Chap. 6: Chain Conditions

Σ = partially ordered set $\leq$

Prop. 6.1: TFAE:

(i) Every increasing sequence $x_1 \leq x_2 \leq \cdots$ in Σ is stationary.

(ii) Every non-empty subset of Σ has a maximal elt.

Proof: i) $\Rightarrow$ ii): Suppose that (ii) is false

$\rightarrow$ $\exists S \subset \Sigma$ with no maximal elt.

Let $x_1 \in S$: As x_1 is not maximal $\exists x_2 \in S$ s.t. $x_1 \leq x_2$ and $x_1 \neq x_2$

x_2 not maximal so $\exists x_3 \in S$ s.t. $x_2 \leq x_3$ and $x_3 \neq x_2$

By induction we can find $x_n \in S, n \geq 1$ s.t. $x_{n-1} \leq x_n$ and $x_n \neq x_{n-1}$

$\rightarrow$ the sequence $(x_n)_{n\geq 1}$ is ascending and non-stationary.

(ii) $\Rightarrow$ (i): Let $(x_n)_{n\geq 1}$ be a sequence in Σ

$S = \{x_n ; n \geq 1\}$ has a maximal elt $x_m \in S$

If $n \geq m+1$, then

$$\left.\begin{cases} x_m \leq x_n \\ x_n \in S \\ x_m \text{ maximal} \end{cases}\right\} \rightsquigarrow x_n = x_m \ \forall n \geq m+1$$

$\rightarrow$ The sequence $(x_n)_{n\geq 1}$ is stationary

Rmk: . $(\Sigma, \leq)$ partially ordered set

. m maximal: $x \geq m \Rightarrow x = m$.

. M module over ring A
. Σ = set of sub-modules of A

If Σ ordered wrt $\subseteq$, then: (i) = ascending chain condition (a.c.c.)
(ii) = maximal condition.
. If Σ ordered wrt $\supseteq$, then: (i) = descending chain cond. (d.c.c.)
(ii) = minimal condition.

. If M satisfies a.c.c, then M is called Noetherian
. ——— d.c.c, then ——— Artinian

Examples:
(1) Finite group satisfies (a.c.c) and (d.c.c)
$\hookrightarrow$ $\mathbb{Z}$-module.

(2) $\mathbb{Z}$ satisfies a.c.c:
$\hookrightarrow$ $\mathbb{Z}$-module = ideals
. ideals of $\mathbb{Z}$: $(m_1) \subseteq (m_2) \iff m_2$ divides m_1
$(m_1) \subseteq (m_2) \subseteq \dots$
m_j divides m_1 $\forall j$ only finitely many possibilities
→ $\mathbb{Z}$ satisfies a.c.c.
. $a \in \mathbb{Z} \setminus \{0\}$, $(a) \supsetneq (a^2) \supsetneq \cdots \supsetneq (a^n) \supsetneq \dots$
↑ non-stationary.
→ $\mathbb{Z}$ does not satisfies d.c.c.
→ $\mathbb{Z}$ is a Noetherian ring, but is not Artinian.

(3) p = prime integer $G = \{ x \in \mathbb{Q}/\mathbb{Z} ; p^n x \equiv 0$ for some $n \geq 1\}$
$G_n = \{ x \in \mathbb{Q}/\mathbb{Z} ; p^n x \equiv 0 \} \subsetneq G_{n+1}$
→ G does not satisfy a.c.c.

- The G_n are the only proper subgroup of G
  → only descending sequences of subgroups are of the form

$$G_{m_1} \supseteq G_{m_2} \supseteq \cdots$$

w/ $m_1 \geq m_2 \geq \cdots$ → $\underbrace{\{m_j ; j \geq 1\}}_{\text{finite.}} \subset \{1, \ldots, m_1\}$ ↑ finite

→ must be stationary.

(4) $H = \{ \frac{m}{p^n} ; m \in \mathbb{Z}, n \in \mathbb{N}_0 \}$ → does not satisfies a.c.c. or d.c.c.

Exact sequence: $0 \to \mathbb{Z} \hookrightarrow H \xrightarrow{\pi} G \to 0$ $\mathbb{Z}$-module maps

- $\mathbb{Z}$ does not satisfy d.c.c. → H does not satisfies d.c.c.

- G does satisfy a.c.c. : $\{\pi^{-1}(G_n)\}$ non-stationary ascending sequence.

(5) $k[x]$ satisfies a.c.c., not d.c.c. same as for $\mathbb{Z}$.

(6) $k[x_1, x_2, \ldots]$ infinite indeterminate:

- a.c.c. fails: $(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, x_2, \ldots, x_n) \subsetneq \cdots$
  non-stationary ascending sequence
- d.c.c fails: $(x_1) \supsetneq (x_1^2) \supsetneq \cdots$
  non-stationary descending sequence

Prop. 6.2: Let M be an A-module. TFAE:
(i) M is Noetherian
(ii) every submodule of M is finitely gen.

Proof: $\Rightarrow$: N = submodule of M $\quad \Sigma = \{$ submodules of $M \}$.
$\Sigma_1 = \{$ finitely generated submodules of $N \} \subseteq \Sigma$

- $0 \in \Sigma \leadsto \Sigma \neq \emptyset$
- let N_0 be a maximal elt of Σ_1 $\leadsto$ exists since M is Noetherian!
  - Assume $N_0 \subsetneq N$: $\leadsto \exists x \in N$ s.t. $x \notin N_0$
    Set $N_1 = N_0 + Ax \leadsto \begin{cases} N_1 \text{ finitely gen. submodule of } N \\ N_0 \subsetneq N_1 \leadsto \text{contradiction.} \end{cases}$

$\Leftarrow$: $M_1 \subseteq M_2 \subseteq \cdots$ ascending sequence of submodules of M
Set $N = \bigcup_{n=1}^{\infty} M_n$ submodule of M $\to$ N is finitely generated by $x_1, \ldots, x_\ell$

For $j = 1, \ldots, \ell$ $\leadsto$ $x_j \in M_{n_j}$. Set $m = \max\{n_1, \ldots, n_\ell\}$.
$\to x_j \in M_m \ \forall j$
$\to N \subseteq M_m \to M_n = M_m \ \forall n \geq m$.
$\to$ M satisfies a.c.c.
$\leadsto$ M is Noetherian.

Prop. 6.3: $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ exact sequence of A-mod.
Then:
(i) M Noetherian iff M' and M'' are Noetherian.
(ii) M Artinian iff M' and M'' are Artinian

Proof: . Proof of (i):
. If $M'_1 \subseteq M'_2 \subseteq \dots$ ascending chain in M' $\Rightarrow$ $\alpha(M'_1) \subseteq \alpha(M'_2) \subseteq \dots$
ascending chain in M

and $\alpha(M'_j) = \alpha(M'_e) \iff M'_j = M'_e$

α injective $\Longrightarrow$ $\alpha^{-1}(\alpha(M'_j)) = M'_j$

$x \in \alpha^{-1}(\alpha(M'_j)) \Rightarrow \exists y \in M'_j$ s.t.
$\alpha(x) = \alpha(y)$
$\longrightarrow x = y \in M'_j$

. If $M''_1 \subseteq M''_2 \subseteq \cdots$ ascending chain in M''
$\Rightarrow \beta^{-1}(M''_1) \subseteq \beta^{-1}(M''_2) \subseteq \dots$ ascending chain in M.

$\beta^{-1}(M''_j) = \beta(M''_e) \iff M''_j = M''_e$

β surjective: $\beta(\beta^{-1}(M''_j)) = M''_j$: $x \in M''_j \Rightarrow \exists y \in M$ s.t.
$x = \beta(y)$
$\longrightarrow y \in \beta^{-1}(M''_j)$
$\longrightarrow x = \beta(y) \in \beta(\beta^{-1}(M''_j))$.
$\longrightarrow M''_j \subseteq \beta(\beta^{-1}(M''_j)) \subseteq M''_j$.

. Assume M is Noetherian:
. $M'_1 \subseteq M'_2 \subseteq \dots$ ascending chain in M'
$\longrightarrow \alpha(M'_1) \subseteq \alpha(M'_2) \subseteq \cdots$ ascending chain in M
M Noetherian $\Rightarrow \exists n$ s.t. $\alpha(M'_j) = \alpha(M'_n)$ $\forall j \geq n$
$\longrightarrow M'_j = M'_n$ $\forall j \geq n$
$\longrightarrow M'$ is Noetherian.

- Likewise M'' is Noetherian:

$M_1'' \subseteq M_2'' \subseteq \cdots$ ascending chain in M''

M Noetherian $\exists n$ s.t. $\beta^{-1}(M_j'') = \beta^{-1}(M_n'')$ $\forall j \geq n$.

$\to M_j'' = M_n''$ $\forall j \geq n$.

$\Leftarrow$: Assume M' and M'' are Noetherian.

$M_1 \subseteq M_2 \subseteq \cdots$ ascending chain in M

$\to \exists n$ s.t. $\alpha^{-1}(M_j) = \alpha^{-1}(M_n)$ $\forall j \geq n$.

$\beta(M_j) = \beta(M_n)$ $\forall j \geq n$.

- $x \in M_j$, $j \geq n \Rightarrow \beta(x) \in \beta(M_j) = \beta(M_n)$

$\Rightarrow \exists y \in M_n$ s.t. $x - y \in \ker \beta = \operatorname{ran} \alpha$

$x - y \in (\operatorname{ran} \alpha) \cap M_j = \alpha(\alpha^{-1}(M_j))$

$\to \exists z \in \alpha^{-1}(M_j) = \alpha^{-1}(M_n)$ s.t.

$x - y = \alpha(z) \in M_n \Rightarrow x \in y + M_n \subseteq M_n$.

$\to M_j = M_n$.

$\alpha(\alpha^{-1}(M_j)) \subset (\operatorname{ran} \alpha) \cap M_j$

$x \in (\operatorname{ran} \alpha) \cap M_j \Rightarrow \exists y \in M'$ s.t. $\alpha(y) = x \in M_j$

$\to y \in \alpha^{-1}(M_j)$

$\to x = \alpha(y) \in \alpha(\alpha^{-1}(M_j))$

Cor. 6.4: If $M_1, \dots, M_n$ are Noetherian (resp., Artinian) A-modules. Then so is $\bigoplus_{i=1}^{n} M_i$.

Proof: Induction on n:

$n=2$: $0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$. exact sequence

$\to$ follows from 6.3. M_1, M_2 Noetherian.

Assume result true for $n-1$:

$$0 \to M_n \to \bigoplus_{j=1}^{n} M_j \to \bigoplus_{j=1}^{n-1} M_j \to 0 \quad \text{exc}$$

Examples:

(1) Any finite dim. vector space V over a field k is both Noetherian ~~and Artinian~~ and Artinian by a dimension argument: $V_1 \subseteq V_2 \subseteq \dots$

$$\dim V_1 \leq \dim V_2 \leq \dots \leq \dim V$$

$$\{\dim V_j ; j \geq 1\} \subseteq \{1, \dots, \dim V\}$$

finite

$$\left.\begin{array}{l}\exists n \text{ s.t. } \dim V_j = \dim V_n \quad \forall j \geq n \\ V_n \subseteq V_j\end{array}\right\} \Rightarrow V_j = V_n \; \forall j \geq n$$

$\to$ In particular, any field is N + A.

(2) Any principal ideal domain A is Noetherian: every ideal is single-generated

i.e. principal ideal $= (x)$ $\qquad$ § 6.2

$= xA$ $\qquad$ A is Noetherian

(3) $A = k[x_1, x_2, \dots]$ not Noetherian $\Rightarrow$ Frac(A) is Noetherian.

↑ integral domain.

$f, g \in A$, $\exists n$ s.t. $f, g \in k[x_1, \dots, x_n]$.

$fg = 0$ $\qquad$ $\Rightarrow fg = 0$ in $k[x_1, \dots, x_n]$ integral domain

$\to f = 0$ or $g = 0$

$\to$ A integral domain.

(4) $X =$ compact Hausdorff space $\leadsto C(X)$ is not N.
infinite

Let $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n \subsetneq \cdots$ strictly increasing sequence of closed sets

$\{x_1\} \subseteq \{x_1, x_2\} \subseteq \cdots \quad x_j \neq x_i \quad i \neq j$

X Hausdorff $\leadsto$ Urysohn Lemma $\Rightarrow \forall j \; \exists \varphi_j \in C(X)$ s.t.

$\varphi_j(x_{j+1}) = 1$ and $\varphi_j(x_\ell) = 0$ for $1 \le \ell \le j$

$\exists x, y \in X,\ x \neq y \leadsto$ Urysohn Lemma $\leadsto \exists$ contin. fct. $g: X \to \mathbb{R}$
Hausdorff
s.t. $g(x) = 1$ and $g(y) = 0$.

$\to [0,1] \subseteq g(X)$ since g is continuous

Set $F_n = g^{-1}([0, \tfrac{1}{n}])$ $\quad F_n$ closed $\quad X$ compact $\leadsto X$ normal
$\to$ Urysohn Lemma applies

- $F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_n \supsetneq \cdots$

$a_n = \{ f \in C(X);\ f|_{F_n} = 0 \}$ ideal of $C(X)$

$f \in a_n \leadsto f|_{F_n} = 0 \leadsto f|_{F_{n+1}} = 0 \leadsto f \in a_{n+1}$
$F_n \supseteq F_{n+1}$
$\leadsto a_n \subseteq a_{n+1}$

$F_{n+1} \subsetneq F_n \leadsto \exists x_n \in F_n$ s.t. $x_n \notin F_{n+1}$

Urysohn Lemma $\exists f \in C(X)$ s.t. $f(x) = 0$ on F_{n+1}

$f(x_n) \neq 0$

$\to f \in a_{n+1},\ f \notin a_n$

$\to a_n \subsetneq a_{n+1}$

- $\underline{X \text{ normal}}$: any disjoint closed sets can be separated by neighborhoods:
A, B

$\underline{\text{Urysohn Lemma}}$: X normal $\Rightarrow \forall A, B$ disjoint closed sets $\exists f \in C(X)$ s.t. $f = 1$ on A and $f = 0$ on B.

Step 1:

- A closed, $y \notin A \rightsquigarrow \exists$ open U, V s.t. $U \cap V = \emptyset$ $A \subset U$ and $y \in V$

$x \in A$,

X Hausdorff $\rightsquigarrow \exists U_x, V_x$ $U_x \ni x$, $V_x \ni y$ $U_x \cap V_x = \emptyset$

— $A \subset \bigcup(U_x)$

— A compact $\rightsquigarrow$ $A \subseteq \bigcup_j (U_{x_j})$ Set $U = \bigcup(U_{x_j}) \rightarrow A \subseteq U$, U op.

$V = \bigcap V_{x_j}$ V open $\ni y$

$$U \cap V = \bigcup_j (U_{x_j} \cap V) = \bigcup_j (U_{x_j} \cap V_{x_j}) = \emptyset$$

Step 2: A, B closed

$\forall y \in B$ $\exists U_y, V_y$ s.t. $A \subseteq U_y$ and $V_y \ni y$, $U_y \cap V_y = \emptyset$

$B \subseteq \bigcup V_y$ B compact $\rightsquigarrow$ $B \subseteq \bigcup_j (V_{y_j})$

Set $V = \bigcup(V_{y_j})$, $U = \bigcap U_{y_j}$ $\rightsquigarrow$ U open set $\supseteq A$

V open set $\supseteq B$

$$U \cap V = \bigcup_j (U \cap V_{y_j}) \subseteq \bigcup_j (U_{y_j} \cap V_{y_j}) = \emptyset$$

$\boxed{\text{Every compact Hausdorff space is normal}}$

Prop. 6.5: A Noetherian (resp., Artinian) ring. If M is a fin. gen. A-mod., then M is N. (resp., A.)

Proof: $\boxed{\text{Prop. 2.3}}$ We have an exact sequence.

$$0 \longrightarrow M' \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

(6.4) $\rightsquigarrow$ A^n is N. (resp., A.)
(6.3) $\rightsquigarrow$ M is N. (resp. A.).

Prop. 6.6: A = N. ring (resp. A.) + $\mathfrak{a}$ ideal of A. Then $A/\mathfrak{a}$ is a N. (resp. A.) ring.

Proof: $0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$ exact sequence of A-modules.
(6.3) $\rightsquigarrow$ $A/\mathfrak{a}$ is a Noeth. (resp., Art.) A-module.
$\rightsquigarrow$ any $A/\mathfrak{a}$-module is an A-mod. $\rightsquigarrow$ $A/\mathfrak{a}$ is Noeth. (resp., Art.) as an $A/\mathfrak{a}$-module.

$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ ascending sequence in $A/\mathfrak{a}$
$\mathfrak{a}_i$ ideals in $A/\mathfrak{a}$ $\rightsquigarrow$ A-modules $\rightsquigarrow$ sequence is stationary.
$\longrightarrow$ $A/\mathfrak{a}$ is N.

Chain of submodules of M = <u>finite</u> descending sequence
$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = \{0\} \quad \text{(strict inclusion)}$$

n = length of chain

composition series = maximal length chain.

$\iff M_{i-1}/M_i$ is simple no submodules except 0 and itself.

<u>Prop. 6.7:</u> M has a composition series of length n. Then:

(a) Every comp. series has length n.

(b) Every chain in M can be extended to a comp. series.

<u>Proof:</u> $\ell(M)$ = minimum length of comp. series of M ($\ell(M) = \infty$ if no comp. series).

(i) <u>$N \subset M \Rightarrow \ell(N) < \ell(M)$:</u>

Let (M_i) = composition series of min. length (length $= \ell(M)$)

$\rightarrow N_i = N \cap M_i$ so $N_{i-1}/N_i \subseteq M_{i-1}/M_i$

$\uparrow$ simple so either $N_{i-1}/N_i = 0$ or $N_{i-1}/N_i = M_{i-1}/M_i$

$\Updownarrow$

$N_{i-1} = N_i$ or N_{i-1}/N_i is simple.

$\rightarrow$ we get a composition series of N by removing repeated terms among the N_i.

$\rightarrow \ell(N) \leq \ell(M)$.

. If $\ell(N) = \ell(M)$, then $N_{i-1}/N_i = M_{i-1}/M_i \quad \forall i = 1, 2, \ldots$

$\rightarrow$. $N_n = N_n = 0 \rightarrow M_{n-1} = M_{n-1}/M_n \quad N_{n-1}/N_n = N_{n-1}$.

$$\begin{tikzcd}
0 \arrow[r] & N_{n-1} \arrow[r] \arrow[d, "="] & N_{n-2} \arrow[r, "\pi"] \arrow[d] & N_{n-2}/N_{n-1} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & M_{n-1} \arrow[r] & M_{n-2} \arrow[r, "\pi"] & M_{n-2}/M_{n-1} &
\end{tikzcd}$$

Five Lemma
Middle vertical arrow is surjective.

$\pi(x) \in M_{n-2}/M_{n-1}$ so $\exists\, y \in N_{n-2}$ s.t. $x - y \in M_{n-1} = N_{n-1} \subset N_{n-2}$

$\rightarrow x \in N_{n-2}$.

By induction $N_i = M_i \ \forall i$ ~ $N_0 = M_0$ ~ $M_0 = M$ (6.7)

$N_0 = N \cap M_0 = N \cap M = N$

$\longrightarrow N = M.$

$\longrightarrow$ If $N \subsetneq M$, then $\ell(N) < \ell(M)$.

(ii) Any chain in M has length $\leq \ell(M)$.

$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0$ chain length k

(i) $\longrightarrow$ $\ell(M_i) < \ell(M_{i-1})$ ~ $\ell(M_i) \leq \ell(M_{i-1}) - 1$

$\longrightarrow \ell(M_i) \leq \ell(M_0) - i = \ell(M) - i$

$0 = \ell(M_k) \leq \ell(M) - k$

$\longrightarrow k \leq \ell(M).$

(iii) Composition series of M ~ length k $\longrightarrow$ $k \leq \ell(M)$

By definition of $\ell(M)$ ~ $\ell(M) \leq k$

$\longrightarrow \ell(M) = k.$

$\longrightarrow$ All composition series have same length.

(iv). From (ii) a chain in M has length $\leq \ell(M)$

~ if it has length $\ell(M)$ then it cannot be extended into a longer chain.

~ it is a composition series

Combining w/ (iii) we get

a chain of length k is a composition series $\iff k = \ell(M)$

Consequence:

. A chain of length $k < \ell(M)$ is not a composition series

~ it can be extended to a chain of length $k+1$

~ By induction it can extend to a ch of lgth l for any $l = k+1, \dots, \ell(M)$.

$\longrightarrow$ the chain of length $\ell(M)$ is a composition series.

Prop. 6.8: Let M be an A-module. TFAE:
(i) M has a composition series
(ii) M satisfies a.c.c. and d.c.c.

Proof: $\Rightarrow$: $\rightsquigarrow$ all chains are bounded
$\rightsquigarrow$ all ascending / descending sequences are stationary.

$\Leftarrow$: M satisfies a.c.c. $\overset{6.1}{\Longleftrightarrow}$ maximal condition for $(\Sigma, \subseteq)$
$\Rightarrow$ M admits a maximal submodule M_1 Σ = submodules of M
$M \supsetneq M_1$

Set $\Sigma = \{$sub-modules of $M\}$, $(\Sigma, \subseteq)$ partially ordered set
(6.1): M satisfies a.c.c. $\Leftrightarrow$ every non-empty subset of Σ has a maximal elt
$\Rightarrow$ Every non-zero submodule M' admits a maximal sub-module $M'' \subsetneq M'$

Set $M_0 = M$, let M_1 be a maximal submodule $M_1 \subsetneq M_0$

Two possibilities: • $M_1 = 0$
• $M_1 \neq 0 \rightsquigarrow \exists$ max. sub-module $M_2 \subsetneq M_1$

Two possibl.: • $M_2 = 0$
• $M_2 \neq 0 \rightsquigarrow \exists$ max. sub-module $M_3 \subsetneq M_2$

Repeating this process we get in two potential states:
(a) The process stops & we get a sequence
$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$ where M_{j+1} is a max. module of M_j
$\Updownarrow$
we cannot insert a module between M_{j+1} and M_j
$\hookrightarrow$ we got a composition series.

(b) The process does not stop: we got a non-stationary sequence
$M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_l \supsetneq \cdots$
Not possible since M satisfies d.c.c.

Conclusion: M has a composition series.

Prop. 6.9: The length $\ell(M)$ is additive on finite-length A-modules, i.e., if $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then
$$\ell(M) = \ell(M') + \ell(M'').$$

Proof: $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ exact sequence M, M', M'' finite length.

$M' = M'_0 \supsetneq \cdots \supsetneq M'_n = 0$ composition series for M' $n = \ell(M')$,
$M'' = M''_0 \supsetneq \cdots \supsetneq M''_m = 0$ ______ M'' $m = \ell(M'')$.

Set:

- $M_j := \beta^{-1}(M''_j)$ $0 \le j \le m$ $\quad M_0 = \beta^{-1}(M''_0) = \beta^{-1}(M'') = M$
  $M_m = \beta^{-1}(M''_m) = \beta^{-1}(0) = \ker\beta$
  $M_i \supsetneq M_{i+1}$ $0 \le i \le m-1$ $\quad = \operatorname{im}\alpha = \alpha(M')$
- $M_{m+j} = \alpha(M'_j)$ $1 \le j \le n$ $\quad$. $M_{m+j} \supsetneq M_{m+j+1}$ $0 \le j \le n-1$ $\quad = \alpha(M'_0)$

We get a chain

$$M_0 = M \supsetneq \cdots \supsetneq \underset{\beta^{-1}(M''_i)}{M_i} \supsetneq \cdots \supsetneq \underset{\substack{\beta^{-1}(M''_0) \\ \alpha(M'_0)}}{M_m} \supsetneq \cdots \supsetneq \underset{\alpha(M'_{m+i})}{M_i} \supsetneq \cdots M_{m+n} = 0$$

- If N is an A-module s.t. $M_i \supsetneq N \supsetneq M_{i+1}$ with $0 \le i \le m-1$, then
  $\beta^{-1}(M''_i) \supsetneq N \supsetneq \beta^{-1}(M''_{i+1})$ $\quad \beta(\beta^{-1}(N'')) = N''$
  $\longrightarrow M''_i \supsetneq \beta(N) \supsetneq M''_{i+1}$ is not possible
  since $\{M''_i\}$ composition series
- If N is an N-module s.t. $\underset{\alpha^{-1}(M'_j)}{M_{m+j}} \supsetneq N \supsetneq \underset{\alpha^{-1}(M'_{j+1})}{M_{m+j+1}}$ $0 \le j \le n-1$:

  $\longrightarrow \alpha(\alpha^{-1}(M'_j)) \supsetneq \alpha^{-1}(N) \supsetneq \alpha(\alpha^{-1}(M'_{j+1}))$
  $M'_j \supsetneq \alpha^{-1}(N) \supsetneq M'_j$ not possible since $\{M'_j\}$ composition series.

$\longrightarrow \{M_i\}_{0 \le i \le n+m}$ composition series $\to \ell(M) = m+n = \ell(M') + \ell(M'')$.

Prop. 6.10: Let V be a vector space over a field k. TFAE

(i) $\dim V < \infty$

(ii) $\ell(V) < \infty$

(iii) V satisfies a.c.c.

(iv) V ——— d.c.c.

Moreover, if these conditions are satisfied, then $\ell(V) = \dim V$.

Proof: (i) $\Rightarrow$ (ii): Set $n = \dim V$ and let $e_1, \dots, e_n$ be a basis of V. Set

$V_i = \text{Span}\{e_{i+1}, \dots, e_n\}$, $0 \le i \le n-1$, $V_n = 0$.

Then $V = V_0 \supsetneq V_1 \supsetneq \dots \supsetneq V_n = 0$

V_{i+1} has codim. 1 in V_i so maximal subspace.

$\rightarrow \{V_i\}$ composition series so $\ell(V) = n = \dim V < \infty$

By 6.8 (ii) $\Rightarrow$ (iii) + (iv) so

(iii) $\Rightarrow$ (i) + (iv) $\Rightarrow$ (i): Assume $\dim V = \infty$

$\rightarrow \exists$ linearly independent sequence $\{e_i ; i \ge 1\}$;

Set $V_i = \text{Span}\{e_j ; j \ge i\}$ $\quad W_i = \text{Span}\{e_j ; j \le i\}$.

Then $V_i \supsetneq V_{i+1}$ $\forall i$ $\quad W_i \subsetneq W_{i+1}$ $\forall i$

$\rightarrow$ non-stationary descending sequence $\quad \rightarrow$ non-stationary ascending sequence.

$\rightarrow$ d.c.c. fails $\quad$ — a.c.c. fails.

Cor. 6.11: Let A be a ring s.t. $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$ where $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are max. ideals. Then:
A Noeth. $\iff$ A is Art.

Proof: Induction on n.

- $n=1$: $\mathfrak{m}_1 = 0 \leadsto 0$ maximal ideal
  $\leadsto$ the only ideals of A are 0 and A
  $\leadsto$ A is a Field $\leadsto$ [Remind §1]

- Assume the result is true for n: $\mathfrak{m}_1 \cdots \mathfrak{m}_n \mathfrak{m}_{n+1} = 0$ $\mathfrak{m}_i$ max. ideal
  Set $\mathfrak{a} = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. We have an exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

$\mathfrak{a} = \mathfrak{m}_1 \cdots \mathfrak{m}_n / 0 = \mathfrak{m}_1 \cdots \mathfrak{m}_n \, / \, \mathfrak{m}_1 \cdots \mathfrak{m}_{n+1}$ = vector space over the field $A/\mathfrak{m}_{n+1}$

6.10 $\leadsto$ $\mathfrak{a}$ Noetherian over $A/\mathfrak{m}_{n+1} \iff \mathfrak{a}$ Artinian over $A/\mathfrak{m}_{n+1}$

$\mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_n$ as ideals over A

M : A-submodule of $\mathfrak{a} \Rightarrow M$ ideal of A contained in $\mathfrak{a}$.
$A \times M \longrightarrow M \leadsto A/\mathfrak{m}_{n+1} \times M \longrightarrow M$
$(\mathfrak{m}_{n+1}\mathfrak{a} = 0 \longrightarrow \mathfrak{m}_{n+1} M = 0$ $\quad \Rightarrow M$ $A/\mathfrak{m}_{n+1}$ - module of A.

M $A/\mathfrak{m}_{n+1}$ - module $\Rightarrow M$ A-module

$\{A/\mathfrak{m}_{n+1}$ - submodules of $\mathfrak{a}\} = \{A$ - submodules of $\mathfrak{a}\} = \{$ideals of A contained in $\mathfrak{a}\}$
" vector subspaces over $k = A/\mathfrak{m}_{n+1}$

In any case $\{\mathfrak{a}$ Noeth. (resp. Art.) over $A/\mathfrak{m}_{n+1}\} \iff \{\mathfrak{a}$ Noeth. (resp. Art.) over $A\}$

$\longrightarrow$ $\mathfrak{a}$ Noeth. over A $\iff$ $\mathfrak{a}$ Art. over A

- let $\overline{m_1}, \dots, \overline{m_n}$ images of $m_1, \dots, m_n$ in $A/\mathfrak{a}$.
  - $\overline{m_i}$ ideal
  - $\overline{m_i} \subseteq \mathfrak{b} \subsetneq A/\mathfrak{a}$ $\mathfrak{b}$ ideal

$\to \pi^{-1}(\overline{m_i}) \subseteq \pi^{-1}(\mathfrak{b}) \subsetneq A$.

$x \in \pi^{-1}(\overline{m_i})$ $\exists y \in m_i$ s.t. $x - y \in \mathfrak{a} \subset m_i$

$\to x \in m_i$.

$m_i \subseteq \pi^{-1}(\mathfrak{b}) \subseteq A$ m_i max as $m_i = \pi^{-1}(\mathfrak{b})$

$\to \overline{m_i} = \pi(\pi^{-1}(\mathfrak{b})) = \mathfrak{b}$

$y \in \mathfrak{b} \to \exists x \in A$ s.t. $y = \pi(x)$ $\to \overline{m_i}$ maximal.

$x \in \pi^{-1}$

As the result is true for $n-1$: $\left\{ \begin{matrix} A/\mathfrak{a} \text{ Noeth.} \\ \text{over } A/\mathfrak{a} \end{matrix} \right\} \iff \left\{ \begin{matrix} A/\mathfrak{a} \text{ Art.} \\ \text{over } A/\mathfrak{a} \end{matrix} \right\}$

ideals of $A/\mathfrak{a} \iff$ A-submodules of $A/\mathfrak{a}$

ideals of $A/\mathfrak{a}$ $\leftarrow$ A-submodule M of $A/\mathfrak{a}$

$\mathfrak{a}M = 0$

$\to$ $A/\mathfrak{a}$ Noeth. over $A \iff A/\mathfrak{a}$ Art. over A.

Exact sequen $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$

(6.3) A Noeth. (over A) $\iff$ $\mathfrak{a}$ and $A/\mathfrak{a}$ are Noeth. over A

A Art. (over A) $\iff$ $\mathfrak{a}$ and $A/\mathfrak{a}$ are Art. over A

$\hookrightarrow$ A Noeth. $\iff$ A Art.