

Differentiable Manifolds

§11. The Rank of a Smooth Map

Sichuan University, Fall 2021

Constant Rank Theorem

Reminder

Let N be a manifold of dimension n and M a manifold of dimension m .

- The rank at $p \in N$ of a smooth map $f : N \rightarrow M$ is the rank of its differential $f_{*,p} : T_p N \rightarrow T_{f(p)} M$.
- The rank is always $\leq \min(m, n)$.

Constant Rank Theorem

Theorem (Constant Rank Theorem; Theorem B.4)

Let $f : U \rightarrow \mathbb{R}^m$ be a C^∞ map, where $U \subset \mathbb{R}^n$ is open. Assume that f has constant rank k near $p \in U$. Then there are:

- A diffeomorphism F from a neighborhood of p onto a neighborhood of $0 \in \mathbb{R}^n$ with $F(p) = 0$,
- A diffeomorphism G from a neighborhood of $f(p)$ onto a neighborhood of $0 \in \mathbb{R}^m$ with $G(f(p)) = 0$,

in such a way that

$$F \circ f \circ G^{-1}(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Remark

If $k = m$, then

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^m).$$

Constant Rank Theorem

Theorem (Constant Rank Theorem for Manifolds; Theorem 11.1)

Suppose that M is a manifold of dimension m and N is a manifold of dimension n . Let $f : N \rightarrow M$ be a smooth map that has constant rank k near a point $p \in N$. Then, there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that, for all $(r^1, \dots, r^n) \in \phi(U)$, we have

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Constant Rank Theorem

Proof of Theorem 11.1.

- Let $(\overline{U}, \overline{\phi})$ and $(\overline{V}, \overline{\psi})$ be charts around p and $q = f(p)$.
- Apply the constant rank theorem to $\overline{\psi} \circ f \circ \overline{\phi}^{-1}$ to get
 - A diffeomorphism $F : W \rightarrow W'$ from a neighborhood W of $\overline{\phi}(p)$ onto a neighborhood of $0 \in \mathbb{R}^n$ with $F(\overline{\phi}(p)) = 0$,
 - A diffeomorphism $G : X \rightarrow X'$ from a neighborhood X of $\overline{\psi}(q)$ onto a neighborhood of $0 \in \mathbb{R}^m$ with $G(\overline{\psi}(q)) = 0$,

in such a way that

$$\begin{aligned}(F \circ \overline{\psi}) \circ f \circ (G \circ \overline{\phi})^{-1}(r^1, \dots, r^n) &= F \circ (\overline{\psi} \circ f \circ \overline{\phi}^{-1}) \circ G^{-1}(r^1, \dots, r^n) \\ &= (r^1, \dots, r^k, 0, \dots, 0).\end{aligned}$$

- Set $U = \overline{\phi}^{-1}(W)$ and $V = \overline{\psi}^{-1}(X)$. Then $(U, G \circ \overline{\phi})$ and $(V, F \circ \overline{\psi})$ are charts centered p and q with the required property.



Constant Rank Theorem

Remark

Suppose that $(U, \phi) = (U, x^1, \dots, x^n)$ is a chart centered at p and $(V, \psi) = (V, y^1, \dots, y^m)$ is a chart centered at $f(p)$ such that

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

- For any $q \in U$, we have $\phi(q) = (x^1(q), \dots, x^n(q))$ and $\psi(f(q)) = (y^1 \circ f(q), \dots, y^m \circ f(q))$.
- Thus,

$$\begin{aligned}(y^1 \circ f(q), \dots, y^m \circ f(q)) &= \psi(f(q)) = (\psi \circ f \circ \phi^{-1})(\phi(q)) \\ &= (\psi \circ f \circ \phi^{-1})(x^1(q), \dots, x^n(q)) \\ &= (x^1(q), \dots, x^k(q), 0, \dots, 0).\end{aligned}$$

- Therefore, relative to the local coordinates (x^1, \dots, x^n) and (y^1, \dots, y^m) the map f is such that

$$(x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^k, 0, \dots, 0).$$

Constant Rank Theorem

A consequence of the constant rank theorem is the following extension of the regular level set theorem (Theorem 9.9).

Theorem (Constant-Rank Level Set Theorem; Theorem 11.2)

Let $N \rightarrow M$ be a smooth map and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N , then $f^{-1}(c)$ is a regular submanifold of codimension k .

Remark

A neighborhood of a subset $A \subset N$ is an open set containing A .

Constant Rank Theorem

Example (Orthogonal group $O(n)$; Example 11.3)

The *orthogonal group* $O(n)$ is the subgroup of $GL(n, \mathbb{R})$ of matrices A such that $A^T A = I_n$ (identity matrix),

- This is the level set $f^{-1}(I_n)$, where $f : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$, $A \rightarrow A^T A$.
- It can be shown that f has constant rank (in fact it has rank $k = \frac{1}{2}n(n+1)$).
- Therefore, by the constant-rank level set theorem $O(n)$ is a regular submanifold of $GL(n, \mathbb{R})$ (of codimension $\frac{1}{2}n(n+1)$).

The Immersion and Submersion Theorems

Reminder

Suppose that M is a manifold of dimension m and N is a manifold of dimension n , and let $f : N \rightarrow M$ be a smooth map.

- f is an *immersion* at p if $f_{*,p} : T_p N \rightarrow T_{f(p)} M$ is injective.
- f is a *submersion* at p if $f_{*,p} : T_p N \rightarrow T_{f(p)} M$ is surjective.

Remark

Equivalently,

f is an immersion at $p \iff (n \leq m \text{ and } \text{rk } f_{*,p} = n),$

f is a submersion at $p \iff (n \geq m \text{ and } \text{rk } f_{*,p} = m).$

As we always have $\text{rk } f_{*,p} \leq \min(m, n)$, we see that

f is an immersion/submersion at $p \iff f_{*,p}$ has maximal rank.

The Immersion and Submersion Theorems

Facts

Set $k = \min(m, n)$, and denote by $\mathbb{R}_{\max}^{m \times n}$ the set of $m \times n$ matrices $A \in \mathbb{R}^{m \times n}$ of maximal rank.

- An $m \times n$ -matrix has maximal rank if and only if it has a non-zero $k \times k$ -minor.
- The minors are polynomials in the coefficients of matrices, and hence are continuous functions.
- Thus, if a matrix A has a non-zero $k \times k$ -minor, then this minor is non-zero for any matrix that is sufficiently close to A , and so those matrices have maximal rank.
- It follows that $\mathbb{R}_{\max}^{m \times n}$ is a neighbourhood of each of its elements, and hence is an open set in $\mathbb{R}^{m \times n}$.

The Immersion and Submersion Theorems

Facts

Suppose that $f : N \rightarrow M$ is a smooth map. Let (U, x^1, \dots, x^n) be chart about p in N and (V, y^1, \dots, y^m) a chart about $f(p)$ in M . Set $U_{\max} = \{q \in U; f_{*,q} \text{ has maximal rank}\}$.

- If $q \in U$, then $f_{*,q} : T_q N \rightarrow T_{f(q)} M$ is represented by the matrix $J(q) := [\partial f^i / \partial x^j(q)]$, with $f^i = y^i \circ f$, and hence $\text{rk } f_{*,q} = \text{rk } J(q)$. Thus,

$$U_{\max} = \{q \in U; J(q) \in \mathbb{R}_{\max}^{m \times n}\} = J^{-1}(\mathbb{R}_{\max}^{m \times n}).$$

- It can be shown that $q \rightarrow J(q)$ is C^∞ , and hence is continuous.
- As $\mathbb{R}_{\max}^{m \times n}$ is open, it follows that U_{\max} is open as well.
- In particular, if f_* has maximal rank at p , then it has maximal rank near p .

The Immersion and Submersion Theorems

As a consequence we obtain:

Proposition (Proposition 11.4)

If a smooth map $f : N \rightarrow M$ is a immersion (resp., a submersion) at a point $p \in N$, then it is an immersion (resp., submersion) near p . In particular, it has constant rank near p .

The Immersion and Submersion Theorems

Combining the previous proposition with the Constant Rank Theorem gives the following result.

Theorem (Theorem 11.5)

Let $f : N \rightarrow M$ be a smooth map.

- ① **Immersion Theorem.** If f is an immersion at p , then there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that near $\phi(p)$ we have

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^n, 0, \dots, 0).$$

- ② **Submersion Theorem.** If f is a submersion at p , then there are a chart (U, ϕ) centered at p in N and a chart (V, ψ) centered at $f(p)$ in M such that near $\phi(p)$ we have

$$(\psi \circ f \circ \phi^{-1})(r^1, \dots, r^m, r^{m+1}, \dots, r^n) = (r^1, \dots, r^m).$$

The Immersion and Submersion Theorems

Remark

- The immersion theorem implies that if $f : N \rightarrow M$ is an immersion then, for every $p \in N$, there are a chart (U, x^1, \dots, x^n) centered at p in N and a chart (V, y^1, \dots, y^m) centered at $f(p)$ in M relative to which f is such that

$$(*) \quad (x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^n, 0, \dots, 0).$$

- Conversely, If f satisfies $(*)$, then, setting $f^i = y^i \circ f$, we have

$$[\partial f^i / \partial x^j] = \begin{bmatrix} \partial x^i / \partial x^j \\ 0_{m-n} \end{bmatrix} = \begin{bmatrix} I_n \\ 0_{m-n} \end{bmatrix}.$$

In particular, $[\partial f^i / \partial x^j]$ has maximal rank, which implies that f is an immersion near p .

The Immersion and Submersion Theorems

Remark

- The submersion theorem implies that if $f : N \rightarrow M$ is a submersion then, for every $p \in N$, there are a chart (U, x^1, \dots, x^n) centered at p in N and a chart (V, y^1, \dots, y^m) centered at $f(p)$ in M relative to which f is such that

$$(x^1, \dots, x^m, x^{m+1}, \dots, x^n) \longrightarrow (x^1, \dots, x^m).$$

- The projection $(x^1, \dots, x^m, x^{m+1}, \dots, x^n) \rightarrow (x^1, \dots, x^m)$ is an open map (see Problem A.7). This implies that f maps any neighborhood of p onto a neighborhood of $f(p)$.
- As this is true for every $p \in N$, we see that f is an open map. Therefore, we obtain:

Corollary (Corollary 11.6)

Every submersion $f : N \rightarrow M$ is an open map.

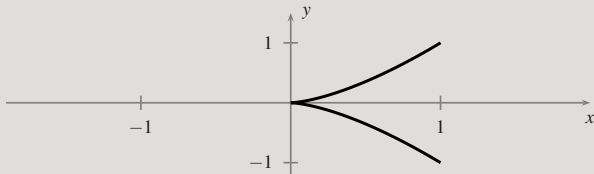
Images of Smooth Maps

Let us look at some examples of smooth maps $f : \mathbb{R} \rightarrow \mathbb{R}^2$.

Example (Example 11.7)

Let $f(t) = (t^2, t^3)$.

- This is a one-to-one map, since $t \rightarrow t^3$ is one-to-one.
- As $f'(0) = (0, 0)$ the differential $f_{*,0}$ is zero, and so f is not an immersion at 0 .
- The image of f is the cuspidal cubic $y^2 = x^3$.



Images of Smooth Maps

Example (Example 11.8)

Let $f(t) = (t^2 - 1, t^3 - t)$.

- As $f'(t) = (2t, 3t^2 - 1) \neq (0, 0)$ the differential f_* is one-to-one everywhere, and hence f is an immersion.
- However, f is not one-one since $f(1) = f(-1) = (0, 0)$.
- The image of f is the nodal cubic $y^2 = x^2(x + 1)$ (see Tu's book).

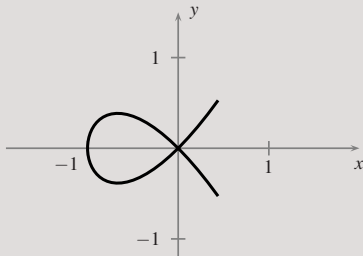
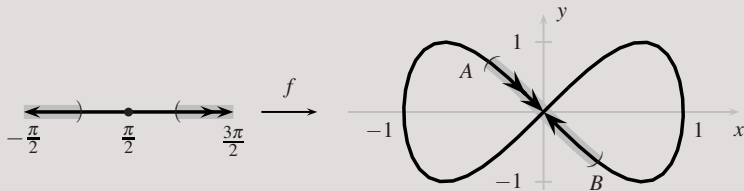


Image of Smooth Maps

Example (The Figure-eight; Example 11.12)

Set $I = (-\pi/2, 3\pi/2)$, and let $f : I \rightarrow \mathbb{R}^2$, $t \rightarrow (\cos t, \sin 2t)$.



- $f'(t) = (-\sin t, 2\cos 2t) \neq (0, 0)$, and so f is an immersion.
- f is one-to-one, and so f is a bijection onto its image $f(I)$.
- The inverse map $f^{-1} : f(I) \rightarrow I$ is not continuous: if $t \rightarrow (3\pi/2)^-$, then $f(t) \rightarrow (0, 0) = f(\pi/2)$, but

$$f^{-1}(f(t)) = t \rightarrow 3\pi/2 \notin I.$$

In particular, $f : I \rightarrow f(I)$ is not a homeomorphism.

Image of Smooth Maps

Summary

As the previous examples show:

- A one-to-one smooth map need not be an immersion.
- An immersion need not be one-to-one.
- A one-to-one immersion need not be a homeomorphism onto its image.

Definition

A smooth map $f : N \rightarrow M$ is called an *embedding* if f is an immersion and a homeomorphism onto its image $f(N)$ with respect to the subspace topology.

Remark

A one-to-one immersion $f : N \rightarrow M$ is an embedding if and only if it is an open map.

Image of Smooth Maps

The importance of embeddings stems from the following result.

Theorem (Theorem 11.13)

If $f : N \rightarrow M$ is an embedding, then its image $f(N)$ is a regular submanifold in M .

Image of Smooth Maps

Proof of Theorem 11.13.

- As f is an immersion, by the immersion theorem, for any $p \in N$, there are a chart (U, x^1, \dots, x^n) centered at p in N and a chart (V, y^1, \dots, y^m) centered at $f(p)$ relative to which f is such that $(x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^n, 0, \dots, 0)$. Thus,

$$f(U) = \{q \in V; y^{n+1}(q) = \dots = y^m(q) = 0\}$$

- As $f : N \rightarrow f(N)$ is a homeomorphism, $f(U)$ is an open set in $f(N)$ with respect to the subspace topology. That is, there is an open V' in M such that $f(U) = V' \cap f(N)$.

- Thus,

$$V \cap V' \cap f(N) = V \cap f(U) = f(U) = \{y^{n+1} = \dots = y^m = 0\}.$$

That is, $(V \cap V', y^1, \dots, y^m)$ is an adapted chart relative to $f(N)$ near $f(p)$ in M .

- This shows that $f(N)$ is a regular submanifold.



We have the following converse of the previous theorem.

Theorem (Theorem 11.14)

If N is a regular submanifold in M , then the inclusion $i : N \rightarrow M$ is an embedding.

Image of Smooth Maps

Proof of Theorem 11.14.

Let N be a regular submanifold in M .

- As N has the subspace topology, the inclusion $i : N \rightarrow M$ is a homeomorphism onto its image.
- As N is a regular submanifold, near every $p \in N$, there is an adapted chart (U, x^1, \dots, x^m) near p in M such that $(U \cap N, x^1, \dots, x^n)$ is a chart in N near p and $U \cap N = \{x^{n+1} = \dots = x^m = 0\}$.
- Therefore, relative to the charts $(U \cap N, x^1, \dots, x^n)$ and (U, x^1, \dots, x^m) the inclusion $i : N \rightarrow M$ is such that
$$(x^1, \dots, x^n) \longrightarrow (x^1, \dots, x^n, 0, \dots, 0).$$
- By a previous remark, it follows that the map $i : N \rightarrow M$ is an immersion near p . □

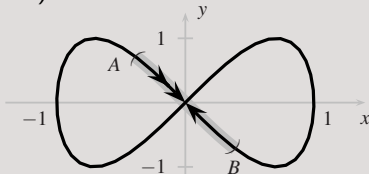
Image of Smooth Maps

Remarks

- 1 The images of smooth embeddings are called *embedded submanifolds*.
- 2 The previous two results show that the regular submanifolds and embedded submanifolds are the same objects.
- 3 The images of one-to-one immersions are called *immersed submanifolds*.

Example

The figure-eight is an immersed submanifold in \mathbb{R}^2 (but this is not a regular submanifold).



Smooth Maps into a Submanifold

Question

Suppose that $f : N \rightarrow M$ is smooth map such that $f(N)$ is contained in a given subset $S \subset M$. If S is manifold, then is the induced map $f : N \rightarrow S$ smooth as well?

Theorem (Theorem 11.15)

Suppose that $f : N \rightarrow M$ is a smooth map whose image is contained in a regular submanifold S in M . Then the induced map $f : N \rightarrow S$ is smooth.

Remarks

- 1 The above result does not hold if S is only an immersed submanifold (see Tu's book).
- 2 The converse holds. As S is a regular submanifold, the inclusion $i : S \rightarrow M$ is smooth. Thus, if $f : N \rightarrow S$ is a smooth map, then $i \circ f : N \rightarrow M$ is a C^∞ map that induces f .

Smooth Maps into a Submanifold

Proof of Theorem 11.15.

Set $m = \dim M$ and $s = \dim S$, and let $p \in N$.

- As S is a regular submanifold and $f(p) \in S$, there is an adapted chart $(V, \psi) = (V, y^1, \dots, y^m)$ near $f(p)$ in M . Then $(V \cap S, \psi_S) = (V \cap S, y^1, \dots, y^s)$ is a chart near $f(p)$ in S .
- As f is a C^∞ -map, the functions $y^i \circ f$ are C^∞ on $U := f^{-1}(V)$ (which is an open neighbourhood of p in N since f is continuous).
- On $f^{-1}(V)$ we have $\psi_S \circ f = (y^1 \circ f, \dots, y^s \circ f)$, and so $\psi_S \circ f : f^{-1}(V) \rightarrow \mathbb{R}^s$ is a smooth map.
- As $(V \cap S, \psi_S)$ is chart for S , it follows from Proposition 6.15 that the induced map $f : f^{-1}(V) \rightarrow S$ is smooth, and hence is smooth near p . □

Smooth Maps into a Submanifold

Example (Multiplication map of $SL(n, \mathbb{R})$; Example 11.16)

$SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ of matrices of determinant 1.

- This is a regular submanifold in $GL(n, \mathbb{R})$ (Example 9.11), and so the inclusion $\iota : SL(n, \mathbb{R}) \hookrightarrow GL(n, \mathbb{R})$ is a smooth map.
- By Example 6.21 we have a smooth multiplication map,

$$\mu : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R}).$$

- We thus get a smooth map,

$$\mu \circ (\iota \times \iota) : SL(n, \mathbb{R}) \times SL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R}).$$

- As it takes values in $SL(n, \mathbb{R})$, and $SL(n, \mathbb{R})$ is a regular submanifold in $GL(n, \mathbb{R})$, we get a smooth multiplication map,

$$SL(n, \mathbb{R}) \times SL(n, \mathbb{R}) \longrightarrow SL(n, \mathbb{R}).$$

Smooth Maps into a Submanifold

Theorem 11.5 and its converse are especially useful when $M = \mathbb{R}^m$. In this case we have:

Corollary

Let S be a regular submanifold in \mathbb{R}^m and $f : N \rightarrow \mathbb{R}^m$ a map such that $f(N) \subset S$. Set $f = (f^1, \dots, f^m)$. Then TFAE:

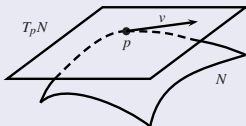
- (i) f is smooth as a map from N to S .
- (ii) f is smooth as a map from N to \mathbb{R}^m .
- (iii) The components f^1, \dots, f^m are smooth functions on N .

The Tangent Space to a Submanifold in \mathbb{R}^m

Facts

Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a smooth function with no critical points on its zero set $N = f^{-1}(0)$.

- By the regular level set theorem N is a regular submanifold in \mathbb{R}^{n+1} of dimension n .
- Then the inclusion $i : N \rightarrow \mathbb{R}^{n+1}$ is an embedding, and so, for every $p \in N$, the differential $i_* : T_p N \rightarrow T_p \mathbb{R}^{n+1}$ is injective.
- We thus can identify the tangent space $T_p N$ with a subspace of $T_p \mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$. More precisely, we regard it as a subspace of \mathbb{R}^{n+1} through p .
- Thus, any $v \in T_p N$, is identified with a vector $\langle v^1, \dots, v^{n+1} \rangle$, which is then identified with the point $x = p + (v^1, \dots, v^{n+1})$.



The Tangent Space to a Submanifold in \mathbb{R}^m

Facts

- Set $p = (p^1, \dots, p^{n+1})$ and $x = (x^1, \dots, x^{n+1})$. Let $c : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1}$ be a smooth curve such that $c(0) = p$, $c'(0) = v$, and $c(t) \in N$, i.e., $f(c(t)) = 0$. Then

$$0 = \left. \frac{d}{dt} \right|_0 f(c(t)) = \sum (c^i)'(0) \frac{\partial f}{\partial x^i}(c(0)) = \sum v^i \frac{\partial f}{\partial x^i}(p).$$

- As $v^i = x^i - p^i$, we see that (x^1, \dots, x^{n+1}) satisfies,

$$(*) \quad \sum \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0.$$

- As p is a regular point, $\frac{\partial f}{\partial x^i}(p) \neq 0$ for some i , and so the solution set of $(*)$ has dimension n .
- As $\dim N = n$, the tangent space $T_p N$ has dimension n , and so it is identified with the full solution set of $(*)$.

Therefore, we obtain the following result:

Proposition

Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a smooth function with no critical points on its zero set $N = f^{-1}(0)$. If $p = (p^1, \dots, p^{n+1})$ is a point in N , then the tangent space $T_p N$ is defined by the equation,

$$(*) \quad \sum \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0.$$

Remark

Equivalently, $T_p N$ is identified with the hyperplane through p that is normal to the gradient vector $\langle \partial f / \partial x^1(p), \dots, \partial f / \partial x^{n+1}(p) \rangle$.

The Tangent Space to a Submanifold in \mathbb{R}^m

Example (Tangent plane to a sphere)

The sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is the zero set of

$$f(x, y, z) = x^2 + y^2 + z^2 - 1.$$

- We have

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z.$$

- Thus, at $p = (a, b, c) \in \mathbb{S}^2$ the tangent plane has equation,

$$\frac{\partial f}{\partial x}(p)(x - a) + \frac{\partial f}{\partial y}(p)(y - b) + \frac{\partial f}{\partial z}(p)(z - c) = 0,$$

$$\iff a(x - a) + b(y - b) + c(z - c) = 0,$$

$$\iff ax + by + cz = a^2 + b^2 + c^2,$$

$$\iff ax + by + cz = 1.$$