Differentiable Manifolds
§10. Categories and Functors

Sichuan University, Fall 2021

Definition (Categories)

A (concrete) category $\mathscr{C}$ consists of the following data:

- A collection $\mathrm{Ob}(\mathscr{C})$ of sets called *objects*.
- For each pair of objects $A, B \in \mathrm{Ob}(\mathscr{C})$ a collection $\mathrm{Mor}(A, B)$ of maps $f : A \to B$ called *morphisms*.

We further require the following properties:

(i) *Identity axiom*. For every object A the identity map $\mathbb{1}_A : A \to A$ is a morphism, i.e., $\mathbb{1}_A \in \mathrm{Mor}(A, A)$. In particular, for any morphisms $f : A \to B$ and $g : B \to A$, we have

$$f \circ \mathbb{1}_A = f \qquad \text{and} \qquad \mathbb{1}_A \circ g = g.$$

(ii) *Associativity axiom*. If $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Example

The category of sets, where:

- The objects are arbitrary sets.
- The morphisms are arbitrary maps.

This category is denoted **Set**.

Example

The category of groups, where:

- The objects are groups.
- The morphisms are group homomorphisms.

This category is denoted **Grp**.

Categories

Example

The category of real vector spaces, where:

- The objects are vector spaces over $\mathbb{R}$.
- The morphisms are $\mathbb{R}$-linear maps.

This category is denoted $\mathbf{Vect}_{\mathbb{R}}$.

Example

The category of real algebras, where:

- The objects are algebras over $\mathbb{R}$.
- The morphisms are algebra homomorphisms.

This category is denoted $\mathbf{Alg}_{\mathbb{R}}$.

Categories

Example

The category of topological spaces (a.k.a. *continuous category*), where:

- The objects are topological spaces.
- The morphisms are continuous maps.

This category is denoted **Top**.

Example

The category of smooth manifolds (a.k.a. *smooth category*), where:

- The objects are smooth manifold.
- The morphisms are smooth maps between manifolds.

This category is denoted $\mathbf{Man}^{\infty}$.

Categories

Example

The category of *pointed manifolds*, where:

- The objects are pointed manifolds, i.e., pairs (M, q) where M is a (smooth) manifold and q is a point of M.
- A morphism $f \in \mathrm{Mor}((N, p), (M, q))$ is a smooth map $F : N \to M$ such that $F(p) = q$.

This category is denoted $\mathbf{Man}^{\infty}_{\bullet}$.

Definition (Definition 10.1)

Let A and B be objects in a given category $\mathscr{C}$.

- We say that a morphism $f : A \to B$ is an *isomorphism* when f is a bijection and $f^{-1} \in \mathrm{Mor}(B, A)$.
- We say that the objects A and B are isomorphic, and write $A \simeq B$, when there is an isomorphism $f : A \to B$.

Examples

1. In the category **Top** the isomorphisms are called homeomorphisms.
2. In the category $\mathbf{Man}^{\infty}$ the isomorphisms are called diffeomorphisms.

Functors

Definition (Functors; Definition 10.2)

Given categories $\mathscr{C}$ and $\mathscr{D}$, a *(covariant) functor* $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ associate to every object A in $\mathscr{C}$ an object $\mathscr{F}(A)$ in $\mathscr{D}$ and associates to every morphism $f : A \to B$ (between objects in $\mathscr{C}$) a morphism $\mathscr{F}(f) : \mathscr{F}(A) \to \mathscr{F}(B)$ in such a way that

$$\mathscr{F}(\mathbb{1}_A) = \mathbb{1}_{\mathscr{F}(A)} \quad \text{and} \quad \mathscr{F}(f \circ g) = \mathscr{F}(f) \circ \mathscr{F}(g).$$

Functors

Example

The tangent space construction gives rise to a functor $\mathscr{F} : \mathbf{Man}^{\infty}_{\bullet} \to \mathbf{Vect}_{\mathbb{R}}$.

- To any pointed manifold (N, p) is associated the tangent space $\mathscr{F}(N) = T_pN$, which is a vector space.
- To any smooth map $f : (N, p) \to (M, q)$ is associated the differential $\mathscr{F}(f) = f_{*,p} : T_pN \to T_qM$, which is a linear map.
- The differential of the identity $\mathbb{1}_N : N \to N$ is the identity map $\mathbb{1}_{T_pN} : T_pN \to T_pN$.
- The functorial property $\mathscr{F}(f \circ g) = \mathscr{F}(f) \circ \mathscr{F}(g)$ is just the Chain Rule,

$$(g \circ f)_{*,p} = g_{*,f(p)} \circ f_{*,p}.$$

Remark

Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a functor and let $f : A \to B$ be an isomorphism between objects in $\mathscr{C}$. We get morphisms,

$$\mathscr{F}(f) : \mathscr{F}(A) \to \mathscr{F}(B), \qquad \mathscr{F}(f^{-1}) : \mathscr{F}(B) \to \mathscr{F}(A).$$

By the functor properties we have

$$\mathscr{F}(f^{-1}) \circ \mathscr{F}(f) = \mathscr{F}(f^{-1} \circ f) = \mathscr{F}(\mathbb{1}_A) = \mathbb{1}_{\mathscr{F}(A)}.$$

Likewise, $\mathscr{F}(f) \circ \mathscr{F}(f^{-1}) = \mathbb{1}_{\mathscr{F}(B)}$. Therefore, we arrive at the following result:

Proposition (Proposition 10.3)

If $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is a functor and $f : A \to B$ is an isomorphism, then the morphism $\mathscr{F}(f) : \mathscr{F}(A) \to \mathscr{F}(B)$ is an isomorphism with inverse $\mathscr{F}(f^{-1}) : \mathscr{F}(B) \to \mathscr{F}(A)$.

Example

Let $f : N \to M$ be a diffeomorphism between manifolds.

- If $p \in N$, then $f : (N, p) \to (M, f(p))$ is an isomorphism in the category $\mathbf{Man}_\bullet^\infty$, and so the differential $\mathscr{F}(f) = f_{*,p} : T_pN \to T_{f(p)}M$ is an isomorphism of vector spaces (Corollary 8.6).
- It follows that $\dim N = \dim M$, i.e., the dimension of a manifold is invariant under diffeomorphisms (Corollary 8.7).

In the definition of functor we may reverse the direction of the arrows.

Definition (Contravariant Functors; Definition 10.4)

Given categories $\mathscr{C}$ and $\mathscr{D}$, a *contravariant functor* $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ associate to every object A in $\mathscr{C}$ an object $\mathscr{F}(A)$ in $\mathscr{D}$ and associate to every morphism $f:A\to B$ (between objects in $\mathscr{C}$) a morphism $\mathscr{F}(f):\mathscr{F}(B)\to\mathscr{F}(A)$ in such a way that

$$\mathscr{F}(\mathbb{1}_A)=\mathbb{1}_{\mathscr{F}(A)} \quad \text{and} \quad \mathscr{F}(f\circ g)=\mathscr{F}(g)\circ\mathscr{F}(f).$$

In the same way as with covariant functors, we have:

Proposition

If $\mathscr{F}:\mathscr{C}\to\mathscr{D}$ is a contravariant functor and $f:A\to B$ is an isomorphism, then the morphism $\mathscr{F}(f):\mathscr{F}(B)\to\mathscr{F}(A)$ is an isomorphism with inverse $\mathscr{F}(f^{-1}):\mathscr{F}(A)\to\mathscr{F}(B)$.

Functors

Definition

Let $F : N \to M$ be a smooth map between manifolds. The *pullback map* $F^* : C^\infty(M) \to C^\infty(N)$ is defined by

$$F^* h = h \circ F, \qquad h \in C^\infty(M).$$

Fact

If $F : N \to M$ and $G : P \to N$ are smooth maps, and $h \in C^\infty(M)$, then we have

$$(F \circ G)^* h = h \circ F \circ G = (F^* h) \circ G = G^*(F^* h) = (G^* \circ F^*) h.$$

Thus,

$$(F \circ G)^* = G^* \circ F^*.$$

Example

The smooth functions on manifolds give rise to a contravariant functor $\mathscr{F} : \mathbf{Man}^\infty \to \mathbf{Alg}_\mathbb{R}$.

- To a manifold M is associated the algebra $\mathscr{F}(M) = C^\infty(M)$.
- To a smooth map $F : N \to M$ is associated the pullback $\mathscr{F}(F) = F^* : C^\infty(M) \to C^\infty(N)$.
- We have $(\mathbb{1}_M)^* = \mathbb{1}_{C^\infty(M)}$.
- If $F : N \to M$ and $G : P \to N$ are smooth maps, then

$$\mathscr{F}(F \circ G) = (F \circ G)^* = G^* \circ F^* = \mathscr{F}(G) \circ \mathscr{F}(F).$$

Therefore, $\mathscr{F}$ is a contravariant functor.

The Dual and Multicovector Functors

Reminder

- If V is a vector space, then $V^\vee = \mathrm{Hom}(V, \mathbb{R})$ is the dual space of V consisting of all linear forms $\alpha : V \to \mathbb{R}$.
- If $\{e_1, \ldots, e_n\}$ is a basis of V, then the dual basis $\{\alpha^1, \ldots, \alpha^n\}$ of $V^\vee$ is given by

$$\alpha^i(e_j) = \delta^i_j, \qquad 1 \leq i, j \leq n.$$

Definition (Dual of a linear map)

If $L : V \to W$ is a linear map, its *dual map* is the linear map $L^\vee : W^\vee \to V^\vee$ defined by

$$L^\vee(\alpha) = \alpha \circ L, \qquad \alpha \in W^\vee.$$

The Dual and Multicovector Functors

Proposition (Proposition 10.5)

1. $(\mathbb{1}_V)^\vee = \mathbb{1}_{V^\vee}$.
2. *If* $f : V \to W$ *and* $g : W \to U$ *are linear maps, then* $(f \circ g)^\vee = g^\vee \circ f^\vee$.

Corollary

The dual construction gives rise to a contravariant functor $\mathscr{F} : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$*:*

- *To each vector space is associated its dual* $\mathscr{F}(V) = V^\vee$.
- *To each linear map* $L : V \to W$ *is associated its dual map* $\mathscr{F}(L) = L^\vee : W^\vee \to V^\vee$.

In particular, if $L : V \to W$ *is an isomorphism, then its dual map* $L^\vee : W^\vee \to V^\vee$ *is an isomorphism as well.*

The Dual and Multicovector Functors

Reminder

If V is a vector space, then $A_k(V)$, $k \geq 1$, is the vector space of k-covectors on V, i.e., alternating k-linear maps $f : V^k \to \mathbb{R}$.

Definition (Pullback by a linear map)

If $L : V \to W$ is a linear map, then its *pullback map* is the linear map $L^* : A_k(W) \to A_k(V)$ defined by

$$(L^* f)(v_1, \ldots, v_k) = f\left(L(v_1), \ldots, L(v_k)\right), \quad f \in A_k(W),\ v_i \in V.$$

The Dual and Multicovector Functors

Proposition (Proposition 10.5)

1. $(\mathbb{1}_V)^* = \mathbb{1}_{A_k(V)}$.
2. *If* $K : U \to V$ *and* $L : V \to W$ *are linear maps, then* $(K \circ L)^* = L^* \circ K^*$.

Corollary

The construction $A_k(\cdot)$ *gives rise to a contravariant functor* $\mathscr{F} : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$*:*

- *To each vector space is associated its space of* k*-covectors* $\mathscr{F}(V) = A_k(V)$.
- *To each linear map* $L : V \to W$ *is associated its pullback map* $\mathscr{F}(L) = L^* : A_k(W) \to A_k(V)$.

In particular, if $L : V \to W$ *is an isomorphism, then its pullback map* $L^* : A_k(W) \to A_k(V)$ *is an isomorphism as well.*