Differentiable Manifolds
§1. Smooth Functions on Euclidean Space

Sichuan University, Fall 2021

C^∞ Functions vs. Analytic Functions

Convention

Coordinates in $\mathbb{R}^n$ are denoted with superscripts indices $x^1, \dots x^n$ (differential geometry convention; see §4.7).

In what follows $p = (p^1, \dots, p^n)$ is a point in open set U in $\mathbb{R}^n$.

Definition (C^k Functions)

Let $f : U \to \mathbb{R}$ be a real-valued function and k an integer ≥ 0.

- We say that f is C^k at p when all partial derivatives $\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}}$ of order $\leq j$ exists and are continuous at p.
- We say that f is C^k on U when it is C^k at every point of U.

C^∞ Functions vs. Analytic Functions

Definition (C^∞ Functions)

Let $f : U \to \mathbb{R}$ be a real-valued function.

- We say that f is C^∞ at p when it is C^k for all k, i.e., partial derivatives of all orders exist and are continuous at p.
- We say that f is C^∞ on U when it is C^∞ at every point of U.

Definition (Vector-Valued Functions)

Let $f : U \to \mathbb{R}^m$ be a vector-valued function with components $f^1, \ldots, f^m$.

- We say that f is C^k (resp., C^∞) at p when all the components $f^1, \ldots, f^m$ are C^k (resp., C^∞).
- We say that f is C^k (resp., C^∞) on U when it is C^k (resp., C^∞) at every point of U, i.e., the components $f^1, \ldots, f^m$ are C^k (resp., C^∞) on U.

C^∞ Functions vs. Analytic Functions

Examples

1. A C^0 function is a continuous function.
2. Polynomial, sine, cosine, exponential functions are C^∞ on the real-line.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = (x) = \sqrt[3]{x}$.

- f is C^0, but it is not C^1 at $x = 0$.
- Set $g(x) = \int_0^x f(t)dt = \frac{4}{3}x^{\frac{4}{3}}$. Then f is C^1 but not C^2 at $x = 0$.
- In the same way, for every $k \geq 0$, we construct an example of a function that is C^k but not C^{k+1}.

Convention

- In Tu's book a neighborhood of p is actually an open neighborhood of p, i.e., an open set that contains p.
- Usually, a neighborhood of p is a set that contains an open set containing p.

C^∞ Functions vs. Analytic Functions

Definition

Let $f : U \to \mathbb{R}$ be C^∞ near p. We say that f is real-analytic at p when there is a neighborhood V of p on which f agrees with its Taylor series,

$$f(x) = f(p) + \sum_{i=1} \frac{\partial f}{\partial x^i}(p)(p^i - x^i) \\ + \frac{1}{2!}\sum_{i,j} \frac{\partial f}{\partial x^i x^j}(p)(p^i - x^i)(p^j - x^j) + \cdots.$$

Remark

- It can be shown that f is C^∞ on V.
- It is even real-analytic at every point of V.

C^∞ Functions vs. Analytic Functions

Remark

There are C^∞-functions that are not real-analytic, i.e., a C^∞-function need not agree with its Taylor series at a given point (*cf.* example below).

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

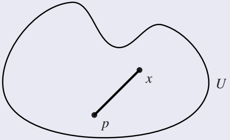
$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then f is a C^∞-function on $\mathbb{R}$ such that $f^{(k)}(0) = 0$ for all $k \geq 0$ (*cf.* Problem 1.2).

Taylor's Theorem with Remainder

Definition

A subset $S \subset \mathbb{R}^n$ is called star-shaped with respect to a given point $p \in S$, when, for every $x \in S$ the segment line from p to x lies in S, i.e., $(1-t)p + tx \in S$ for all $t \in [0,1]$.



Example

For any $\epsilon > 0$ the open ball

$$B(p,\epsilon) := \{x \in \mathbb{R}^n;\ \|x-p\| < \epsilon\}$$

is star-shaped with respect to p.

Taylor's Theorem with Remainder

Lemma (Taylor's Theorem with Remainder; see Tu's book)

Let $f : U \to \mathbb{R}$ be a C^∞-function on an open set U which is star-shaped with respect to $p = (p^1, \ldots, p^n) \in U$. Then there are functions $g_1(x), \ldots, g_n(x)$ in $C^\infty(U)$ such that

$$f(x) = f(p) + \sum_{i=1}^{n} (x^i - p^i) g_i(x), \qquad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Remarks

1. There are higher order versions of Taylor's Theorem with Remainder (see Problem 1.6 for the 2nd order version).
2. If U is not star-shaped we always can restricts f to a ball $B(p, \epsilon) \subset U$ (provided that ϵ is small enough). Taylor's theorem then can be applied, since $B(p, \epsilon)$ is a star-shaped neighborhood of p.