Commutative Algebra
Chapter 6: Chain Conditions

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Chain Conditions

Reminder

- An order relation on a set Σ is a reflexive and transitive relation $\leq$ on Σ such that

$$x \leq y \text{ and } y \leq x \implies x = y.$$

- A set equipped with an order relation is called *partially ordered*.

Examples

Let Σ be the set of submodules of a module M over a ring A. Then $(\Sigma, \subseteq)$ and $(\Sigma, \supseteq)$ are both partially ordered sets.

Chain Conditions

Proposition (Proposition 6.1)

Let $(\Sigma, \leq)$ be a partially ordered set. TFAE:

(i) *Every increasing sequence $x_1 \leq x_2 \leq \cdots \leq x_j \leq \cdots$ is stationary (i.e., there is m such that $x_j = x_m$ for all $j \geq m$).*

(ii) *Every non-empty subset of Σ has a maximal element.*

Definition

Let M be a module over a ring A and denote by Σ be the set of its submodules.

- For the partially ordered set $(\Sigma, \subseteq)$ the condition (i) is called the *ascending chain condition* (a.c.c.) and the condition (ii) is called the *maximal condition*.
- For the partially ordered set $(\Sigma, \supseteq)$ the condition (i) is called the *descending chain condition* (d.c.c.) and the condition (ii) is called the *minimal condition*.

Chain Conditions

Definition

Let M be a module over A.

1. We say that M is *Noetherian* (after Emmy Noether) if M satisfies a.c.c. or the maximal condition.
2. We say that M is *Artinian* (after Emil Artin) if M satisfies d.c.c. or the minimal condition.

Remark

Let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M = 0$. Then:

- M is an $A/\mathfrak{a}$-module.
- M is Noetherian (resp., Artinian) over A if and only if it is Noetherian (resp., Artinian) over $A/\mathfrak{a}$.

Example

Any finite group is both Noetherian and Artinian (as a module over $\mathbb{Z}$).

Example (see Proposition 6.10)

Let V be a vector space over a field k. Then:

1. If $\dim V < \infty$, then V is both Noetherian and Artinian.
2. If $\dim V = \infty$, then V is neither Noetherian nor Artinian.

Example

The ring $\mathbb{Z}$ (as a module over itself) is Noetherian, but is not Artinian.

Example

Let p be a prime integer, and let G be the subgroup of $\mathbb{Q}/\mathbb{Z}$ defined by

$$G = \{x \in \mathbb{Q}/\mathbb{Z};\ p^n x = 0 \text{ for some } n \geq 1\}.$$

Then G is Artinian, but it is not Noetherian.

Example

Set $H = \{mp^n;\ m \in \mathbb{Z},\ n \in \mathbb{Z}_+\}$. Then H is a subgroup of $\mathbb{Z}$ and we have an exact sequence,

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow G \longrightarrow 0.$$

The group H is neither Noetherian nor Artinian.

Example

If k is a field, then the polynomial ring $k[x]$ (seen as a module over itself) is Noetherian, but not Artinian.

Example

If k is a field, then the polynomial ring $k[x_1, x_2, \ldots]$ with an infinite number of variables is neither Noetherian nor Artinian.

Example (see Theorem 8.5)

If A is a ring and is Artinian (as a module over itself), then it is Noetherian as well.

Proposition (Proposition 6.2)

Let M be a module over A. TFAE:

(i) *M is Noetherian.*

(ii) *Every submodule of M is finitely generated.*

Proposition (Proposition 6.3)

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules.

(i) *M is Noetherian if and only if M' and M'' are both Noetherian.*

(ii) *M is Artinian if and only if M' and M'' are both Artinian.*

Corollary (Corollary 6.4)

If $M_1, \ldots, M_n$ are Noetherian (resp., Artinian) modules over A, then so is their direct sum $\bigoplus_{i=1}^{n} M_i$.

Definition

A ring A is called *Noetherian* (resp., *Artinian*) when it is Noetherian (resp., Artinian) as a module over itself, i.e., it satisfies a.c.c. (resp., d.c.c.) on *ideals*.

Examples

1. Any field k is both Noetherian and Artinian (see slide 5).
2. The ring $\mathbb{Z}$ is Noetherian, but not Artinian (see slide 5).
3. Any principal ideal domain is Noetherian (this follows from Proposition 6.2).

Example

Let $A = k[x_1, x_2, \ldots]$ be the polynomial ring with infinite variables over a field k.

- A is not Noetherian (see slide 7).
- It is an integral domain, and so its fraction field $k = \mathrm{Frac}(A)$ is Noetherian.
- This shows that a subring of a Noetherian ring need not be Noetherian.

Example

Let X be a compact Hausforff space. Then the ring $C(X)$ of continuous functions on X is not Noetherian.

Chain Conditions

Proposition (Proposition 6.5)

Let A be a Noetherian (resp., Artinian) ring and M a finitely generated A-module. Then M is Noetherian (resp., Artinian).

Proposition (Proposition 6.6)

Let A be a Noetherian (resp., Artinian) ring and $\mathfrak{a}$ an ideal of A. Then $A/\mathfrak{a}$ is a Noetherian (resp., Artinian) ring.

Chain Conditions

Definition

Let M be a module over A.

- A *chain* of submodules of M is a strictly descending finite sequence of the form,

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n = 0.$$

- We call n the *length* of the chain.
- A *composition series* is a maximal chain, i.e., we cannot insert any module between M_i and M_{i+1}.

Remarks

- The composition series condition is equivalent to requiring each module M_i/M_{i+1} to be *simple*, i.e., it has no submodules but 0 and itself.
- If $M = A$ and the M_i are ideals, then this is equivalent to saying that M_i/M_{i+1} is a field.

Proposition (Proposition 6.7)

Suppose that M has a composition series of length n. Then:

(i) *Every composition series has length n.*

(ii) *Every chain in M can be extended to a composition series.*

Definition

- If M has a composition series, then we denote by $\ell(M)$ the length of any composition series.
- Otherwise we set $\ell(M) = \infty$.
- We call $\ell(M)$ the *length* of M.

Chain Conditions

Proposition (Proposition 6.8)

Let M be a module over A. TFAE:

(i) *M has a composition series.*

(ii) *M is both Noetherian and Artinian.*

Definition

A module M that has a composition series is called a *module of finite length*.

Remark (Jordan-Hölder Theorem; see Carlson + Gaillard)

Let M be an A-module of finite length n. If $(M_i)_{0\leq i\leq n}$ and $(M_i')_{0\leq i\leq n}$ are two composition series for M. Then the quotients M_{i-1}/M_i and M_{i-1}'/M_i' are isomorphic for $i=1,\ldots,n$,

Proposition (Proposition 6.9)

The length is an additive function on finite-length A-modules. That is, if $0\to M'\to M\to M''\to 0$ is any exact sequence of finite-length A-modules, then

$$\ell(M)=\ell(M')+\ell(M'').$$

Chain Conditions

Proposition (Proposition 6.10)

Let V be a vector space over a field k. TFAE:

1. *V has finite dimension.*
2. *V has finite length.*
3. *V is Noetherian.*
4. *V is Artinian.*

Furthermore, if these conditions hold, then $\ell(V) = \dim V$.

Corollary (Corollary 6.11)

Let A be a ring such that there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ so that $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$. Then A is Noetherian if and only if it is Artinian.