Differentiable Manifolds
§16. Lie Algebras

Sichuan University, Fall 2020

Tangent Space at the Identity of a Lie Group

Reminder (see Section 15)

Let G be a Lie group with unit e.

- Given any $g \in G$, the left-multiplication $\ell_g : G \rightarrow G$, $x \rightarrow gx$ is a diffeomorphism such that $\ell_g(e) = g$.
- Thus, the differential $(\ell_g)_{*,e} : T_eG \rightarrow T_gG$ is a linear isomorphism.

Consequence

Describing T_eG allows us to describe T_gG for every $g \in G$.

Tangent Space at the Identity of a Lie Group

Example (Tangent space of $\mathrm{GL}(n,\mathbb{R})$ at I)

$\mathrm{GL}(n,\mathbb{R})$ is an open subset of the vector space $\mathbb{R}^{n\times n}$. Thus,

$$T_I\,\mathrm{GL}(n,\mathbb{R}) = T_I\mathbb{R}^{n\times n} = \mathbb{R}^{n\times n}.$$

Consequence

For any Lie subgroup $G \subset \mathrm{GL}(n,\mathbb{R})$ the tangent space $T_I G$ is a linear subspace of $\mathbb{R}^{n\times n}$.

Tangent Space at the Identity of a Lie Group

Reminder (see Section 15)

- If $X \in \mathbb{R}^{n\times n}$, then

$$\det\left(e^{X}\right) = e^{\mathrm{tr}[X]}.$$

- The differential $\det_{*,I} : \mathbb{R}^{n\times n} \to \mathbb{R}$ is given by

$$\det_{*,I}(X) = \mathrm{tr}(X), \qquad X \in \mathbb{R}^{n\times n}.$$

Tangent Space at the Identity of a Lie Group

Proposition (Tangent Space Criterion)

Let G be an embedded Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ and V a subspace of $\mathbb{R}^{n\times n}$ such that

$$\dim V = \dim G \quad \textit{and} \quad e^X \in G\ \forall X \in V.$$

Then $T_I G = V$.

Proof.

- Let $X \in V$. Then $c(t) = e^{tX}$, $t \in \mathbb{R}$, is a smooth curve in $\mathrm{GL}(n,\mathbb{R})$ with values in G such that $c(0) = I$ and $c'(0) = X$.
- As G is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$, it follows that $c(t)$ is a smooth curve in G, and hence $X = c'(0) \in T_I G$.
- Thus, V is a subspace of $T_I G$. As $\dim V = \dim G = \dim T_I G$ it follows that $T_I G = V$.

The result is proved. □

Example (Tangent space of $\mathrm{SL}(n,\mathbb{R})$ at I; Example 16.2)

- Let $X \in \mathbb{R}^{n\times n}$. As $\det(e^X) = e^{\mathrm{tr}(X)}$, we have

$$e^X \in \mathrm{SL}(n,\mathbb{R}) \Longleftrightarrow \det\left(e^X\right) = e^{\mathrm{tr}(X)} = 1 \Longleftrightarrow \mathrm{tr}(X) = 0.$$

- Set $V = \{X \in \mathbb{R}^{n\times n}; \mathrm{tr}(X) = 0\}$. Then

$$e^X \in \mathrm{SL}(n,\mathbb{R})\, \forall X \in V \quad \text{and} \quad \dim V = n^2 - 1 = \dim \mathrm{SL}(n,\mathbb{R}).$$

Thus,

$$T_I\, \mathrm{SL}(n,\mathbb{R}) = V = \left\{X \in \mathbb{R}^{n\times n}; \mathrm{tr}(X) = 0\right\}.$$

Tangent Space at the Identity of a Lie Group

Example (Tangent space of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ at I; Example 16.4)

- Let K_n be the space of skew-symmetric $n \times n$ matrices, i.e.,
$$K_n = \left\{X \in \mathbb{R}^{n\times n};\ X^T = -X\right\}.$$
- If $X \in K_n$, then $(e^X)^T = e^{X^T} = e^{-X} = (e^X)^{-1}$, and hence
$$\left(e^X\right)^T e^X = \left(e^X\right)^{-1} e^X = I.$$
Thus, $e^X \in \mathrm{O}(n)$ for all $X \in K_n$.
- As $\dim K_n = \frac{1}{2}n(n-1) = \dim \mathrm{O}(n)$, we deduce that
$$T_I\, \mathrm{O}(n) = K_n.$$
- As $\mathrm{SO}(n)$ is an open set in $\mathrm{O}(n)$, we have
$$T_I\, \mathrm{SO}(n) = T_I\, \mathrm{O}(n) = K_n.$$

Tangent Space at the Identity of a Lie Group

Example (Tangent space of $U(n)$ at I; Problem 16.2)

- Let L_n be the space of skew-Hermitian $n \times n$ matrices, i.e.,
$$L_n = \{X \in \mathbb{C}^{n\times n};\ X^* = -X\}.$$
- If $X \in L_n$, then $(e^X)^* = e^{X^*} = e^{-X} = (e^X)^{-1}$, and hence
$$(e^X)^* e^X = (e^X)^{-1} e^X = I.$$
Thus, $e^X \in U(n)$ for all $X \in L_n$.
- As $\dim L_n = n^2 = \dim U(n)$ (see Problem 16.1) we get
$$T_I\, U(n) = L_n.$$

Tangent Space at the Identity of a Lie Group

Example (Tangent space of SU(n) at I)

- Define
$$L_n^0 = \left\{X \in L_n;\ \operatorname{tr}(X) = 0\right\}.$$
- If $X \in L_n^0$, then $e^X \in \mathrm{U}(n)$, and
$$\det\left(e^X\right) = e^{\operatorname{tr}(X)} = e^0 = 1.$$
Thus, $e^X \in \mathrm{SU}(n)$ for all $X \in L_n^0$.
- As $\dim L_n^0 = n^2 - 2 = \dim \mathrm{SU}(n)$, we deduce that
$$T_I\,\mathrm{SU}(n) = L_n^0 = \left\{X \in \mathbb{C}^{n\times n}; X^* = -X,\ \operatorname{tr}(X) = 0\right\}.$$

Left-Invariant Vector Fields on a Lie Group

Definition

A vector field X on a Lie group G is called *left-invariant* if

$$(\ell_g)_* X = X \qquad \forall g \in G.$$

We denote by $L(G)$ the space of left-invariant vector fields on G.

Remark

Let X be a vector field on G. Given any $g \in G$, we have

$$\big[(\ell_g)_* X\big]_h = (\ell_g)_{*,g^{-1}h}\left(X_{g^{-1}h}\right), \qquad h \in G.$$

Thus, X is left-invariant if and only if

$$(\ell_g)_{*,g^{-1}h}(X_{g^{-1}h}) = X_h \qquad \forall g, h \in G.$$

Equivalently,

$$(\ell_g)_{*,h}(X_h) = X_{gh} \qquad \forall g, h \in G.$$

Remark

Let X be a left-invariant vector field. Then

$$(\ell_g)_{*,h}(X_h) = X_{gh} \qquad \forall g, h \in G.$$

In particular, for $h = e$ we get

$$X_g = \left(\ell_g\right)_{*,e}(X_e) \qquad \forall g \in G.$$

Thus, X is uniquely determined by X_e.

Left-Invariant Vector Fields on a Lie Group

Definition

For any tangent vector $A \in T_e G$, we let $\tilde{A}$ be the vector field on G defined by

$$\tilde{A}_g = (\ell_g)_{*,e}(A) \qquad \forall g \in G.$$

Proposition

Let $A \in T_e G$. Then $\tilde{A}$ is a left-invariant vector field on G.

Proof.

Let $g, h \in G$. Then by the chain rule we have

$$(\ell_g)_{*,h}(\tilde{A}_h) = (\ell_g)_{*,h} \circ (\ell_h)_{*,e}(A) = (\ell_{gh})_{*,e}(A) = \tilde{A}_{gh}.$$

It follows that $\tilde{A}$ is left-invariant (*cf.* slide 10). □

Left-Invariant Vector Fields on a Lie Group

Remarks

- We call $\tilde{A}$ the *left-invariant vector field generated by A*.
- As $\ell_e = \mathbb{1}_M$, and hence $(\ell_e)_* = \mathbb{1}_{T_eG}$, we have
$$\tilde{A}_e = (\ell_e)_{*,e}(A) = \mathbb{1}_{T_eG}(A) = A.$$
- Conversely, if $A = X_e$, where X is a left-invariant vector field, then
$$\tilde{A}_g = (\ell_g)_{*,e}(X_e) = X_g.$$
That is, $\tilde{A} = X$.

Therefore, we obtain:

Proposition

The map $X \to X_e$ is a linear isomorphism from $L(G)$ onto T_eG with inverse $A \to \tilde{A}$.

Reminder (see Problem 8.2)

- Given any $p \in \mathbb{R}^n$, we have $T_p\mathbb{R}^n = \mathbb{R}^n$ under the identification,

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \longleftrightarrow (a^1, \ldots, a^n).$$

- If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then under the identifications $T_p\mathbb{R}^n = \mathbb{R}^n$ and $T_{F(p)}\mathbb{R}^m = \mathbb{R}^m$, the differential $L_{*,p}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- In fact (see Problem 8.2), we have

$$L_{*,p} = L \qquad \forall p \in \mathbb{R}^n.$$

Left-Invariant Vector Fields on a Lie Group

Example (Left-invariant vector fields on $\mathrm{GL}(n,\mathbb{R})$)

- If $g \in \mathrm{GL}(n,\mathbb{R})$, then $T_g \,\mathrm{GL}(n,\mathbb{R}) = T_g\mathbb{R}^{n\times n} = \mathbb{R}^{n\times n}$ under the identification,
$$\sum_{i,j} a_{ji} \frac{\partial}{\partial x_{ij}}\bigg|_g \longleftrightarrow [a_{ij}].$$
- If $g \in \mathrm{GL}(n,\mathbb{R})$, then the left-multiplication $\ell_g : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $A \to gA$ is a linear map.
- Under the identifications $T_I \,\mathrm{GL}(n,\mathbb{R}) = T_g \,\mathrm{GL}(n,\mathbb{R}) = \mathbb{R}^{n\times n}$ we then have
$$(\ell_g)_{*,e} = \ell_g \qquad \forall g \in G.$$
- Thus, if $A = [a_{ij}] \in \mathbb{R}^{n\times n} = T_I \,\mathrm{GL}(n,\mathbb{R})$, then
$$\tilde{A}_g = (\ell_g)_{*,e}\bigg(\sum_{i,j} a_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_e\bigg) = \sum_{i,j} (gA)_{ij} \frac{\partial}{\partial x_{ij}}\bigg|_g.$$

Example (continued)

- If we use the coordinates $g = (x_{ijj})$, then $(gA)_{ij} = \sum_k x_{ik} a_{kj}$, we get
$$\tilde{A}_g = \sum_{i,j} \left(\sum_k x_{ik} a_{kj} \right) \frac{\partial}{\partial x_{ij}} \bigg|_g .$$
- In other words, the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by A is just
$$A = \sum_{i,j,k} x_{ik} a_{kj} \frac{\partial}{\partial x_{ij}} .$$
All the left-invariant vector fields on $\mathrm{GL}(n, \mathbb{R})$ are of this form.

Reminder (Proposition 8.17 and Proposition 14.3)

- A vector field X on a manifold M is smooth if and only if $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$.
- Let $X_p \in T_pM$ and $c : (-\epsilon, \epsilon) \to M$ a smooth curve such that $c(0) = p$ and $c'(0) = X$. Then

$$X_p f = \frac{d}{dt}\bigg|_{t=0} f \circ c(t) \qquad \forall f \in C_p^\infty(M).$$

- If $(U, x^1, \ldots, x^n)$ is a chart for M and $f \in C^\infty(M)$, then the partial derivatives $\partial f/\partial x^1, \ldots, \partial f/\partial x^n$ are smooth functions on U (see §§6.6).

Left-Invariant Vector Fields on a Lie Group

Proposition (Proposition 16.8)

Every left-invariant vector field X on G is smooth.

The following result is proved in Lee's book:

Proposition

Every left-invariant vector field on G is complete, i.e., its flow is defined on all $\mathbb{R} \times M$.

Reminder (Lie algebras; see Section 14)

A *Lie algebra over a field* $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ together with an alternating bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ satisfying Jacobi's identity,

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$

Definition

A *Lie subalgebra* of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is a vector subspace $\mathfrak{h}$ which closed under the Lie bracket $[\cdot,\cdot]$, i.e.,

$$[X,Y] \in \mathfrak{h} \qquad \forall X, Y \in \mathfrak{h}.$$

Remark

Any Lie subalgebra is a Lie algebra with respect to the original bracket $[\cdot,\cdot]$.

The Lie algebra of a Lie group

Definition

Let $\mathfrak{h}$ and $\mathfrak{g}$ be Lie algebras.

- A *Lie algebra homomorphism* $f : \mathfrak{h} \to \mathfrak{g}$ is a linear map such that
$$f\big([X, Y]\big) = \big[f(X), f(Y)\big] \qquad \forall X, Y \in \mathfrak{h}.$$
- A *Lie algebra isomorphism* $f : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism which is a bijection.

Remark

If $f : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra isomorphism, then $f^{-1} : \mathfrak{g} \to \mathfrak{h}$ is automatically a Lie algebra homomorphism.

Reminder (see Section 14)

Let M be a smooth manifold.

- The space $\mathscr{X}(M)$ of smooth vector fields is a Lie algebra under the Lie bracket of vector fields.
- If $F : M \to N$ is a diffeomorphism and X and Y are smooth vector fields on M, then

$$F_*([X, Y]) = [F_*X, F_*Y].$$

The Lie Algebra of a Lie Group

Proposition (see Proposition 16.9)

If X and Y are left-invariant vector fields, then their Lie bracket $[X, Y]$ is left-invariant as well.

Proof.

Let $g \in G$. As $\ell_g : G \to G$ is a diffeomorphism, we have

$$(\ell_g)_*([X, Y]) = \left[(\ell_g)_* X, (\ell_g)_* Y\right] = [X, Y].$$

Thus, the vector field $[X, Y]$ is left-invariant. □

Corollary

The space $L(G)$ of left-invariant vector fields on G is a Lie subalgebra of $\mathscr{X}(G)$. In particular, this is a Lie algebra under the Lie bracket of vector fields.

The Lie Algebra of a Lie Group

Remarks

- We know that $A \to \tilde{A}$ is a vector space isomorphism from $T_e G$ onto $L(G)$.
- We can use this isomorphism to pullback the Lie algebra structure of $L(G)$ to $T_e G$.

Definition

If $A, B \in T_e G$, then their Lie bracket $[A, B] \in T_e G$ is defined by

$$[A, B] = \left[\tilde{A}, \tilde{B}\right]_e.$$

The Lie Algebra of a Lie Group

Proposition (see Proposition 16.10)

$(T_eG, [\cdot,\cdot])$ *is a Lie algebra which is isomorphic to* $L(G)$ *as a Lie algebra. In particular,*

$$\widetilde{[A,B]} = [\tilde{A}, \tilde{B}] \qquad \forall A, B \in T_eG.$$

Definition

$(T_eG, [\cdot,\cdot])$ is called the *Lie algebra of G* and is often denoted $\mathfrak{g}$.

Remarks

- For instance, the Lie algebras of $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $SO(n)$, $U(n)$, $SU(n)$ are denoted $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{so}(n)$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, etc..
- Some authors defines the Lie algebra of G to be the Lie algebra $L(G)$.

The Lie Bracket of $\mathfrak{gl}(n, \mathbb{R})$

Proposition (Proposition 16.4; see Tu's book)

Under the identification $\mathfrak{gl}(n, \mathbb{R}) = T_I \,\mathrm{GL}(n, \mathbb{R}) \simeq \mathbb{R}^{n \times n}$ *the Lie bracket of* $\mathfrak{gl}(n, \mathbb{R})$ *is given by*

$$[A, B] = AB - BA, \qquad A, B \in \mathbb{R}^{n \times n}.$$

Reminder (Problem 14.2)

If $X = \sum a^i(x)\partial/\partial x^i$ and $X = \sum b^i(x)\partial/\partial x^i$ are smooth vector fields on $\mathbb{R}^n$, then

$$[X, Y] = \sum_i c^i \frac{\partial}{\partial x^i}, \quad \text{where } c^i = \sum_j \left(a^j \frac{\partial b^i}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \right).$$

Pushforward of Left-Invariant Vector Fields

Reminder (see Section 14)

- Let $F : N \to M$ be a smooth map. A smooth vector field X on N and a smooth vector field $\tilde{X}$ on M are F-related when
$$F_{*,p}(X_p) = \tilde{X}_{F(p)} \qquad \forall p \in N.$$
- If F is a diffeomorphism, then F_*X is unique vector field on M which is F-related to X.
- In general we cannot define the pushforward F_*X if F is not a diffeomorphism.

Pushforward of Left-Invariant Vector Fields

Definition

Let $F : H \to G$ be a Lie group homomorphism and X a left-invariant vector field on H. The *pushforward* $F_* X$ is the left-invariant vector field on G generated by $F_{*,e}(X_e)$. That is,

$$F_* X = F_{*,e}(X_e)^{\sim}$$

Proposition (Proposition 16.12)

Let $F : H \to G$ be a Lie group homomorphism and X a left-invariant vector field on H. Then $F_* X$ is F-related to X.

Pushforward of Left-Invariant Vector Fields

Proof of Proposition 16.12.

- As F_*X is the left-invariant vector field generated by $F_{*,e}(X_e)$,

$$(F_*X)_g = (\ell_g)_{*,e}\big(F_{*,e}(X_e)\big) \qquad \forall g \in G.$$

- Here $F(e) = e$, so the chain rule gives

$$(F_*X)_g = (\ell_g)_{*,F(e)} \circ F_{*,e}(X_e) = (\ell_g \circ F)_{*,e}(X_e).$$

- As F is a Lie group homomorphism, $\ell_{F(h)} \circ F = F \circ \ell_h$. Thus, for $g = F(h)$ we get

$$(F_*X)_{F(h)} = \big(F \circ \ell_h\big)_{*,e} = F_{*,\ell_h(e)} \circ (\ell_h)_{*,e}(X_e).$$

- As X is left-invariant, $(\ell_h)_{*,e}(X_e) = X_h$. Thus,

$$(F_*X)_{F(h)} = F_{*,h}(X_h) \qquad \forall h \in H.$$

This shows that F_*X is F-related to X. □

Pushforward of Left-Invariant Vector Fields

Remark

F_*X is the unique left-invariant vector field on G which is F-related to X.

Proof.

Let $\tilde{X}$ be a left-invariant vector field on G which is F-related to X.

- As $\tilde{X}$ and F_*X are left-invariant, their uniquely determined by $\tilde{X}_e$ and $F_*(X)_e = F_{*,e}(X_e)$. Thus, to show that $\tilde{X} = F_*X$ we only need to show that $\tilde{X}_e = F_{*,e}(X_e)$.
- As $\tilde{X}$ is F-related to X, we have $\tilde{X}_e = F_{*,e}(X_e)$, and hence $\tilde{X} = F_*X$.

This prove the result. □

The Differential as a Lie Algebra Homomorphism

Reminder (Proposition 14.17)

Suppose that $F : N \to M$ is a smooth map. Let X and Y be smooth vector fields on N which are F-related to smooth vector fields $\tilde{X}$ and $\tilde{Y}$ on M. Then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.

The Differential as a Lie Algebra Homomorphism

Proposition

Let $F : H \to G$ be a Lie group homomorphism, and let X and Y be left-invariant vector fields on H. Then

$$F_*\big([X, Y]\big) = \big[F_*X, F_*Y\big].$$

Proof.

- As F_*X and F_*Y are F-related to X and Y, their Lie bracket $[F_*X, F_*Y]$ is F-related to $[X, Y]$.
- As F_*X and F_*Y are left-invariant, $[F_*X, F_*Y]$ is left-invariant.
- By the previous slide $F_*[X, Y]$ is the unique left-invariant vector field on G which is F-related to $[X, Y]$.
- It follows that $[F_*X, F_*Y] = F_*([X, Y])$.

The proof is complete. □

The Differential as a Lie Algebra Homomorphism

Corollary

Let $F : H \to G$ be a Lie group homomorphism. Then the pushforward of left-invariant vector field gives rise to a Lie algebra homomorphism,

$$F_* : L(H) \longrightarrow L(G), \qquad X \longrightarrow F_* X.$$

Corollary (Proposition 16.14)

If $F : H \to G$ is a Lie group homomorphism, then its differential at the identity is a Lie algebra homomorphism,

$$F_{*,e} : T_e H \longrightarrow T_e G, \qquad F_{*,e}\big([A, B]\big) = \big[F_{*,e} A, F_{*,e} B\big].$$

Proof of Proposition 16.14.

- We have a commutative diagram,

$$\begin{array}{ccc} L(H) & \xrightarrow{F_*} & L(G) \\ \downarrow \wr & & \downarrow \wr \\ T_eH & \xrightarrow{F_{*,e}} & T_eG. \end{array}$$

- The upper horizontal arrow is a Lie algebra homomorphism.
- The vertical arrows are Lie algebra isomorphisms.
- Therefore, the lower horizontal arrow is a Lie algebra homomorphism.

The proof is complete. □

The Differential as a Lie Algebra Homomorphism

Reminder (see Section 15)

A subgroup H of a Lie group G is called a *Lie subgroup* if

- H is an immersed submanifold in G.
- The multiplication and inversion maps of H are smooth.

Remark

Let H be a Lie subgroup of a Lie group G.

- As H is an immersed submanifold, the inclusion $\iota : H \rightarrow G$ is an immersion.
- Thus, the differential $\iota_{*,e} : T_eH \rightarrow T_eG$ is injective.
- This allows us to identify T_eH with a subspace of T_eG.

The Differential as a Lie Algebra Homomorphism

Proposition

Let H be a Lie subgroup of a Lie group G. Then the Lie bracket of its Lie algebra T_eH agrees with the Lie bracket of T_eG on its domain.

Proof.

- The inclusion $\iota : H \to G$ is a Lie group homomorphism, since it is a smooth map and a group homomorphism.
- Thus, the differential $\iota_{*,e} : T_eH \to T_eG$ is a Lie group homomorphism.
- This implies that the Lie bracket of its Lie algebra T_eH agrees with the Lie bracket of T_eG.

The result is proved. □

The Differential as a Lie Algebra Homomorphism

Corollary

Let H be a Lie subgroup of a Lie group G. Let $\mathfrak{g} = T_eG$ be the Lie algebra of G. Then the Lie algebra $\mathfrak{h} = T_eH$ of H is a Lie subalgebra of $\mathfrak{g}$.

Remarks (see Tu's book)

- Conversely, it can be shown that every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the Lie algebra of a unique connected Lie subgroup of G.
- This gives a one-to-one correspondence between Lie subalgebras of $\mathfrak{g}$ and (connected) Lie subgroups H of G.
- In particular, under this correspondence a Lie subalgebra may correspond to a non-embedded Lie subgroup.

Example

- The Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n \times n}$ equipped with the matrix Lie bracket,

$$[A, B] = AB - BA, \qquad A, B \in \mathbb{R}^{n \times n}.$$

- The following are Lie subalgebras of $\mathfrak{gl}(n, \mathbb{R})$:

$$\mathfrak{sl}(n, \mathbb{R}) = \left\{A \in \mathbb{R}^{n \times n};\ \mathrm{tr}(A) = 0\right\},$$
$$\mathfrak{o}(n) = \mathfrak{so}(n) = \left\{A \in \mathbb{R}^{n \times n};\ A^T = -A\right\}.$$

  There are the respective Lie algebras of $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(n)$, and $\mathrm{SO}(n)$.

The Differential as a Lie Algebra Homomorphism

Example

- The Lie algebra of $\mathrm{GL}(n,\mathbb{C})$ is $\mathfrak{gl}(n,\mathbb{C}) = \mathbb{C}^{n\times n}$ equipped with the matrix Lie bracket.
- The following are Lie subalgebras of $\mathfrak{gl}(n,\mathbb{C})$:

$$\mathfrak{sl}(n,\mathbb{C}) = \left\{A \in \mathbb{C}^{n\times n};\ \mathrm{tr}(A) = 0\right\},$$
$$\mathfrak{u}(n) = \left\{A \in \mathbb{R}^{n\times n};\ A^* = -A\right\},$$
$$\mathfrak{su}(n) = \left\{A \in \mathbb{R}^{n\times n};\ A^* = -A,\ \mathrm{tr}(A) = 0\right\}.$$

  There are the respective Lie algebras of $\mathrm{SL}(n,\mathbb{C})$, $U(n)$, and $\mathrm{SU}(n)$.