

Differentiable Manifolds

§12. The Tangent Bundle

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The Tangent Bundle as a Manifold

Objective

Let M be smooth manifold of dimension n . We would like to bundle together all the tangent spaces $T_p M$ so as to get a smooth manifold, called the *tangent bundle*.

Definition

As a set, the *tangent bundle* of M is the disjoint union,

$$TM := \bigsqcup_{p \in M} T_p M = \{(p, v); p \in M, v \in T_p M\}.$$

Remarks

- ① For $p \in M$ we identify the subset $\{p\} \times T_p M$ with the tangent space $T_p M$. This allows us to see $T_p M$ as a subset of TM .
- ② In particular, we write an element of TM either as (p, v) with $p \in M$ and $v \in T_p M$, or simply as v .

The Tangent Bundle as a Manifold

Remark

Let U be an open set in M . If $p \in U$, then $T_p U = T_p M$. Thus,

$$TU = \bigsqcup_{p \in U} T_p U = \bigsqcup_{p \in U} T_p M.$$

Definition

The *canonical map* $\pi : TM \rightarrow M$ is defined by

$$\pi((p, v)) = v, \quad p \in M, \quad v \in T_p M.$$

Remarks

- ① The map $\pi : TM \rightarrow M$ is onto.
- ② If $p \in M$, then $\pi^{-1}(p) = T_p M$.

The Tangent Bundle as a Manifold

Example

Let U be an open in \mathbb{R}^n . If $p \in U$, then $T_p U = T_p \mathbb{R}^n = \mathbb{R}^n$. Recall that, if (r^1, \dots, r^n) are the standard coordinates on \mathbb{R}^n , then we identify

$$T_p \mathbb{R}^n \ni v = \sum v^i \frac{\partial}{\partial r^i} \Big|_p \longleftrightarrow \langle v^1, \dots, v^n \rangle \in \mathbb{R}^n.$$

Thus, the pair (p, v) is naturally identified with (p, v^1, \dots, v^n) . Therefore, we have

$$TU = \bigsqcup_{p \in U} T_p U = \bigsqcup_{p \in U} \mathbb{R}^n = U \times \mathbb{R}^n.$$

The Tangent Bundle as a Manifold

Aim

Equip TM with a smooth structure. That is, we need to:

- ① Define a topology on TM .
- ② Construct a C^∞ -atlas for TM .

The Tangent Bundle as a Manifold

Facts

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart in M .

- Here $\phi(U)$ is an open in \mathbb{R}^n , and so $T(\phi(U)) = \phi(U) \times \mathbb{R}^n$.
- For every $p \in U$, the differential $\phi_{*,p}$ is an isomorphism from $T_p M = T_p U$ onto $T_{\phi(p)}(\phi(U)) = \mathbb{R}^n$.
- Therefore, we define a map $\tilde{\phi} : TU \rightarrow \phi(U) \times \mathbb{R}^n$ by

$$\tilde{\phi}(p, v) = (\phi(p), \phi_{*,p}v), \quad p \in U, v \in T_p U.$$

- This is a bijection with inverse $(x, v) \rightarrow (\phi^{-1}(x), \phi_{*,\phi^{-1}(x)}^{-1}v)$.
- This allows us to define a topology on TU by pulling back the topology of $\phi(U) \times \mathbb{R}^n$:

$$W \subset TU \text{ is open} \iff \tilde{\phi}(W) \text{ is open in } \phi(U) \times \mathbb{R}^n.$$

- With respect to this topology $\tilde{\phi}$ is a homeomorphism.

The Tangent Bundle as a Manifold

Remark

- If $p \in U$, then $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is a basis of $T_p M$.
- The differential $\phi_{*,p} : T_p U \rightarrow T_{\phi(p)} V = \mathbb{R}^n$ maps $\frac{\partial}{\partial x^i} \Big|_p$ to $\frac{\partial}{\partial r^i} \Big|_{\phi(p)}$. Thus,

$$\sum v^i \frac{\partial}{\partial x^i} \Big|_p \xrightarrow{\phi_*} \sum v^i \frac{\partial}{\partial r^i} \Big|_{\phi(p)} \longleftrightarrow \langle v^1, \dots, v^n \rangle \in \mathbb{R}^n.$$

- If $\phi(p) = (x^1(p), \dots, x^n(p))$ and $v = \sum v^i \frac{\partial}{\partial x^i} \Big|_p \in T_p M$, then

$$\tilde{\phi}(p, v) = (\phi(p), \phi_{*,p} v) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n).$$

In particular, this defines coordinates on TU .

The Tangent Bundle as a Manifold

Facts

- Let $(V, \psi) = (V, y^1, \dots, y^n)$ be another chart of M such that $U \cap V \neq \emptyset$. Define $\tilde{\psi} : TV \rightarrow \psi(V) \times \mathbb{R}^n$ by

$$\tilde{\psi}(p, v) = (\psi(p), \psi_{*,p}v), \quad p \in V, v \in T_p M.$$

- On $T(U \cap V) = (TU) \cap (TV)$ we have two topologies induced by the respective topologies of TU and TV .
- On $\tilde{\phi}(TU \cap TV) = \phi(U \cap V) \times \mathbb{R}^n$ we have

$$\tilde{\psi} \circ \tilde{\phi}^{-1}(r, v) = (\psi \circ \phi^{-1}(r), \psi_* \circ \phi_*^{-1}v) = (\psi \circ \phi^{-1}(r), (\psi \circ \phi^{-1})_{*,r}v).$$

- Here $(\psi \circ \phi^{-1})_{*,r}$ is the multiplication by the Jacobian matrix $J_{\psi \circ \phi^{-1}}(r) = [\partial(y^j \circ \phi^{-1}) / \partial r^i(r)]$ whose entries depends smoothly on r .
- Therefore, $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \rightarrow \psi(U \cap V) \times \mathbb{R}^n$ is smooth map. Its inverse map $\tilde{\phi} \circ \tilde{\psi}^{-1}$ is smooth as well, and so $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is a diffeomorphism.

The Tangent Bundle as a Manifold

Facts (Continued)

- $T(U \cap V)$ is open in TU and in TV .
- As $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \rightarrow \psi(U \cap V) \times \mathbb{R}^n$ is a diffeomorphism, this is a homeomorphism.
- If $W \subset T(U \cap V)$, then

$$\begin{aligned} W \text{ open in } TU &\iff \tilde{\phi}(W) \text{ open in } \phi(U) \times \mathbb{R}^n \\ &\iff \tilde{\psi} \circ \tilde{\phi}^{-1}[\tilde{\phi}(W)] \text{ open in } \psi(U) \times \mathbb{R}^n \\ &\iff \tilde{\psi}(W) \text{ open in } \psi(U) \times \mathbb{R}^n \\ &\iff W \text{ open in } TV \end{aligned}$$

Thus, TU and TV induce the same topology on $T(U \cap V)$.

- It follows that, if W is open in TU and X is open in TV , then $W \cap X$ is open in $T(U \cap V)$.

The Tangent Bundle as a Manifold

Summary

- (a) If (U, ϕ) is a chart for M , then $\tilde{\phi} : TU \rightarrow \phi(U) \times \mathbb{R}^n$ allows us to define a topology on TU by pulling back the topology of $\phi(U) \times \mathbb{R}^n$. This map then becomes a homeomorphism.
- (b) If (V, ψ) is another chart for M , then the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \rightarrow \psi(U \cap V) \times \mathbb{R}^n$ is a diffeomorphism.
- (c) TU and TV induce the same topology on $U \cap V$. In particular, if W is open in TU and X is open in TV , then $W \cap X$ is open in $T(U \cap V)$.

In particular, (b) would allow us to get a C^∞ atlas for TM provided we can define a topology on TM by patching together the TU -topologies.

The Tangent Bundle as a Manifold

Reminder (Topological Bases; see Appendix A)

Let X be a topological space. A *basis for the topology of X* is a collection \mathcal{B} of open sets such that, for every open $U \subset X$ and every $p \in U$, there is an open set $V \in \mathcal{B}$ such that $p \in V \subset U$.

Remark

If \mathcal{B} is a basis for the topology of X , then every open set is the union of open sets in \mathcal{B} . We say that \mathcal{B} generates the topology of X .

The Tangent Bundle as a Manifold

Proposition (Proposition A.8)

Let X be a set and \mathcal{B} a collection of subsets such that:

- (i) $X = \bigcup_{V \in \mathcal{B}} V$.
- (ii) If $V_1, V_2 \in \mathcal{B}$ and $p \in V_1 \cap V_2$, then there is $W \in \mathcal{B}$ such that $p \in W \subset V_1 \cap V_2$.

Then:

- 1 \mathcal{B} is a basis for a unique topology on X .
- 2 The open sets for this topology consists of unions of sets in \mathcal{B} .

Remark

The condition (ii) holds automatically if \mathcal{B} is closed under finite intersection.

The Tangent Bundle as a Manifold

Facts

Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas of M (which defines its smooth structure). Define

$$\mathcal{B} = \bigcup_{\alpha} \{W; W \text{ is an open in } TU_\alpha\}.$$

Note that $TU_\alpha \in \mathcal{B}$.

- As $\cup U_\alpha = M$, we have

$$\bigcup_{\alpha} TU_\alpha = \bigsqcup_{p \in \cup U_\alpha} T_p M = \bigsqcup_{p \in \cup M} T_p M = TM.$$

- If W_α is an open in TU_α and W_β is an open in TU_β , then $W_1 \cap W_2$ is open in $T(U_\alpha \cap U_\beta)$, and hence is contained in \mathcal{B} .

It follows that \mathcal{B} satisfies the conditions (i) and (ii) of Proposition A.8, and so it's a basis for a unique topology on TM .

The Tangent Bundle as a Manifold

Definition

The topology of TM is the topology generated by \mathcal{B} . The open sets are unions of sets in \mathcal{B} .

Remark

Each TU_α is open in TM , since it is contained in \mathcal{B} .

Proposition (Proposition 12.4)

As a topological space TM is Hausdorff.

Remark

Each open TU_α is Hausdorff since it is homeomorphic to the open set $\phi_\alpha(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$. This can be used to show that TM is Hausdorff.

The Tangent Bundle as a Manifold

Proposition (Proposition 12.3)

The topology of TM is second countable.

Remarks

- It can be shown that the topology of M admits a countable basis $\{U_i\}_{i \in I}$ consisting of domains of charts (cf. Lemma 12.2 of Tu's book).
- Each TU_i is second countable since it is homeomorphic to an open in $\mathbb{R}^n \times \mathbb{R}^n$.
- If for each $i \in I$, we let $\{W_{i,j}\}_{j \in \mathbb{N}}$ be a countable basis for the topology of TU_i , then $\{W_{i,j}; i \in I, j \in \mathbb{N}\}$ is a countable basis for the topology of TM (see Tu's book).

The Tangent Bundle as a Manifold

Facts

- Each TU_α is an open in TM .
- We know that $TM = \cup_\alpha TU_\alpha$.
- The local trivializations $\tilde{\phi}_\alpha : TU_\alpha \rightarrow \phi_\alpha(U_\alpha) \times \mathbb{R}^n$ are homeomorphisms onto open sets in $\mathbb{R}^n \times \mathbb{R}^n$.
- All the transition maps $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ are smooth.

Proposition

The collection $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas for TM , and hence TM is a smooth manifold of dimension $2n$.

Remark

If $\{(V_\beta, \psi_\beta)\}$ is any C^∞ -atlas for M , then we also get a C^∞ atlas $\{(TV_\beta, \tilde{\psi}_\beta)\}$ for TM . It is compatible with the atlas $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$, and so it defines the same smooth structure.

The Tangent Bundle as a Manifold

Facts

- The canonical map $\pi : TM \rightarrow M$ is such that $\pi(v) = p$ if $v \in T_p M$. It is onto.
- Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart for M . Then

$$\begin{aligned}\phi \circ \pi \circ \tilde{\phi}^{-1}(r^1, \dots, r^n, v^1, \dots, v^n) \\ = \phi \circ \pi \left[\phi^{-1}(r^1, \dots, r^n), \sum v^i \partial / \partial x^i \right] \\ = \phi \circ \phi^{-1}(r^1, \dots, r^n) = (r^1, \dots, r^n).\end{aligned}$$

- As (U, ϕ) and $(TU, \tilde{\phi})$ are charts this shows that π is C^∞ .
- By the converse of the submersion theorem (exercise!) this also shows that π is a submersion.

Proposition

The canonical projection $\pi : TM \rightarrow M$ is a surjective submersion.

Definition

A *vector bundle of rank r* over a manifold M is a smooth manifold E together with a surjective smooth map $\pi : E \rightarrow M$ such that:

- (i) For every $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a vector space of dimension r .
- (ii) For each $p \in M$ there is an open neighborhood U of p in M and a diffeomorphism $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^r$ (called *trivialization of E over U*) such that
 - $\pi \circ \phi(q, \xi^1, \dots, \xi^r) = q$ for all $q \in U$ and $(\xi^1, \dots, \xi^r) \in \mathbb{R}^r$.
 - For each $q \in U$, the restriction of ϕ to E_q is a vector space isomorphism from E_q onto $\{q\} \times \mathbb{R}^r$.

Remarks

- We sometimes write a vector bundle $E \xrightarrow{\pi} M$.
- We may also think of a vector bundle as a triple (E, M, π) . In this picture E is called the *total space*, M is called the *base space*, and π is called the *projection*.

Remark

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle and S a regular submanifold in M . Then $\pi^{-1}(S) \xrightarrow{\pi|_S} S$ is a smooth vector bundle over S denoted $E|_S$ and called the *restriction of E to S* .

Vector Bundles

Example

- A *trivial vector bundle* is of the form $E = M \times \mathbb{R}^r$.
- In this case the projection $\pi : M \times \mathbb{R}^r \rightarrow M$ is just the projection on the first factor.

Example

- The tangent bundle TM is a vector bundle of rank n .
- If (U, x^1, \dots, x^n) is a chart, then a trivialization of TM over U is the map $\psi : TU \rightarrow U \times \mathbb{R}^n$ given by

$$\psi \left(\sum v^i \frac{\partial}{\partial x^i} \Big|_p \right) = (p, v^1, \dots, v^n), \quad p \in U, \quad v^i \in \mathbb{R}.$$

In particular, $(\phi \times 1_{\mathbb{R}^n}) \circ \psi = \tilde{\phi}$.

Remark

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle. Suppose that $(U, \psi) = (U, x^1, \dots, x^n)$ is a chart for M and we have local trivialization,

$$\phi : E|_U \longrightarrow U \times \mathbb{R}^r, \quad \phi(\xi) = (\pi(\xi), c^1(\xi), \dots, c^r(\xi)).$$

Then $(\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi : E|_U \rightarrow \psi(U) \times \mathbb{R}^r$ is a diffeomorphism, year

$$\begin{aligned} (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi &= (\psi \times \mathbb{1}_{\mathbb{R}^r})(\pi, c^1, \dots, c^r) \\ &= (x^1, \dots, x^1, c^1, \dots, c^r). \end{aligned}$$

In particular, $(\pi^{-1}(U), (\psi \times \mathbb{1}_{\mathbb{R}^r}) \circ \phi)$ is a chart for E . We call x^1, \dots, x^n the *base coordinates* and c^1, \dots, c^n the *fiber coordinates*

Vector Bundles

Definition (Bundle Maps)

Let $\pi_E : E \rightarrow M$ and $\pi_F : F \rightarrow N$ be smooth vector bundles. A *bundle map* from E to F is given by a pair of smooth maps (f, \tilde{f}) , $f : M \rightarrow N$, $\tilde{f} : E \rightarrow F$ such that:

(i) $\pi_F \circ \tilde{f} = f \circ \pi_E$, i.e., we have a commutative diagram,

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & F \\ \pi_E \downarrow & & \downarrow \pi_F \\ M & \xrightarrow{f} & N'. \end{array}$$

(ii) For every $p \in M$, the map \tilde{f} restricts to a linear map $\tilde{f}_p : E_p \rightarrow F_{f(p)}$.

Vector Bundles

Example

Any smooth map $f : M \rightarrow N$ gives rise to a bundle map (f, \tilde{f}) from TM to TN with $\tilde{f} = f_*$. Namely,

$$\tilde{f}(v) = f_{*,p}(v) \quad p \in M, v \in T_p M.$$

Remarks

- The smooth vector bundles define a category where the objects are smooth vector bundles and the morphisms are bundle maps.
- From this point of view, the tangent bundle construction defines a functor from the category of smooth manifolds to the category of smooth vector bundles.

Remark

- We may also consider the category of vector bundles over a fixed manifold M .
- In this case the morphisms are bundle maps (f, \tilde{f}) with $f = \mathbb{1}_M$.

Smooth Sections

Definition (Section of a Vector Bundle)

Let $E \xrightarrow{\pi} M$ be a smooth vector bundle.

- A *section* of E is any map $s : M \rightarrow E$ such that $\pi \circ s = \mathbb{1}_M$, i.e., $s(p) \in E_p$ for all $p \in M$,
- A *smooth section* is a section which is smooth as a map from M to E .

Remarks

- The set of smooth sections of E is denoted $\Gamma(E)$ or $\Gamma(M, E)$.
- If U is an open subset of M , we denote by $\Gamma(U, E)$ the set of smooth sections of $E|_U$.
- Sections of $E|_U$ are called *local sections*, whereas sections defined on the entire manifold M are called *global sections*.

Definition (Vector Field)

- A *vector field* is a section of the tangent bundle TM .
- A *smooth vector field* is a smooth section of TM .

Remark

In other words, a vector field $X : M \rightarrow TM$ assigns to each $p \in M$ a tangent vector $X_p \in T_p M$.

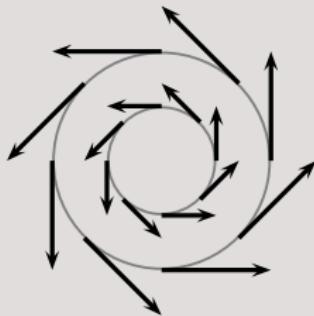
Smooth Sections

Example

On \mathbb{R}^2

$$X_{(x,y)} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = \langle -y, x \rangle$$

is a smooth vector field on \mathbb{R}^2 .



Smooth Sections

Proposition (Proposition 12.9)

Let E be a vector bundle over M . Then its set of smooth sections $\Gamma(E)$ is a module over the ring $C^\infty(M)$ with respect to the addition and scalar multiplication given by

$$(s_1 + s_2)(p) = s_1(p) + s_2(p), \quad s_i \in \Gamma(E), \quad p \in M,$$
$$(fs)(p) = f(p)s(p), \quad f \in C^\infty(M), \quad s \in \Gamma(E), \quad p \in M.$$

Remarks

- Here $s_1(p) + s_2(p)$ and $f(p)s(p)$ make sense as elements of the fiber E_p , because E_p is a vector space.
- If U is an open set, then $\Gamma(U, E)$ is a module over $C^\infty(U)$.

Smooth Frames

Definition (Frames of Vector Bundles)

Let E be a smooth vector bundle of rank r over M .

- A *frame* of E over an open $U \subset M$ is given by sections s_1, \dots, s_r such that $\{s_1(p), \dots, s_r(p)\}$ is a basis of the fiber E_p for every $p \in U$.
- We say that the frame $\{s_1, \dots, s_r\}$ is *smooth* when the sections s_1, \dots, s_r are smooth.

Remarks

- A frame of the tangent bundle is called a *tangent frame*, or simply a *frame*.
- For instance, $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is a smooth tangent frame over \mathbb{R}^2 .

Example

Let e_1, \dots, e_r be the canonical basis of \mathbb{R}^r . For $i = 1, \dots, r$, define $\tilde{e}_i : M \rightarrow M \times \mathbb{R}^r$ by

$$\tilde{e}_i(p) = (p, e_i), \quad p \in M.$$

- Each map \tilde{e}_i is a smooth section of the trivial bundle $M \times \mathbb{R}^r$.
- If $p \in M$, then $\{\tilde{e}_1(p), \dots, \tilde{e}_r(p)\}$ is a basis of $\{p\} \times \mathbb{R}^r$.

Therefore, $\{\tilde{e}_1, \dots, \tilde{e}_r\}$ is a smooth frame of $M \times \mathbb{R}^r$ over M .

Smooth Frames

Example (Frame of a trivialization)

Suppose E is a smooth vector bundle of rank r over M . Let $\phi : E|_U \rightarrow U \times \mathbb{R}^r$ be a trivialization over an open $U \subset M$.

- From the previous example $\{\tilde{e}_1, \dots, \tilde{e}_r\}$ is a smooth frame of $U \times \mathbb{R}^r$ over U .
- As ϕ is smooth, $t_i = \phi^{-1} \circ \tilde{e}_i$ is a smooth map from U to $E|_U$.
- If $p \in U$, then $t_i(p) = \phi(\tilde{e}_i(p)) = \phi(p, e_i) \in E_p$, so t_i is a smooth section of E .
- The trivialization ϕ induces a linear isomorphism from E_p to $\{p\} \times \mathbb{R}^r$. It pullbacks the basis $\{\tilde{e}_1(p), \dots, \tilde{e}_r(p)\}$ of $\{p\} \times \mathbb{R}^r$ to $\{t_1(p), \dots, t_r(p)\}$, so the latter is a basis of E_p .
- Therefore, $\{t_1, \dots, t_r\}$ is a smooth frame of E over U . It is called the *frame of the trivialization* (U, ϕ) .

Smooth Frames

Facts

Let s be a section of E over U . If $p \in U$, then $s(p) \in E_p$ and $\{t_1(p), \dots, t_r(p)\}$ is a basis of E_p . Thus, we may write

$$s(p) = \sum b^i(p)t_i(p), \quad b^i(p) \in \mathbb{R}.$$

- If the coefficients $b_i(p)$ depends smoothly on p , then s is smooth.
- Conversely, if s is a smooth section, then $\phi \circ s : U \rightarrow U \times \mathbb{R}^r$ is a smooth map.
- If $p \in U$, then $\phi \circ s(p) = \phi \left[\sum b^i(p)t_i(p) \right] = \sum b^i(p)\phi[t_i(p)]$.
- As $\phi[t_i(p)] = \phi[\phi^{-1}(\tilde{e}_i(p))] = \tilde{e}_i(p) = (p, e_i)$, we get

$$\phi \circ s(p) = \sum b^i(p)(p, e_i) = (p, b^1(p), \dots, b^r(p)).$$

- As $\phi \circ s$ is a smooth map, the components $b^1(p), \dots, b^r(p)$ must be smooth functions.

Smooth Frames

From the previous slide we obtain:

Lemma (Lemma 12.11)

Let $\phi : E|_U \rightarrow U \times \mathbb{R}^r$ be a trivialization of E over an open $U \subset M$ with frame $\{t_1, \dots, t_n\}$. A section $s = \sum b^i t_i$ of E over U is smooth if and only if b^1, \dots, b^r are smooth functions.

More generally, we have:

Proposition (Proposition 12.12; see Tu's book)

Let $\{s_1, \dots, s_r\}$ be a smooth frame of E over an open $U \subset M$. A section $s = \sum c^i s_i$ of E over U is smooth if and only if c^1, \dots, c^r are smooth functions.

Corollary

If $\{s_1, \dots, s_r\}$ is a smooth frame of E over an open $U \subset M$, then this is a $C^\infty(U)$ -basis of the $C^\infty(U)$ -module $\Gamma(U, E)$.

Smooth Frames

Remark

Let $\{s_1, \dots, s_r\}$ be a smooth frame of E over an open $U \subset M$.

Define $\sigma : U \times \mathbb{R}^r \rightarrow E|_U$ by

$$\sigma(p, \xi^1, \dots, \xi^r) = \sum \xi^i s_i(p), \quad p \in U, \quad \xi^i \in \mathbb{R}.$$

- The map σ is a smooth bijection that induces a linear isomorphism from $\{p\} \times \mathbb{R}^r$ onto E_p .
- It can be shown that the inverse map $\phi = \sigma^{-1} : E|_U \rightarrow U \times \mathbb{R}^r$ is smooth, and so this is a trivialization of E over U .
- The frame of (ϕ, U) is $\{s_1, \dots, s_r\}$, since

$$\phi^{-1}(\tilde{e}_i(p)) = \sigma(e_i(p)) = s_i(p).$$

It follows that we have a one-to-one correspondence between trivializations and smooth frames.

Smooth Frames

Example

Let (U, x^1, \dots, x^n) be a local chart for M .

- We know that (U, x^1, \dots, x^n) gives rise to the trivialization $\psi : TU \rightarrow U \times \mathbb{R}^n$ given by

$$\psi(v) = (p, v^1, \dots, v^n) \quad \text{if } v = \sum v^i \frac{\partial}{\partial x^i} \Big|_p \in T_p M, \quad p \in U.$$

- In particular, as $\psi\left(\frac{\partial}{\partial x^i} \Big|_p\right) = (p, e_i) = \tilde{e}_i(p)$, we have

$$t_i(p) = \psi^{-1}(\tilde{e}_i(p)) = \frac{\partial}{\partial x^i} \Big|_p.$$

Thus, $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ is the frame of the trivialization (U, ψ) . In particular, this is a smooth tangent frame over U .