Commutative Algebra
Chapter 7: Noetherian Rings

Sichuan University, Fall 2020

Reminder from Chapter 6

Definition

A module M over a ring A is *Noetherian* if it satisfies on of the following equivalent conditions:

(i) *Ascending chain condition (a.c.c.):* Every ascending sequence of submodules $M_1 \subseteq M_2 \subseteq \cdots$ is sationnary.

(ii) *Maximal condition:* Every non-empty set of submodules of M has a maximal element.

Definition

A ring A is *Noetherian* if it is Noetherian as a module over itself, i.e., it satisfies a.c.c. and the maximal condition on ideals.

Reminder from Chapter 6

Examples

1. Any field k is Noetherian.
2. The ring $\mathbb{Z}$ is Noetherian.
3. Any principal ideal domain is Noetherian (this follows from Proposition 6.2).

Reminder from Chapter 6

Proposition (Proposition 6.2)

Let M be a module over A. TFAE:

(i) *M is Noetherian.*

(ii) *Every submodule of M is finitely generated.*

In particular, for $M = A$ we get:

Corollary

A ring A is Noetherian if and only if every ideal of A is finitely generated (as an A-module).

Reminder from Chapter 6

Proposition (Proposition 6.5)

Let A be a Noetherian ring and M a finitely generated A-module. Then M is Noetherian.

Proposition (Proposition 6.6)

Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. Then $A/\mathfrak{a}$ is a Noetherian ring.

Noetherian Rings

Proposition (Proposition 7.1)

Let $\phi : A \to B$ be a surjective ring homomorphism. If A is Noetherian, then so is B.

Proposition (Proposition 7.2)

Let $A \subseteq B$ be rings such that A is Noetherian and B is finitely generated as an A-module. Then B is Noetherian (as a ring).

Example

The ring of Gaussian integer $B = \mathbb{Z}[i]$ is Noetherian.

Reminder (see Proposition 3.11)

Let A be a ring and S a multiplicatively closed subset of A.

- If $\mathfrak{a}$ is an ideal of A such that $\mathfrak{a} \cap S = \emptyset$, then its extension in $S^{-1}A$ is $S^{-1}\mathfrak{a}$.
- If $\mathfrak{b}$ is an ideal of $S^{-1}A$, then its contraction in A is
$$\mathfrak{b}^c = \left\{x \in A; x/1 \in \mathfrak{b}\right\}.$$
- Every ideal $\mathfrak{b}$ of $S^{-1}A$ is an extended ideals, and hence $\mathfrak{b} = S^{-1}(\mathfrak{b}^c)$.

Noetherian Rings

Proposition (Proposition 7.3)

If A is a Noetherian ring and $S \subseteq A$ is closed under multiplication, then the fraction ring $S^{-1}A$ is Noetherian.

Corollary (Corollary 7.4)

If A is a Noetherian ring and $\mathfrak{p}$ is a prime ideal of A, then the local ring $A_{\mathfrak{p}}$ is Noetherian.

Noetherian Rings

Theorem (Hilbert's Basis Theorem; Theorem 7.5)

If A is a Noetherian ring, then the polynomial ring $A[x]$ is Noetherian as well.

Remark

If A is a Noetherian ring, then it can be also shown that the ring of formal power series $A[[x]]$ is Noetherian (see Carlson's notes; see also Corollary 10.27).

Noetherian Rings

Corollary (Corollary 7.6)

If A is a Noetherian ring, then the ring $A[x_1, \dots, x_n]$ is Noetherian.

Corollary (Corollary 7.7)

If A is Noetherian, then every finitely generated A-algebra is a Noetherian ring. In particular, every finitely generated ring and every finitely generated algebra over a field are Noetherian rings.

Noetherian Rings

Reminder (see Proposition 5.1)

Let $A \subseteq B$ be rings and $x \in B$. TFAE:

(i) x is integral over A.

(ii) $A[x]$ is a finitely generated A-module.

(iii) $A[x]$ is contained in a subring C of B such that C is a finitely generated A-module.

Reminder (Corollary 5.2)

Let $x_1, \ldots, x_n$ be elements of B that are integral over A. Then the ring $A[x_1, \ldots, x_n]$ is a finitely generated A-module.

Proposition (Proposition 7.8)

Let $A \subseteq B \subseteq C$ be rings. Suppose that A is Noetherian and C is finitely generated as an A-algebra. Assume further that one of the following two conditions holds:

(i) *C is finitely generated as a B-module.*

(ii) *C is integral over B.*

Then B is finitely generated as an A-algebra.

Remark

In this situation the condition (i) and (ii) are equivalent.

Proposition (Proposition 7.9)

Let k be a field and E a finitely generated k-algebra. If E is a field, then this is a finite algebraic extension of k.

Corollary (Weak Nullstellensatz; Corollary 7.10)

Let k be a field, A a finitely generated k-algebra, and $\mathfrak{m}$ a maximal ideal of A. Then the field $A/\mathfrak{m}$ is a finite algebraic extension of k. In particular, if k is algebraically closed, then $A/\mathfrak{m} \simeq k$.

Remark

The weak Nullstellensatz allows us to get the strong form of Hilbert's Nullstellensatz (see Problem 7.14 and Carlson's notes).

Primary Decomposition in Noetherian Rings

Definition

An ideal $\mathfrak{a}$ of a ring A is called *irreducible* if

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies \left(\mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c}\right).$$

Lemma (Lemma 7.11)

In A is a Noetherian ring A, every ideal is finite intersection of irreducible ideals.

Lemma (Lemma 7.12)

In a Noetherian ring A, every irreducible ideal is primary.

Primary Decomposition in Noetherian Rings

From the previous two lemmas we immediately obtain:

Theorem (Theorem 7.13)

In a Noetherian ring, every ideal has a primary decomposition.

Consequence

All the results of Chapter 4 apply to Noetherian rings.

Primary Decomposition in Noetherian Rings

Proposition (Proposition 7.14)

In a Noetherian ring A, every ideal contains a power of its radical.

Corollary (Corollary 7.15)

In a Noetherian ring the nilradical is nilpotent.

Primary Decomposition in Noetherian Rings

Corollary (Corollary 7.16)

Suppose that A is a Noetherian ring. Let $\mathfrak{m}$ be a maximal ideal of A and let $\mathfrak{q}$ be any ideal. TFAE:

(i) $\mathfrak{q}$ *is* $\mathfrak{m}$*-primary.*

(ii) $r(\mathfrak{q}) = \mathfrak{m}$.

(iii) $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ *for some* $n \geq 1$.

Reminder (Proposition 4.2)

If $\mathfrak{a}$ is an ideal in A whose radical $r(\mathfrak{a})$ is maximal, then $\mathfrak{a}$ is primary. In particular, every power of a maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-primary.

Primary Decomposition in Noetherian Rings

Reminder (1st Uniqueness Theorem; Theorem 4.5)

Let $\mathfrak{a}$ be a decomposable ideal and $\mathfrak{a} = \cap_{i=1}^{n} \mathfrak{q}_i$ a primary decomposition. Set $p_i = r(\mathfrak{q}_i)$, $i = 1, \ldots, n$. Then the $\mathfrak{p}_i$ are exactly the prime ideals of the form $r(\mathfrak{a} : x)$, $x \in A$. In particular, they don't depend on the primary decomposition of $\mathfrak{a}$.

Remarks

1. The proof of the uniqueness theorem shows that, for each i, if $x \in \cap_{j \neq i} \mathfrak{q}_j$, then $(\mathfrak{a} : x)$ is $\mathfrak{p}_i$-primary.
2. If $\mathfrak{a} = (0)$, then the $\mathfrak{p}_i$ are exactly the prime ideals of the form $\mathrm{Ann}(x)$, $x \in A$.

Proposition (Proposition 7.17)

Let A be a Noetherian ring and $\mathfrak{a} \subsetneq A$ an ideal of A. Then the prime ideals which belong to $\mathfrak{a}$ are exactly the prime ideals of the form $(\mathfrak{a} : x)$, $a \in A$.