mathbb{R}$ be k -linear. Then

$$\tau(\sigma f) = (\tau\sigma)f \quad \forall \sigma, \tau \in S_k.$$

Definition

Given a set X and a group G with identity e , a *left action* is a map $G \times X \ni (g, x) \rightarrow gx \in X$ such that

- ① $ex = x$ for all $x \in X$.
- ② $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Fact

The map $(\sigma, f) \rightarrow \sigma f$ is a left action of the symmetric group S_k on the vector space $L_k(V)$ of k -linear functions.

Remark

Each permutation $\sigma \in S_k$ acts linearly on $L_k(V)$.

Remarks

- ① We also define a *right action* as a map $X \times G \ni (x, g) \rightarrow xg \in X$ such that
 - ① $ex = x$ for all $x \in X$.
 - ② $(xg)h = x(gh)$ for all $g, h \in G$ and $x \in X$.
- ② If $(g, x) \rightarrow gx$ is a left action, then $(x, g) \rightarrow xg^{-1}$ is a right action.

Symmetrizing and Alternating Operators

Definition (Symmetrizing Operator)

The *symmetrizing operator* $S : L_k(V) \rightarrow L_k(V)$ is defined by

$$Sf = \sum_{\sigma \in S_k} \sigma f.$$

That is,

$$(Sf)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} f(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Definition (Alternating Operator)

The *alternating operator* $A : L_k(V) \rightarrow L_k(V)$ is defined by

$$Af = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma f.$$

Symmetrizing and Alternating Operators

Proposition (Proposition 3.12)

Let $f : V^k \rightarrow \mathbb{R}$ be k -linear. Then

- ① Sf is symmetric.
- ② Af is alternating.

Lemma (Lemma 3.14)

If $f : V^k \rightarrow \mathbb{R}$ is k -linear and alternating, then $Af = (k!)f$.

Remark

Let $f : V^k \rightarrow \mathbb{R}$ be k -linear. Then

$$\begin{aligned} f \text{ is symmetric} &\iff Sf = (k!)f, \\ f \text{ is alternating} &\iff Af = (k!)f. \end{aligned}$$

The Tensor Product

Definition

Let $f : V^k \rightarrow \mathbb{R}$ be k -linear and let $g : V^\ell \rightarrow \mathbb{R}$ be ℓ -linear. Their *tensor product* is the $(k + \ell)$ -linear function $f \otimes g : V^{k+\ell} \rightarrow \mathbb{R}$ defined by

$$(f \otimes g)(v_1, \dots, v_{k+\ell}) = f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}), \quad v_i \in V.$$

Example

Let e_1, \dots, e_n be a basis of V and $\alpha^1, \dots, \alpha^n$ its dual basis. Given a bilinear map $g : V \times V \rightarrow \mathbb{R}$, set $g_{ij} = g(e_i, e_j)$. Then

$$g = \sum g_{ij} \alpha^i \otimes \alpha^j.$$

Exercise (Exercise 3.17, Associativity of the tensor product)

If f , g and h are multilinear functions on V , then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

The Wedge Product

Definition

Given alternating functions $f \in A_k(V)$ and $g \in A_\ell(V)$, their *wedge product* (or *exterior product*) is the alternating function $f \wedge g \in A_{k+\ell}(V)$ defined by

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g).$$

That is,

$$(f \wedge g)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

The Wedge Product

Example (Example 3.19; wedge product of covectors)

If $f \in A_1(V)$ and $g \in A_1(V)$, then

$$(f \wedge g)(v_1, v_2) = f(v_1)g(v_2) - f(v_2)g(v_1).$$

Example (Example 3.18)

If $f \in A_2(V)$ and $g \in A_1(V)$, then

$$(f \wedge g)(v_1, v_2, v_3) = \\ f(v_1, v_2)g(v_3) - f(v_1, v_3)g(v_2) + f(v_2, v_3)g(v_1).$$

The Wedge Product

Definition

A permutation $\sigma \in S_{k+\ell}$ is called a (k, ℓ) -shuffle when

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+\ell).$$

Fact

It can be shown that if $f \in A_k(V)$ and $g \in A_\ell(V)$, then

$$(f \wedge g)(v_1, \dots, v_{k+\ell}) = \sum_{\substack{(k, \ell)\text{-shuffles} \\ \sigma}} \text{sgn}(\sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

Anticommutativity of the Wedge Product

Proposition (Proposition 3.21)

The wedge product is anticommutative, i.e., if $f \in A_k(V)$ and $g \in A_\ell(V)$, then

$$g \wedge f = (-1)^{k\ell} f \wedge g.$$

Corollary (Corollary 3.23)

If $f \in A_k(V)$ and k is odd, then $f \wedge f = 0$.

Associativity of the Wedge Product

Lemma (Lemma 3.24)

Let $f \in A_k(V)$ and $g \in A_\ell(V)$. Then

- (i) $A(A(f) \otimes g) = k!A(f \otimes g)$.
- (ii) $A(f \otimes A(g)) = \ell!A(f \otimes g)$.

Proposition (Proposition 3.25; Associativity of the wedge product)

Let $f \in A_k(V)$, $g \in A_\ell(V)$, and $h \in A_m(V)$. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

Consequence

We may omit parantheses in wedge products and write $(f \wedge g) \wedge h$ and $f \wedge (g \wedge h)$ simply as $f \wedge g \wedge h$.

Associativity of the Wedge Product

Corollary (Corollary 3.26; corollary of the proof of Proposition 3.25)

If $f \in A_k(V)$, $g \in A_\ell(V)$, and $h \in A_m(V)$, then

$$f \wedge g \wedge h = \frac{1}{k!\ell!m!} A(f \otimes g \otimes h).$$

Fact

More generally, it can be shown by induction that if $f_i \in A_{k_i}(V)$, $i = 1, \dots, r$, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{k_1! \cdots k_r!} A(f_1 \otimes \cdots \otimes f_r).$$

Associativity of the Wedge Product

Proposition (Proposition 3.27; wedge product of 1-covectors)

If $\alpha^1, \dots, \alpha^k$ are in $A_1(V) = V^\vee$ and v_1, \dots, v_k are vectors in V , then

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det [\alpha^i(v_j)] .$$

Associativity of the Wedge Product

Definition (Graded Algebras)

- 1 An algebra A over a field \mathbb{K} called *graded* when it can be decomposed as

$$A = \bigoplus_{k=0}^{\infty} A^k,$$

where the A^k are subspaces such that the multiplication maps $A^k \times A^\ell$ to $A^{k+\ell}$.

- 2 We then say that A is *anticommutative* (or *graded commutative*) when

$$ba = (-1)^{k\ell} ab \quad \text{for all } a \in A^k \text{ and } b \in A^\ell.$$

Associativity of the Wedge Product

Definition

Given graded algebras $A = \bigoplus A^k$ and $B = \bigoplus B^k$, a *graded algebra homomorphism* $f : A \rightarrow B$ is an algebra homomorphism that preserves the degree, i.e.,

$$f(A^k) \subset B^k \quad \text{for all } k \geq 0.$$

Example

- The polynomial algebra $A = \mathbb{R}[x, y]$ is graded by degree.
- Here A^k consists of homogeneous polynomials of total degree k , i.e., linear combinations of monomials $x^p y^q$ with $p + q = k$.

Associativity of the Wedge Product

Definition (Exterior Algebra)

If V is vector space of dimension n , its *exterior algebra* (or *Grassmann algebra*) is the vector space,

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^n A_k(V).$$

Fact

With respect to the wedge product $A_*(V)$ is an anticommutative graded algebra.

Basis of k -Covectors

Convention

- V is an n -dimensional vector space with basis e_1, \dots, e_n .
- $\alpha^1, \dots, \alpha^n$ is the dual basis of V^\vee .

Definition

Given any multi-index $I = (i_1, \dots, i_k)$ we define

$$e_I = (e_{i_1}, \dots, e_{i_k}) \quad \text{and} \quad \alpha^I = \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}.$$

Remark

Here $e_I \in V^k$ and $\alpha^I \in A_k(V)$.

Basis of k -Covectors

Observation

By multi-linearity a k -linear function $f : V^k \rightarrow \mathbb{R}$ is completely determined by its values on the k -tuples $(e_{i_1}, \dots, e_{i_k})$.

Facts

Let $f \in A_k$ and let (i_1, \dots, i_k) be a multi-index. Then

- ① $f(e_{i_1}, \dots, e_{i_k}) = 0$ whenever $i_p = i_q$ for some $p \neq q$.
- ② If all the i_p are distinct, then there is a unique permutation $\sigma \in S_k$ such that $i_{\sigma(1)} < \dots < i_{\sigma(k)}$.
- ③ We then have

$$f(e_{i_1}, \dots, e_{i_k}) = \text{sgn}(\sigma) f(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}).$$

Consequence

If $f \in A_k(V)$, then it is completely determined by its values on k -uples of the form,

$$(e_{i_1}, \dots, e_{i_k}) \quad \text{with } 1 \leq i_1 < i_2 < \dots < i_k \leq n.$$

Basis of k -Covectors

Lemma (Lemma 3.28)

Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ be strictly ascending multi-indices, i.e., $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. Then

$$\alpha^I(e_J) = \delta_J^I = \begin{cases} 1 & \text{for } I = J, \\ 0 & \text{for } I \neq J. \end{cases}$$

Proposition (Proposition 3.29)

The k -covectors α^I , $I = (i_1 < \dots < i_k)$, form a basis of $A_k(V)$.

Corollary (Corollary 3.30 + Corollary 3.31)

- 1 If $0 \leq k \leq n$, then $\dim A_k(V) = \binom{n}{k}$.
- 2 If $k > n$, then $A_k(V) = \{0\}$.