

# Differentiable Manifolds

## §11. The Rank of a Smooth Map

Sichuan University, Fall 2021

# Constant Rank Theorem

## Reminder

Let  $N$  be a manifold of dimension  $n$  and  $M$  a manifold of dimension  $m$ .

- The rank at  $p \in N$  of a smooth map  $f : N \rightarrow M$  is the rank of its differential  $f_{*,p} : T_p N \rightarrow T_{f(p)} M$ .
- The rank is always  $\leq \min(m, n)$ .

# Constant Rank Theorem

Theorem (Constant Rank Theorem; Theorem B.4)

Let  $f : U \rightarrow \mathbb{R}^m$  be a  $C^\infty$  map, where  $U \subset \mathbb{R}^n$  is open. Assume that  $f$  has constant rank  $k$  near  $p \in U$ . Then there are:

- A diffeomorphism  $F$  from a neighborhood of  $p$  onto a neighborhood of  $0 \in \mathbb{R}^n$  with  $F(p) = 0$ ,
- A diffeomorphism  $G$  from a neighborhood of  $f(p)$  onto a neighborhood of  $0 \in \mathbb{R}^m$  with  $G(f(p)) = 0$ ,

in such a way that

$$F \circ f \circ G^{-1}(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Remark

If  $k = m$ , then

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^m).$$

## Constant Rank Theorem

Theorem (Constant Rank Theorem for Manifolds; Theorem 11.1)

Suppose that  $M$  is a manifold of dimension  $m$  and  $N$  is a manifold of dimension  $n$ . Let  $f : N \rightarrow M$  be a smooth map that has constant rank  $k$  near a point  $p \in N$ . Then, there are a chart  $(U, \phi)$  centered at  $p$  in  $N$  and a chart  $(V, \psi)$  centered at  $f(p)$  in  $M$  such that, for all  $(r^1, \dots, r^n) \in \phi(U)$ , we have

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

# Constant Rank Theorem

## Proof of Theorem 11.1.

- Let  $(\bar{U}, \bar{\phi})$  and  $(\bar{V}, \bar{\psi})$  be charts around  $p$  and  $q = f(p)$ .
- Apply the constant rank theorem to  $\bar{\psi} \circ f \circ \bar{\phi}^{-1}$  to get
  - A diffeomorphism  $F : W \rightarrow W'$  from a neighborhood  $W$  of  $\bar{\phi}(p)$  onto a neighborhood of  $0 \in \mathbb{R}^n$  with  $F(\bar{\phi}(p)) = 0$ ,
  - A diffeomorphism  $G : X \rightarrow X'$  from a neighborhood  $X$  of  $\bar{\psi}(q)$  onto a neighborhood of  $0 \in \mathbb{R}^m$  with  $G(\bar{\psi}(q)) = 0$ ,

in such a way that

$$\begin{aligned}(F \circ \bar{\psi}) \circ f \circ (G \circ \bar{\phi})^{-1}(r^1, \dots, r^n) &= F \circ (\bar{\psi} \circ f \circ \bar{\phi}^{-1}) \circ G^{-1}(r^1, \dots, r^n) \\ &= (r^1, \dots, r^k, 0, \dots, 0).\end{aligned}$$

- Set  $U = \bar{\phi}^{-1}(W)$  and  $V = \bar{\psi}^{-1}(X)$ . Then  $(U, G \circ \bar{\phi})$  and  $(V, F \circ \bar{\psi})$  are charts centered  $p$  and  $q$  with the required property.

□

# Constant Rank Theorem

## Remark

Suppose that  $(U, \phi) = (U, x^1, \dots, x^n)$  is a chart centered at  $p$  and  $(V, \psi) = (V, y^1, \dots, y^m)$  is a chart centered at  $f(p)$  such that

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

- For any  $q \in U$ , we have  $\phi(q) = (x^1(q), \dots, x^n(q))$  and  $\psi(f(q)) = (y^1 \circ f(q), \dots, y^m \circ f(q))$ .
- Thus,

$$\begin{aligned} (y^1 \circ f(q), \dots, y^m \circ f(q)) &= \psi(f(q)) = (\psi \circ f \circ \phi^{-1})(\phi(q)) \\ &= (\psi \circ f \circ \phi^{-1})(x^1(q), \dots, x^n(q)) \\ &= (x^1(q), \dots, x^k(q), 0, \dots, 0). \end{aligned}$$

- Therefore, relative to the local coordinates  $(x^1, \dots, x^n)$  and  $(y^1, \dots, y^m)$  the map  $f$  is such that

$$(x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^k, 0, \dots, 0).$$

# Constant Rank Theorem

A consequence of the constant rank theorem is the following extension of the regular level set theorem (Theorem 9.9).

**Theorem (Constant-Rank Level Set Theorem; Theorem 11.2)**

*Let  $N \rightarrow M$  be a smooth map and  $c \in M$ . If  $f$  has constant rank  $k$  in a neighborhood of the level set  $f^{-1}(c)$  in  $N$ , then  $f^{-1}(c)$  is a regular submanifold of codimension  $k$ .*

**Remark**

A neighborhood of a subset  $A \subset N$  is an open set containing  $A$ .

# Constant Rank Theorem

## Example (Orthogonal group $O(n)$ ; Example 11.3)

The *orthogonal group*  $O(n)$  is the subgroup of  $GL(n, \mathbb{R})$  of matrices  $A$  such that  $A^T A = I_n$  (identity matrix),

- This is the level set  $f^{-1}(I_n)$ , where  $f : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ ,  $A \mapsto A^T A$ .
- It can be shown that  $f$  has constant rank (in fact it has rank  $k = \frac{1}{2}n(n + 1)$ ).
- Therefore, by the constant-rank level set theorem  $O(n)$  is a regular submanifold of  $GL(n, \mathbb{R})$  (of codimension  $\frac{1}{2}n(n + 1)$ ).

# The Immersion and Submersion Theorems

## Reminder

Suppose that  $M$  is a manifold of dimension  $m$  and  $N$  is a manifold of dimension  $n$ , and let  $f : N \rightarrow M$  be a smooth map.

- $f$  is an *immersion* at  $p$  if  $f_{*,p} : T_p N \rightarrow T_{f(p)} M$  is injective.
- $f$  is a *submersion* at  $p$  if  $f_{*,p} : T_p N \rightarrow T_{f(p)} M$  is surjective.

## Remark

Equivalently,

$f$  is an immersion at  $p \iff (n \leq m \text{ and } \text{rk } f_{*,p} = n)$ ,

$f$  is a submersion at  $p \iff (n \geq m \text{ and } \text{rk } f_{*,p} = m)$ .

As we always have  $\text{rk } f_{*,p} \leq \min(m, n)$ , we see that

$f$  is an immersion/submersion at  $p \iff f_{*,p}$  has maximal rank.

## Facts

Set  $k = \min(m, n)$ , and denote by  $\mathbb{R}_{\max}^{m \times n}$  the set of  $m \times n$  matrices  $A \in \mathbb{R}^{m \times n}$  of maximal rank.

- An  $m \times n$ -matrix has maximal rank if and only if it has a non-zero  $k \times k$ -minor.
- The minors are polynomials in the coefficients of matrices, and hence are continuous functions.
- Thus, if a matrix  $A$  has a non-zero  $k \times k$ -minor, then this minor is non-zero for any matrix that is sufficiently close to  $A$ , and so those matrices have maximal rank.
- It follows that  $\mathbb{R}_{\max}^{m \times n}$  is a neighbourhood of each of its elements, and hence is an open set in  $\mathbb{R}^{m \times n}$ .

# The Immersion and Submersion Theorems

## Facts

Suppose that  $f : N \rightarrow M$  is a smooth map. Let  $(U, x^1, \dots, x^n)$  be a chart about  $p$  in  $M$  and  $(V, y^1, \dots, y^m)$  a chart about  $f(p)$  in  $M$ . Set  $U_{\max} = \{q \in U; f_{*,q} \text{ has maximal rank}\}$ .

- If  $q \in U$ , then  $f_{*,q} : T_q M \rightarrow T_{f(q)} M$  is represented by the matrix  $J(q) := [\partial f^i / \partial x^j(q)]$ , with  $f^i = y^i \circ f$ , and hence  $\text{rk } f_{*,q} = \text{rk } J(q)$ . Thus,

$$U_{\max} = \{q \in U; J(q) \in \mathbb{R}_{\max}^{m \times n}\} = J^{-1}(\mathbb{R}_{\max}^{m \times n}).$$

- It can be shown that  $q \rightarrow J(F)(q)$  is  $C^\infty$ , and hence is continuous.
- As  $\mathbb{R}_{\max}^{m \times n}$  is open, it follows that  $U_{\max}$  is open as well.
- In particular, if  $f_*$  has maximal rank at  $p$ , then it has maximal rank near  $p$ .

# The Immersion and Submersion Theorems

As a consequence we obtain:

Proposition (Proposition 11.4)

*If a smooth map  $f : N \rightarrow M$  is a immersion (resp., a submersion) at a point  $p \in N$ , then it is an immersion (resp., submersion) near  $p$ . In particular, it has constant rank near  $p$ .*

# The Immersion and Submersion Theorems

Combining the previous proposition with the Constant Rank Theorem gives the following result.

## Theorem (Theorem 11.5)

Let  $f : N \rightarrow M$  be a smooth map.

- ① **Immersion Theorem.** If  $f$  is an immersion at  $p$ , then there are a chart  $(U, \phi)$  centered at  $p$  in  $N$  and a chart  $(V, \psi)$  centered at  $f(p)$  in  $M$  such that near  $\phi(p)$  we have

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^n, 0, \dots, 0).$$

- ② **Submersion Theorem.** If  $f$  is a submersion at  $p$ , then there are a chart  $(U, \phi)$  centered at  $p$  in  $N$  and a chart  $(V, \psi)$  centered at  $f(p)$  in  $M$  such that near  $\phi(p)$  we have

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^m, r^{m+1}, \dots, r^n) = (r^1, \dots, r^m).$$

# The Immersion and Submersion Theorems

## Remark

- The immersion theorem implies that if  $f : N \rightarrow M$  is an immersion then, for every  $p \in N$ , there are a chart  $(U, x^1, \dots, x^n)$  centered at  $p$  in  $N$  and a chart  $(V, y^1, \dots, y^m)$  centered at  $f(p)$  in  $M$  relative to which  $f$  is such that

$$(*) \quad (x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^n, 0, \dots, 0).$$

- Conversely, If  $f$  satisfies  $(*)$ , then, setting  $f^i = y^i \circ f$ , we have

$$[\partial f^i / \partial x^j] = \begin{bmatrix} \partial x^i / \partial x^j \\ 0_{m-n} \end{bmatrix} = \begin{bmatrix} I_n \\ 0_{m-n} \end{bmatrix}.$$

In particular,  $[\partial f^i / \partial x^j]$  has maximal rank, which implies that  $f$  is an immersion near  $p$ .

# The Immersion and Submersion Theorems

## Remark

- The submersion theorem implies that if  $f : N \rightarrow M$  is a submersion then, for every  $p \in N$ , there are a chart  $(U, x^1, \dots, x^n)$  centered at  $p$  in  $N$  and a chart  $(V, y^1, \dots, y^m)$  centered at  $f(p)$  in  $M$  relative to which  $f$  is such that
$$(x^1, \dots, x^m, x^{m+1}, \dots, x^n) \longrightarrow (x^1, \dots, x^m).$$
- The projection  $(x^1, \dots, x^m, x^{m+1}, \dots, x^n) \rightarrow (x^1, \dots, x^m)$  is an open map (see Problem A.7). This implies that  $f$  maps any neighborhood of  $p$  onto a neighborhood of  $f(p)$ .
- As this is true for every  $p \in N$ , we see that  $f$  is an open map. Therefore, we obtain:

## Corollary (Corollary 11.6)

*Every submersion  $f : N \rightarrow M$  is an open map.*

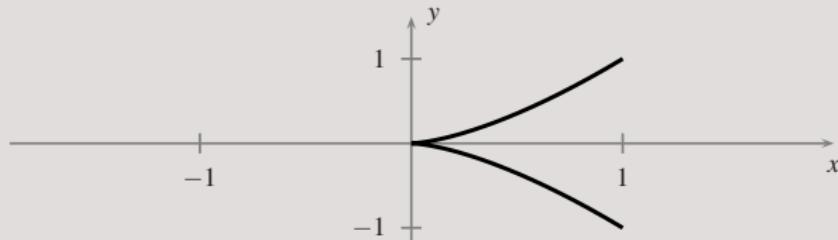
# Images of Smooth Maps

Let us look at some examples of smooth maps  $f : \mathbb{R} \rightarrow \mathbb{R}^2$ .

## Example (Example 11.7)

Let  $f(t) = (t^2, t^3)$ .

- This is a one-to-one map, since  $t \rightarrow t^3$  is one-to-one.
- As  $f'(0) = (0, 0)$  the differential  $f_{*,0}$  is zero, and so  $f$  is not an immersion at 0.
- The image of  $f$  is the cuspidal cubic  $y^2 = x^3$ .

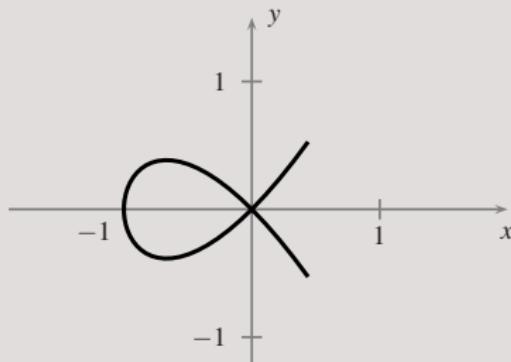


# Images of Smooth Maps

## Example (Example 11.8)

Let  $f(t) = (t^2 - 1, t^3 - t)$ .

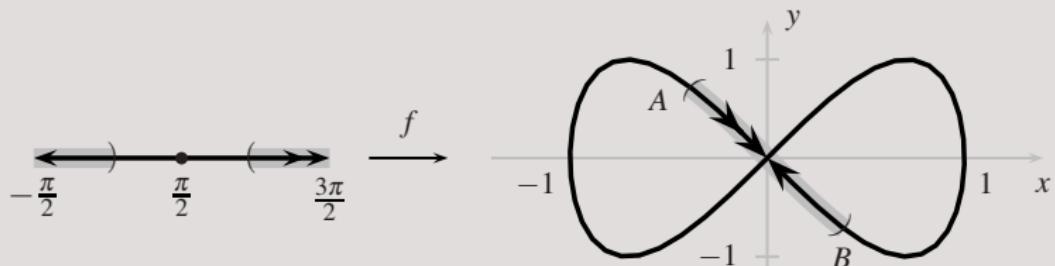
- As  $f'(t) = (2t, 3t^2 - 1) \neq (0, 0)$  the differential  $f_*$  is one-to-one everywhere, and hence  $f$  is an immersion.
- However,  $f$  is not one-one since  $f(1) = f(-1) = (0, 0)$ .
- The image of  $f$  is the nodal cubic  $y^2 = x^2(x + 1)$  (see Tu's book).



# Image of Smooth Maps

Example (The Figure-eight; Example 11.12)

Set  $I = (-\pi/2, 3\pi/2)$ , and let  $f : I \rightarrow \mathbb{R}^2$ ,  $t \mapsto (\cos t, \sin 2t)$ .



- $f'(t) = (-\sin t, 2\cos 2t) \neq (0, 0)$ , and so  $f$  is an immersion.
- $f$  is one-to-one, and so  $f$  is a bijection onto its image  $f(I)$ .
- The inverse map  $f^{-1} : f(I) \rightarrow I$  is not continuous: if  $t \rightarrow (3\pi/2)^-$ , then  $f(t) \rightarrow (0, 0) = f(\pi/2)$ , but

$$f^{-1}(f(t)) = t \rightarrow 3\pi/2 \notin I.$$

In particular,  $f : I \rightarrow f(I)$  is not a homeomorphism.

# Image of Smooth Maps

## Summary

As the previous examples show:

- A one-to-one smooth map need not be an immersion.
- An immersion need not be one-to-one.
- A one-to-one immersion need not be a homeomorphism onto its image.

## Definition

A smooth map  $f : N \rightarrow M$  is called an *embedding* if  $f$  is an immersion and a homeomorphism onto its image  $f(N)$  with respect to the subspace topology.

## Remark

A one-to-one immersion  $f : N \rightarrow M$  is an embedding if and only if it is an open map.

# Image of Smooth Maps

The importance of embeddings stems from the following result.

Theorem (Theorem 11.13)

*If  $f : N \rightarrow M$  is an embedding, then its image  $f(N)$  is a regular submanifold in  $M$ .*

# Image of Smooth Maps

## Proof of Theorem 11.13.

- As  $f$  is an immersion, by the immersion theorem, for any  $p \in N$ , there are a chart  $(U, x^1, \dots, x^n)$  centered at  $p$  in  $N$  and a chart  $(V, y^1, \dots, y^m)$  centered at  $f(p)$  relative to which  $f$  is such that  $(x^1, \dots, x^n) \rightarrow (x^1, \dots, x^n, 0, \dots, 0)$ . Thus,

$$f(U) = \{q \in V; y^{n+1}(q) = \dots = y^m(q) = 0\}$$

- As  $f : N \rightarrow f(N)$  is a homeomorphism,  $f(U)$  is an open set in  $f(N)$  with respect to the subspace topology. That is, there is an open  $V'$  in  $M$  such that  $f(U) = V' \cap f(N)$ .

- Thus,

$$V \cap V' \cap f(N) = V \cap f(U) = f(U) = \{y^{n+1} = \dots = y^m = 0\}.$$

That is,  $(V \cap V', y^1, \dots, y^m)$  is an adapted chart relative to  $f(N)$  near  $f(p)$  in  $M$ .

- This shows that  $f(N)$  is a regular submanifold.



# Image of Smooth Maps

We have the following converse of the previous theorem.

Theorem (Theorem 11.14)

*If  $N$  is a regular submanifold in  $M$ , then the inclusion  $i : N \rightarrow M$  is an embedding.*

# Image of Smooth Maps

## Proof of Theorem 11.14.

Let  $N$  be a regular submanifold in  $M$ .

- As  $N$  has the subspace topology, the inclusion  $i : N \rightarrow M$  is a homeomorphism onto its image.
- As  $N$  is a regular submanifold, near every  $p \in N$ , there is an adapted chart  $(U, x^1, \dots, x^m)$  near  $p$  in  $M$  such that  $(U \cap N, x^1, \dots, x^n)$  is a chart in  $N$  near  $p$  and  $U \cap N = \{x^{n+1} = \dots = x^m = 0\}$ .
- Therefore, relative to the charts  $(U \cap N, x^1, \dots, x^n)$  and  $(U, x^1, \dots, x^m)$  the inclusion  $i : N \rightarrow M$  is such that 
$$(x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^n, 0, \dots, 0).$$
- By a previous remark, it follows that the map  $i : N \rightarrow M$  is an immersion near  $p$ . □

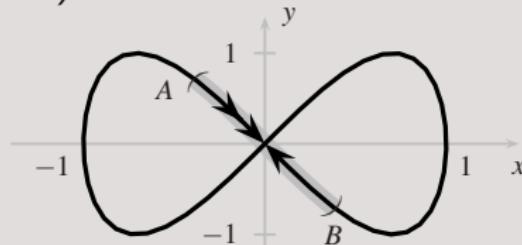
# Image of Smooth Maps

## Remarks

- ① The images of smooth embeddings are called *embedded submanifolds*.
- ② The previous two results show that the regular submanifolds and embedded submanifolds are the same objects.
- ③ The images of one-to-one immersions are called *immersed submanifolds*.

## Example

The figure-eight is an immersed submanifold in  $\mathbb{R}^2$  (but this is not a regular submanifold).



# Smooth Maps into a Submanifold

## Question

Suppose that  $f : N \rightarrow M$  is smooth map such that  $f(N)$  is contained in a given subset  $S \subset M$ . If  $S$  is manifold, then is the induced map  $f : N \rightarrow S$  smooth as well?

## Theorem (Theorem 11.15)

Suppose that  $f : N \rightarrow M$  is a smooth map whose image is contained in a regular submanifold  $S$  in  $M$ . Then the induced map  $f : N \rightarrow S$  is smooth.

## Remarks

- ① The above result does not hold if  $S$  is only an immersed submanifold (see Tu's book).
- ② The converse holds. As  $S$  is a regular submanifold, the inclusion  $i : S \rightarrow M$  is smooth. Thus, if  $f : N \rightarrow S$  is a smooth map, then  $i \circ f : N \rightarrow M$  is a  $C^\infty$  map that induces  $f$ .

# Smooth Maps into a Submanifold

## Proof of Theorem 11.15.

Set  $m = \dim M$  and  $s = \dim S$ , and let  $p \in N$ .

- As  $S$  is a regular submanifold and  $f(p) \in S$ , there is an adapted chart  $(V, \psi) = (V, y^1, \dots, y^m)$  near  $f(p)$  in  $M$ . Then  $(V \cap S, \psi_S) = (V \cap S, y^1, \dots, y^s)$  is a chart near  $f(p)$  in  $S$ .
- As  $f$  is a  $C^\infty$ -map, the functions  $y^i \circ f$  are  $C^\infty$  on  $U := f^{-1}(V)$  (which is an open neighbourhood of  $p$  in  $N$  since  $f$  is continuous).
- On  $f^{-1}(V)$  we have  $\psi_S \circ f = (y^1 \circ f, \dots, y^s \circ f)$ , and so  $\psi_S \circ f : f^{-1}(V) \rightarrow \mathbb{R}^s$  is a smooth map.
- As  $(V \cap S, \psi_S)$  is chart for  $S$ , it follows from Proposition 6.15 that the induced map  $f : f^{-1}(V) \rightarrow S$  is smooth, and hence is smooth near  $p$ . □

# Smooth Maps into a Submanifold

Example (Multiplication map of  $SL(n, \mathbb{R})$ ; Example 11.16)

$SL(n, \mathbb{R})$  is the subgroup of  $GL(n, \mathbb{R})$  of matrices of determinant 1.

- This is a regular submanifold in  $GL(n, \mathbb{R})$  (Example 9.11), and so the inclusion  $\iota : SL(n, \mathbb{R}) \hookrightarrow GL(n, \mathbb{R})$  is a smooth map.
- By Example 6.21 we have a smooth multiplication map,

$$\mu : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R}).$$

- We thus get a smooth map,

$$\mu \circ (\iota \times \iota) : SL(n, \mathbb{R}) \times SL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R}).$$

- As it takes values in  $SL(n, \mathbb{R})$ , and  $SL(n, \mathbb{R})$  is a regular submanifold in  $GL(n, \mathbb{R})$ , we get a smooth multiplication map,

$$SL(n, \mathbb{R}) \times SL(n, \mathbb{R}) \longrightarrow SL(n, \mathbb{R}).$$

# Smooth Maps into a Submanifold

Theorem 11.5 and its converse are especially useful when  $M = \mathbb{R}^m$ . In this case we have:

## Corollary

Let  $S$  be a regular submanifold in  $\mathbb{R}^m$  and  $f : N \rightarrow \mathbb{R}^m$  a map such that  $f(N) \subset S$ . Set  $f = (f^1, \dots, f^m)$ . Then TFAE:

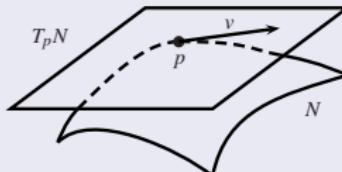
- (i)  $f$  is smooth as a map from  $N$  to  $S$ .
- (ii)  $f$  is smooth as a map from  $N$  to  $\mathbb{R}^m$ .
- (iii) The components  $f^1, \dots, f^m$  are smooth functions on  $N$ .

# The Tangent Space to a Submanifold in $\mathbb{R}^m$

## Facts

Let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a smooth function with no critical points on its zero set  $N = f^{-1}(0)$ .

- By the regular level set theorem  $N$  is a regular submanifold in  $\mathbb{R}^{n+1}$  of dimension  $n$ .
- Then the inclusion  $i : N \rightarrow \mathbb{R}^{n+1}$  is an embedding, and so, for every  $p \in N$ , the differential  $i_* : T_p N \rightarrow T_p \mathbb{R}^{n+1}$  is injective.
- We thus can identify the tangent space  $T_p N$  with a subspace of  $T_p \mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$ . More precisely, we regard it as a subspace of  $\mathbb{R}^{n+1}$  through  $p$ .
- Thus, any  $v \in T_p N$ , is identified with a vector  $\langle v^1, \dots, v^{n+1} \rangle$ , which is then identified with the point  $x = p + (v^1, \dots, v^{n+1})$ .



# The Tangent Space to a Submanifold in $\mathbb{R}^m$

## Facts

- Set  $p = (p^1, \dots, p^{n+1})$  and  $x = (x^1, \dots, x^{n+1})$ . Let  $c : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$  be a smooth curve such that  $c(0) = p$ ,  $c'(0) = v$ , and  $c(t) \in N$ , i.e.,  $f(c(t)) = 0$ . Then

$$0 = \frac{d}{dt} \Big|_0 f(c(t)) = \sum (c^i)'(0) \frac{\partial f}{\partial x^i}(c(0)) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

- As  $v^i = x^i - p^i$ , we see that  $(x^1, \dots, x^{n+1})$  satisfies,

$$(*) \quad \sum \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0.$$

- As  $p$  is a regular point,  $\frac{\partial f}{\partial x^i}(p) \neq 0$  for some  $i$ , and so the solution set of  $(*)$  has dimension  $n$ .
- As  $\dim N = n$ , the tangent space  $T_p N$  has dimension  $n$ , and so it is identified with the full solution set of  $(*)$ .

Therefore, we obtain the following result:

### Proposition

Let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a smooth function with no critical points on its zero set  $N = f^{-1}(0)$ . If  $p = (p^1, \dots, p^{n+1})$  is a point in  $N$ , then the tangent space  $T_p N$  is defined by the equation,

$$(*) \quad \sum \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0.$$

### Remark

Equivalently,  $T_p N$  is identified with the hyperplane through  $p$  that is normal to the gradient vector  $\langle \partial f / \partial x^1(p), \dots, \partial f / \partial x^{n+1}(p) \rangle$ .

# The Tangent Space to a Submanifold in $\mathbb{R}^m$

## Example (Tangent plane to a sphere)

The sphere  $\mathbb{S}^2 \subset \mathbb{R}^3$  is the zero set of

$$f(x, y, z) = x^2 + y^2 + z^2 - 1.$$

- We have

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z.$$

- Thus, at  $p = (a, b, c) \in \mathbb{S}^2$  the tangent plane has equation,

$$\frac{\partial f}{\partial x}(p)(x - a) + \frac{\partial f}{\partial y}(p)(y - b) + \frac{\partial f}{\partial z}(p)(z - c) = 0,$$

$$\iff a(x - a) + b(y - b) + c(z - c) = 0,$$

$$\iff ax + by + cz = a^2 + b^2 + c^2,$$

$$\iff ax + by + cz = 1.$$